

# Stability and Instability Linearization for Integral Equations

Hideaki Matsunaga<sup>1</sup>, Satoru Murakami<sup>2</sup> and Minh Van Nguyen<sup>3</sup>

<sup>1</sup>*Department of Mathematical Sciences, Osaka Prefecture University, Sakai 599-8531, Japan*

<sup>2</sup>*Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan*

<sup>3</sup>*Department of Mathematics, University of West Georgia, 1601 Maple St, Carrollton, GA 30118, USA*

## 1 Introduction

The principle of linearized stability has been widely used as an effective tool for the stability and instability analysis of autonomous equations such as ordinary differential equations, functional differential equations and others; e.g., see [3, 4, 5, 7, 8, 9, 10]. Recently, Diekmann and Gyllenberg [2] have established the principle of linearized stability for autonomous integral equations (with infinite delay). Motivated by [2], in this paper we treat nonlinear integral equation with infinite delay of the form

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + g(t, x_t) \quad (1)$$

with time-dependent "high order term"  $g$  satisfying  $g(t, 0) \equiv 0$ , and under some conditions on  $g$  we establish the stability result (Theorem 4) and the instability result (Theorem 5) for the zero solution of Eq. (1) in terms of stability properties for the associated linear equation

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds. \quad (2)$$

As stated in Section 3 (Corollary), the stability analysis for an equilibrium point of autonomous integral equations can be reduced to the one for the zero solution of Eq. (1) with an appropriate  $g$  which satisfies the conditions imposed in Theorem 4 and Theorem 5; hence our results may be considered as an extension of the principle of linearized stability for autonomous equations to nonautonomous equations.

A key tool for the establishment of our results is the variation-of-constants formula (VCF) in the phase space for integral equations in [14]. In this paper, combining VCF with an idea in Henry's book [10, Chapter 5] we will overcome several difficulties which arise from "nonautonomousness" of the equations.

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*E-mail addresses:* hideaki@ms.osakafu-u.ac.jp (H. Matsunaga), murakami@youhei.xmath.ous.ac.jp (S. Murakami), vnguyen@westga.edu (N. V. Minh)

## 2 Preparatory results for integral equations and abstract equations

In this section, following [14, Sections 2–5], we summarize several results which are essentially used in the development of the paper.

Let  $\mathbb{N}$ ,  $\mathbb{R}^-$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}$  and  $\mathbb{C}$  be the sets of natural numbers, nonpositive real numbers, nonnegative real numbers, real numbers and complex numbers, respectively. For an  $m \in \mathbb{N}$ , we denote by  $\mathbb{C}^m$  (resp.  $\mathbb{R}^m$ ) the space of all  $m$ -column vectors, whose components are complex (resp. real) numbers, with the Euclidean norm  $|\cdot|$ . For any  $m \times m$  matrix  $M$ , the norm  $\|M\|$  is the operator norm of  $M$  which is defined as  $\|M\| = \sup\{|M\alpha|/|\alpha| : \alpha \in \mathbb{C}^m, \alpha \neq 0\}$ .

Let  $\rho$  be a fixed positive constant, and consider the space  $X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$  defined by

$$L^1_\rho := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m) = \{\varphi : \mathbb{R}^- \rightarrow \mathbb{C}^m \mid \varphi(\theta)e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-\}$$

(or more precisely, the equivalent classes of these functions) equipped with norm

$$(\|\varphi\| :=) \|\varphi\|_{1,\rho} = \int_{-\infty}^0 |\varphi(\theta)|e^{\rho\theta} d\theta \quad (\forall \varphi \in X).$$

Clearly,  $(X, \|\cdot\|)$  is a (complex) Banach space.

For any function  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  and  $t < a$ , we define a function  $x_t : \mathbb{R}^- \rightarrow \mathbb{C}^m$  by  $x_t(\theta) = x(t + \theta)$  for  $\theta \in \mathbb{R}^-$ . Let us consider an abstract equation

$$x(t) = F(t, x_t), \tag{3}$$

where  $F : [b, \infty) \times X \rightarrow \mathbb{C}^m$  is a continuous function. For any given  $\varphi \in X$  and  $\sigma \in [b, \infty)$ , we treat the initial value problem for Eq. (3) with the initial condition

$$x_\sigma \equiv \varphi \text{ on } \mathbb{R}^-, \text{ that is, } x(\sigma + \theta) = \varphi(\theta) \text{ for all } \theta \in \mathbb{R}^-. \tag{4}$$

Throughout this paper, we say that a function  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  is a solution of the initial value problem (3)-(4) on  $(\sigma, a)$  if  $x$  satisfies the following three conditions (cf. [6, Sections 2.3, 12.2]);

- (i)  $x_\sigma \equiv \varphi$  on  $\mathbb{R}^-$ ;
- (ii)  $x \in L^1_{loc}[\sigma, a)$ ; that is,  $x$  is locally integrable on  $[\sigma, a)$ ;
- (iii)  $x(t) = F(t, x_t)$  for  $t \in (\sigma, a)$ .

Observing that  $x_t$  is continuous on  $[\sigma, a)$  as an  $X$ -valued function of  $t$  whenever  $x$  satisfies (i)–(ii), one can deduce the following lemma:

**Lemma 1.** *A function  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  is a solution of the initial value problem (3)-(4) on  $(\sigma, a)$  if and only if  $x$  satisfies the conditions (i) and (iii) together with the condition*

(ii)\*  *$x$  is continuous on  $(\sigma, a)$ , the limit  $x(\sigma^+) := \lim_{t \rightarrow +0} x(\sigma + t)$  exists, and the relation  $x(\sigma^+) = F(\sigma, \varphi)$  holds true.*

Let  $x$  be a solution of the initial value problem (3)-(4) on  $(\sigma, a)$ . If there exists another solution  $z$  of (3)-(4) on  $(\sigma, c)$  with some  $c > a$  which satisfies  $x(t) \equiv z(t)$  on  $(\sigma, a)$ , the solution  $x$  is said to be extendable, and the solution  $z$  is called an extension of  $x$ . If  $x$  has no extensions of (3)-(4), then  $x$  is called a noncontinuable solution of (3)-(4).

For any  $\varepsilon > 0$  and  $(\sigma, \varphi) \in \mathbb{R} \times X$ , we set

$$O_\varepsilon(\sigma, \varphi) := \{(t, \psi) \in \mathbb{R} \times X \mid |t - \sigma| < \varepsilon, \|\psi - \varphi\|_{1,\rho} < \varepsilon\}.$$

Now, let  $F : [b, \infty) \times X \rightarrow \mathbb{C}^m$  be any continuous function satisfying the (local) Lipschitz condition (with respect to the second variable); that is, for any  $(\sigma, \varphi) \in [b, \infty) \times X$  there exist positive constants  $\varepsilon := \varepsilon(\sigma, \varphi)$  and  $l := l(\sigma, \varphi)$  such that

$$|F(t, \psi_1) - F(t, \psi_2)| \leq l \|\psi_1 - \psi_2\|_{1,\rho} \quad (5)$$

whenever  $(t, \psi_i) \in O_\varepsilon(\sigma, \varphi) \cap ([b, \infty) \times X)$  for  $i = 1, 2$ . Utilizing Lemma 1 and applying the contraction mapping principle as well as the Zorn lemma, one can establish the result on the existence and uniqueness of (local) solutions for the initial value problem (3)-(4), as well as results on extendable solutions, noncontinuable solutions and globally defined solutions.

**Proposition 1.** ([14, Propositions 1-3]) *Assume that  $F : [b, \infty) \times X \rightarrow \mathbb{C}^m$  is a continuous function which satisfies the condition (5). Then, for any given  $(\sigma, \varphi) \in [b, \infty) \times X$  the following statements hold true:*

- (i) *There exists a  $\delta := \delta(\sigma, \varphi) > 0$  with the property that there is one and only one solution of (3)-(4) on  $(\sigma, \sigma + \delta)$ ;*
- (ii) *If  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  be a solution of (3)-(4) on  $(\sigma, a)$  with  $\sigma < a < \infty$  and if  $\sup_{\sigma < t < a} |x(t)| < \infty$ , then, the limit  $x(a^-) := \lim_{t \rightarrow +0} x(a - t)$  exists, and  $x$  is extended to a solution of (3)-(4) on  $(\sigma, a + \delta_1)$  for some  $\delta_1 > 0$ ;*
- (iii) *If there exist nonnegative continuous functions  $l(\cdot)$  and  $h(\cdot)$  such that*

$$|F(t, \varphi)| \leq l(t) \|\varphi\|_{1,\rho} + h(t), \quad \forall t \geq b, \varphi \in X, \quad (6)$$

*then, the (noncontinuable) solution of (3)-(4) is globally defined.*

Let us consider functional equations of the form

$$x(t) = L(x_t) + p(t), \quad t > \sigma, \quad (7)$$

where  $L : X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m) \rightarrow \mathbb{C}^m$  is a bounded linear operator and  $p \in C(\mathbb{R}^+; \mathbb{C}^m)$ , the space of all continuous functions mapping  $\mathbb{R}^+$  into  $\mathbb{C}^m$ . Given  $\varphi \in X$  and  $\sigma \geq 0$ , there exists a unique globally defined solution, say  $x$ , of Eq. (7) satisfying the initial condition  $x_\sigma \equiv \varphi$  on  $\mathbb{R}^-$ . Indeed, if one defines  $F : \mathbb{R}^+ \times X \rightarrow \mathbb{C}^m$  by  $F(t, \psi) = L(\psi) + p(t)$  for  $(t, \psi) \in \mathbb{R}^+ \times X$ , then  $F$  satisfies the Lipschitz condition as well as the condition (6); and hence the existence result on the (unique) globally defined solutions for the initial value problems is a direct consequence of Proposition 1–(iii). In what follows, we call  $x$  the solution of Eq. (7) through  $(\sigma, \varphi)$ , and write it as  $x(\cdot; \sigma, \varphi, p)$ . If  $\varphi = \psi$  in  $X$  from the uniqueness obtained in Proposition 1 it follows that  $x_t(\sigma, \varphi, p) = x_t(\sigma, \psi, p)$  in  $X$  for any  $t \geq \sigma$  whenever  $\varphi = \psi$  in  $X$ , and consequently  $x_t(\sigma, \varphi, p)$  can be considered as a function mapping  $X$  into  $X$ .

Now, for any  $t \geq 0$  and  $\varphi \in X$ , we define  $T(t)\varphi \in X$  by

$$(T(t)\varphi)(\theta) := x_t(\theta; 0, \varphi, 0) = \begin{cases} x(t + \theta; 0, \varphi, 0), & -t < \theta \leq 0 \\ \varphi(t + \theta), & \theta \leq -t. \end{cases}$$

As noted in the preceding paragraph,  $T(t)\varphi = T(t)\psi$  in  $X$  whenever  $\varphi = \psi$  in  $X$ ; in other words,  $T(t)$  defines a mapping on  $X$ . Indeed,  $T(t)$  is a bounded linear operator on  $X$ . Recall that  $x(\cdot; 0, \varphi, 0)$  is the solution of the homogeneous linear equation

$$x(t) = L(x_t) \quad (8)$$

through  $(0, \varphi)$ . We call  $T(t)$  the “solution operator” for Eq. (8). In fact,  $\{T(t)\}_{t \geq 0}$  is a strongly continuous semigroup of bounded linear operators on  $X$ , which is called the solution semigroup for Eq. (8).

Recall that a family of bounded linear operators  $\{T(t)\}_{t \geq 0}$  in  $X$  is said to be a strongly continuous semigroup of (bounded) linear operators in  $X$  if it satisfies

- (i)  $T(0) = Id$ ;
- (ii)  $T(t)T(s) = T(t + s)$  for all  $t \geq 0, s \geq 0$ ;
- (iii)  $\lim_{t \rightarrow 0^+} T(t)x = x$  for each  $x \in X$ .

The generator  $A$  of a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  is defined to be a closed linear operator with dense domain

$$\mathcal{D}(A) := \{\varphi \in X \mid \lim_{h \rightarrow 0^+} (1/h)(T(h)\varphi - \varphi) \text{ exists in } X\}$$

in which

$$A\varphi := \lim_{h \rightarrow 0^+} (1/h)(T(h)\varphi - \varphi), \quad \varphi \in \mathcal{D}(A).$$

For the number  $\rho$  we set

$$\mathbb{C}_{-\rho} := \{z \in \mathbb{C} \mid \operatorname{Re} z > -\rho\},$$

and consider a function  $\omega_\lambda$  defined by

$$\omega_\lambda(\theta) := e^{\lambda\theta}, \quad \forall \theta \leq 0$$

for each  $\lambda \in \mathbb{C}_{-\rho}$ . One can easily check that if  $\lambda \in \mathbb{C}_{-\rho}$  and  $\alpha \in \mathbb{C}^m$ , then, the function  $\omega_\lambda \alpha$  defined by  $(\omega_\lambda \alpha)(\theta) = \omega_\lambda(\theta)\alpha$ ,  $\theta \leq 0$  belongs to the space  $X$  with norm  $\|\omega_\lambda \alpha\| \leq |\alpha|/(\operatorname{Re} \lambda + \rho)$ . In particular,  $\omega_\lambda \mathbf{e}_i \in X$  and hence  $L(\omega_\lambda \mathbf{e}_i) \in \mathbb{C}^m$  for each  $i = 1, \dots, m$ , where  $\mathbf{e}_i$  is the vector in  $\mathbb{C}^m$  whose  $j$ -th component is 1 if  $j = i$  and 0 otherwise. Notice that  $E := (\mathbf{e}_1, \dots, \mathbf{e}_m)$  is the  $m \times m$  unit matrix. Set  $L(\omega_\lambda E) = (L(\omega_\lambda \mathbf{e}_1), \dots, L(\omega_\lambda \mathbf{e}_m))$ . Then  $L(\omega_\lambda E)$  is an  $m \times m$  matrix, and it satisfies the relation

$$L(\omega_\lambda E)\alpha = L(\omega_\lambda \alpha), \quad \forall \alpha \in \mathbb{C}^m.$$

Let us define a subset  $\tilde{X}$  of  $X$  by

$$\begin{aligned} \tilde{X} = \{ \tilde{\varphi} \in X \mid \tilde{\varphi} \text{ is locally absolutely continuous on } \mathbb{R}^-, \\ (d/d\theta)\tilde{\varphi} \in X \text{ and } \tilde{\varphi}(0) = L(\tilde{\varphi}) \}. \end{aligned}$$

With the above notations, we have the following result on a characterization of the generator  $A$  and the spectrum  $\sigma(A)$ .

**Proposition 2.** ([14, Propositions 4–5]) *The generator  $A$  of the solution semigroup for Eq. (8) and its domain  $\mathcal{D}(A)$  are given by*

$$\begin{aligned} \mathcal{D}(A) &= \{ \varphi \in X \mid \varphi(\theta) = \tilde{\varphi}(\theta) \text{ a.e. } \theta \in \mathbb{R}^- \text{ for some } \tilde{\varphi} \in \tilde{X} \}, \\ A\varphi &= (d/d\theta)\tilde{\varphi}, \quad \varphi \in \mathcal{D}(A). \end{aligned}$$

Also, the relation holds true:

$$\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{ \lambda \in \mathbb{C}_{-\rho} \mid \det(E - L(\omega_\lambda E)) = 0 \}.$$

In the remainder of this paper, we always assume (without stating explicitly) that  $K$  is a (measurable)  $m \times m$  matrix valued function with complex components satisfying the conditions

$$\|K\|_{1,\rho,+} := \int_0^\infty \|K(\tau)\| e^{\rho\tau} d\tau < \infty, \quad (9)$$

$$\|K\|_{\infty,\rho,+} := \operatorname{ess\,sup}\{\|K(\tau)\| e^{\rho\tau} \mid \tau \geq 0\} < \infty; \quad (10)$$

here,  $\rho$  is a (fixed) positive constant. In what follows the notations  $\|\cdot\|_{1,\rho,+}$  and  $\|\cdot\|_{\infty,\rho,+}$  will often be shortened as  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$ , respectively. To the function  $K$ , let us associate a function  $L$  defined on the space  $X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$  by

$$L(\varphi) = \int_{-\infty}^0 K(-\theta)\varphi(\theta)d\theta, \quad \forall \varphi \in X.$$

Then,  $L : X \rightarrow \mathbb{C}^m$  is a bounded linear operator with norm  $\|L\| \leq \|K\|_\infty$ , because of the inequality

$$\begin{aligned} |L(\varphi)| &\leq \int_{-\infty}^0 \|K(-\theta)\| e^{-\rho\theta} |\varphi(\theta)| e^{\rho\theta} d\theta \\ &\leq \|K\|_{\infty,\rho,+} \int_{-\infty}^0 |\varphi(\theta)| e^{\rho\theta} d\theta = \|K\|_\infty \|\varphi\|_{1,\rho} \end{aligned}$$

for any  $\varphi \in X$ .

We now consider linear integral equations of the form

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t), \quad t > \sigma, \quad (11)$$

where  $p$  is an element in  $C(\mathbb{R}; \mathbb{C}^m)$  (the space of all continuous functions mapping  $\mathbb{R}$  into  $\mathbb{C}^m$ ). Eq. (11) can be viewed as the functional equation (7) on the space  $X$ . As in the previous paragraph, one can conclude that given  $\varphi \in X$  there exists a unique globally defined solution  $x$  of Eq. (11) satisfying  $x_\sigma \equiv \varphi$  on  $\mathbb{R}^-$ , that is,  $x$  satisfies Eq. (11) on  $(\sigma, \infty)$  together with the initial condition  $x(\sigma + \theta) = \varphi(\theta)$  for all  $\theta \leq 0$ . In the following, as a notation of the solution for Eq. (11) we will employ the same notation  $x(\cdot; \sigma, \varphi, p)$  as the one for Eq. (7). Similarly, we treat the solution semigroup and its generator with the notations  $\{T(t)\}_{t \geq 0}$  and  $A$  for the homogeneous linear integral equation

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds, \quad t > 0. \quad (12)$$

In particular, by virtue of Proposition 2 we get the following result on the spectrum  $\sigma(A)$  of the generator  $A$  of the solution semigroup for Eq. (12):

$$\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} \mid \det \Delta(\lambda) = 0\}, \quad (13)$$

where

$$\Delta(\lambda) := E - \int_0^\infty K(t)e^{-\lambda t} dt \quad \text{for } \operatorname{Re} \lambda > -\rho.$$

Below we will establish a decomposition of the phase space  $X$  corresponding to a set of several eigenvalues of  $A$  that does not intersect its essential spectrum  $\operatorname{ess}(A)$ . Recall that the essential spectrum  $\operatorname{ess}(T)$  of a closed linear operator  $T : X \rightarrow X$  with dense domain  $\mathcal{D}(T)$  is the set of all  $\lambda$  in  $\sigma(T)$ , for which at least one of the following holds;

- (i) the set  $\mathcal{R}(T - \lambda I) := \{(T - \lambda I)\varphi \mid \varphi \in \mathcal{D}(T)\}$  is not closed;
- (ii) the point  $\lambda$  is a limit point of  $\sigma(T)$ ;
- (iii) the generalized eigenspace  $\mathcal{G}_\lambda(T) := \cup_{k \geq 1} \mathcal{N}((T - \lambda I)^k)$  for  $\lambda$  is infinite dimensional;

where  $\mathcal{N}((T - \lambda I)^k)$  is the null set of the operator  $(T - \lambda I)^k$ ; for details, see, e.g., [1, 11, 16]. A complex number  $\lambda \in \sigma(T) \setminus \text{ess}(T)$  is called a normal eigenvalue of  $T$ . If  $\lambda$  is a normal eigenvalue, then it is in  $P_\sigma(T)$  with finite dimensional generalized eigenspace  $\mathcal{N}((T - \lambda I)^k)$  for some natural number  $k$ , and  $X$  can be represented as the direct sum of  $\mathcal{N}((T - \lambda I)^k)$  and  $\mathcal{R}((T - \lambda I)^k)$ ;  $X = \mathcal{N}((T - \lambda I)^k) \oplus \mathcal{R}((T - \lambda I)^k)$ . We define the essential spectral radius of  $T$  by

$$r_e(T) = \sup\{|\lambda| \mid \lambda \in \text{ess}(T)\}.$$

If a bounded linear operator  $U : X \rightarrow X$  is compact, then the relation  $r_e(T + U) = r_e(T)$  holds true; see, e.g., [16].

The following result yields an estimate on the essential spectral radius of the solution operator  $T(t)$  for Eq. (12).

**Proposition 3.** ([14, Theorem 1]) *Assume that the function  $K$  in Eq. (12) satisfies condition (9), and let  $T(t)$  be the solution operator for Eq. (12). Then,*

$$r_e(T(t)) \leq e^{-\rho t}, \quad \forall t \geq 0. \quad (14)$$

By virtue of [16, Chapter 4, Proposition 4.13], the relation

$$\{e^{\lambda t} \mid \lambda \in \text{ess}(A)\} \subset \text{ess}(T(t)), \quad t > 0$$

holds true; consequently (14) gives the following relation concerning the essential spectrum of the generator  $A$  of the solution semigroup  $\{T(t)\}_{t \geq 0}$ ;

$$\sup_{\lambda \in \text{ess}(A)} \text{Re } \lambda \leq -\rho. \quad (15)$$

Let  $c$  be a (fixed) constant such that  $c > -\rho$ . Define

$$\overline{\mathbb{C}}_c := \{z \in \mathbb{C} \mid \text{Re } z \geq c\}.$$

We consider the set  $\sigma(A) \cap \overline{\mathbb{C}}_c =: \Sigma_c^U$ . By virtue of (13) and (15), we see that if  $\lambda_0 \in \Sigma_c^U$ , then  $\lambda_0 \notin \text{ess}(A)$  and  $\det \Delta(\lambda_0) = 0$ . Therefore, since  $\det \Delta(z)$  is an analytic function of  $z$  in the domain  $\mathbb{C}_{-\rho}$ ,  $\Sigma_c^U$  is (at most) a finite set which consists of normal eigenvalues of  $A$ . Then, from the well known result on the strongly continuous semigroup (see, e.g., [11, Section 5.3], [16, Chapter 4]) or periodic evolutionary process (see, e.g., [7, 10, 11]) one can get the following result on the decomposition of the phase space  $X$ :

**Theorem 1.** ([14, Theorem 2]) For any real  $c > -\rho$ , let  $\Sigma_c^U := \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq c\}$ . Then,  $X$  is decomposed as a direct sum of closed subspaces  $U$  and  $S$

$$X = U \oplus S$$

with the following properties:

- (i)  $\dim U < \infty$ ;
- (ii)  $T(t)U \subset U$ ,  $T(t)S \subset S$  ( $\forall t \geq 0$ );
- (iii)  $\sigma(A|_U) = \Sigma_c^U$ ,  $\sigma(A|_{S \cap \mathcal{D}(A)}) = \sigma(A) \setminus \Sigma_c^U =: \Sigma_c^S$ ;
- (iv)  $T^U(t) := T(t)|_U$  is extendable for  $t \in (-\infty, \infty)$ , as a group of bounded linear operators on  $U$ ;
- (v)  $T^S(t) := T(t)|_S$  is a strongly continuous semigroup of bounded linear operators on  $S$ , and its generator is identical with the operator  $A|_{S \cap \mathcal{D}(A)}$ ;
- (vi) for sufficiently small  $\varepsilon > 0$  there exists a  $\gamma(\varepsilon) > 0$  such that

$$\begin{aligned} \|T^U(t)\| &\leq \gamma(\varepsilon)e^{(c-\varepsilon)t}, \quad \forall t \leq 0 \\ \|T^S(t)\| &\leq \gamma(\varepsilon)e^{(c+\varepsilon)t}, \quad \forall t \geq 0. \end{aligned}$$

We now introduce a continuous function  $\Gamma^n : \mathbb{R}^- \rightarrow \mathbb{R}^+$  for each natural number  $n$  which is of compact support with *support*  $\Gamma^n \subset [-1/n, 0]$  and satisfies  $\int_{-\infty}^0 \Gamma^n(\theta) d\theta = 1$ . Notice that  $\Gamma^n \beta \in X$  for any  $\beta \in \mathbb{C}^m$ . Let us recall that  $x(\cdot; \sigma, \varphi, p)$  is the (unique) solution of Eq. (16)

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t), \quad t > \sigma \quad (16)$$

through  $(\sigma, \varphi)$ ; here  $\varphi \in X$ . We will establish a representation formula for  $x_t(\sigma, \varphi, p)$  (variation-of-constants formula) in the space  $X$  by using  $T(t)$ ,  $\varphi$  and  $p$ .

**Theorem 2.** ([14, Theorem 3]) Let  $p \in C([\sigma, \infty); \mathbb{C}^m)$ . Then

$$x_t(\sigma, \varphi, p) = T(t-\sigma)\varphi + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p(s))ds, \quad \forall t \geq \sigma \quad (17)$$

in  $X$ .

Let us consider a subset  $\bar{X}$  consisting of all elements  $\phi \in X$  which are continuous on  $[-\varepsilon_\phi, 0]$  for some  $\varepsilon_\phi > 0$ , and set

$$X_0 = \{\varphi \in X \mid \varphi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \bar{X}\}.$$

For any  $\varphi \in X_0$ , we define the value of  $\varphi$  at zero by

$$\varphi[0] = \phi(0),$$

where  $\phi$  is an element belonging to  $\bar{X}$  satisfying  $\phi = \varphi$  a.e. on  $\mathbb{R}^-$ . We note that the value  $\varphi[0]$  is well-defined; that is, it does not depend on the particular choice of  $\phi$  since  $\phi(0) = \psi(0)$  for any other  $\psi \in \bar{X}$  such that  $\phi = \psi$  a.e. on  $\mathbb{R}^-$ . It is clear that  $X_0$  is a normed space equipped with norm

$$\|\varphi\|_{X_0} := \|\varphi\|_X + |\varphi[0]|, \quad \forall \varphi \in X_0.$$

Also, by virtue of Lemma 1, the solution  $x(\cdot; \sigma, \psi, p)$  of Eq. (16) through  $(\sigma, \psi) \in \mathbb{R} \times X$  satisfies the relation  $x_t(\sigma, \psi, p) \in X_0$  with  $(x_t(\sigma, \psi, p))[0] = x(t; \sigma, \psi, p)$  whenever  $t > \sigma$ .

If a function  $u : \mathbb{R} \rightarrow \mathbb{C}^m$  satisfies the relations  $u_\sigma \in X$  and  $u(t) \equiv x(t; \sigma, u_\sigma, p)$  on  $(\sigma, \infty)$  for any  $\sigma \in \mathbb{R}$ , we call  $u$  a solution of Eq. (16) on  $\mathbb{R}$ . Of course, if  $u$  is a solution of Eq. (16) on  $\mathbb{R}$ , then it satisfies Eq. (16) for any  $t \in \mathbb{R}$ ; that is,

$$u(t) = \int_{-\infty}^t K(t-s)u(s)ds + p(t), \quad \forall t \in \mathbb{R}.$$

The following result yields an intimate relation between solutions of Eq. (16) and  $X$ -valued functions satisfying an integral equation which arises from the variation-of-constants formula in the phase space. In particular, the latter part of the theorem will essentially be used for the establishment of Theorem 5 in the next section.

**Theorem 3.** ([14, Theorem 4]) Let  $p \in C(\mathbb{R}; \mathbb{C}^m)$ .

(i) If  $x(t)$  is a solution of Eq. (16) on the entire  $\mathbb{R}$ , then the  $x$ -valued function  $\xi(t) := x_t$  satisfies the relations

- (a)  $\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2$  with  $t \geq \sigma$ , in  $X$ ;
- (b)  $\xi \in C(\mathbb{R}; X_0)$ .

(ii) Conversely, if a function  $\xi : \mathbb{R} \rightarrow X$  satisfies the relation

$$\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma,$$

then

- (c)  $\xi \in C(\mathbb{R}; X_0)$ ;
- (d) if we set

$$u(t) = (\xi(t))[0], \quad \forall t \in \mathbb{R},$$

then  $u \in C(\mathbb{R}; \mathbb{C}^m)$ ,  $u_t = \xi(t)$  (in  $X$ ) for any  $t \in \mathbb{R}$  and  $u$  is a solution of Eq. (16) on  $\mathbb{R}$ .

### 3 Linearized stability and instability property for nonautonomous integral equations

In this section, we consider nonlinear integral equations of the form

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + g(t, x_t), \quad (18)$$

where the function  $K$  satisfies the conditions (9) and (10), and  $g : \mathbb{R} \times X \rightarrow \mathbb{C}^m$  is a continuous function which satisfies condition (5), together with  $g(t, 0) \equiv 0$ . Clearly, the zero function is a solution of Eq. (18). In the following, applying the results stated in the preceding section we will establish the main results of this paper (Theorem 4 and Theorem 5) on the local stability or instability property of the zero solution of Eq. (18) under the condition that the term  $g$  is small (in a sense) on a cylindrical neighborhood of  $R \times \{0\}$  in  $\mathbb{R} \times X$ . In fact, as stated in Corollary below, our results may be viewed as an extension of *the principle of linearized stability* known for various autonomous equations (such as ordinary differential equations, functional differential equations and so on) to nonautonomous integral equations.

Recall that given  $(\sigma, \varphi) \in \mathbb{R} \times X$ , there is a unique (noncontinuable) solution of Eq. (18) through  $(\sigma, \varphi)$  (which we will denote by  $x(\cdot, \sigma, \varphi, g)$ ).

**Theorem 4.** *In addition to the above-mentioned conditions, assume further that  $g(t, \varphi) = o(\|\varphi\|)$  as  $\|\varphi\| \rightarrow 0$ , uniformly for  $t \in \mathbb{R}$ , i.e.*

$$\lim_{\|\varphi\| \rightarrow 0} \left( \sup_{t \in \mathbb{R}} \frac{|g(t, \varphi)|}{\|\varphi\|} \right) = 0. \quad (19)$$

*Then, the zero solution of Eq. (18) is exponentially stable provided  $\det \Delta(z) \neq 0$  for all  $z$  with  $\operatorname{Re} z \geq 0$ ; that is, there exist positive constants  $\delta, M, \nu$  with the property that if  $(\sigma, \varphi) \in \mathbb{R} \times X$  with  $\|\varphi\| < \delta$ , then, the solution  $x(\cdot, \sigma, \varphi, g)$  of Eq. (18) exists on  $(\sigma, \infty)$ , and satisfies the estimate*

$$\|x_t(\sigma, \varphi, g)\| \leq M e^{-\nu(t-\sigma)} \|\varphi\|, \quad \forall t \geq \sigma.$$

**Theorem 5.** *In addition to the above-mentioned conditions, assume further that*

$$|g(t, \varphi) - g(t, \psi)| \leq k(\mu) \|\varphi - \psi\| \quad \text{for } \|\varphi\| \leq \mu < \mu_0, \|\psi\| \leq \mu < \mu_0, \quad (20)$$

*where  $k : (0, \mu_0] \rightarrow (0, \infty)$  ( $\mu_0$  is a constant) is a function such that  $k(\mu) \rightarrow 0$  as  $\mu \rightarrow 0$ . Then, the zero solution of Eq. (18) is unstable provided  $\det \Delta(z_0) = 0$  for some  $z_0$  with  $\operatorname{Re} z_0 > 0$ ; more precisely, there are a constant  $c_0 > 0$  and a sequence  $\{\varphi^{(l)}\}$  in  $X$  with the property that  $\|\varphi^{(l)}\| \rightarrow 0$  as  $l \rightarrow \infty$  and  $\sup_{t \geq 0} \|x_t(0, \varphi^{(l)}, g)\| \geq c_0$  for  $l = 1, 2, \dots$*

**[Proof of Theorem 4]:** Since  $\Sigma_0^U = \emptyset$ , by Theorem 1, we get  $U = \{0\}$  and  $X = S$ . Consequently, there exist constants  $C \geq 1$  and  $w > 0$  such that  $\|T(t)\| \leq Ce^{-wt}$ ,  $\forall t \geq 0$ . Let us take a constant  $\mu_0 > 0$  such that  $\sup_{t \in \mathbb{R}} |g(t, \varphi)| \leq (w/2C)\|\varphi\|$  whenever  $\varphi \in X$  with  $\|\varphi\| \leq \mu_0$ , and set  $\delta = \mu_0/C$ . Let any  $\varphi \in X$  such that  $\|\varphi\| < \delta$  be given, and let  $(\sigma, a)$  be the existence interval of the (noncontinuable) solution  $x(\cdot, \sigma, \varphi, g)$ . We will verify that  $a = \infty$ , that is, the solution  $x(\cdot, \sigma, \varphi, g)$  exists on  $(\sigma, \infty)$  and that the solution satisfies  $\|x_t(\sigma, \varphi, g)\| < \mu_0$  for all  $t \geq \sigma$ . Indeed, if  $a < \infty$ , applying Proposition 1 –(ii) we see that  $\|x_t(\sigma, \varphi, g)\| < \mu_0$  on  $[\sigma, \tau)$  and  $\|x_\tau(\sigma, \varphi, g)\| = \mu_0$  for some  $\tau \in (\sigma, a)$ . Notice that

$$x_t(\sigma, \varphi, g) = T(t - \sigma)\varphi + \lim_{n \rightarrow \infty} \int_\sigma^t T(t - s)(\Gamma^n p(s))ds, \quad \forall t \in [\sigma, \tau],$$

by Theorem 2, where  $p(t) := g(t, x_t(\sigma, \varphi, g))$ . Since  $|p(t)| \leq (w/2C)\|x_t(\sigma, \varphi, g)\|$  on  $[\sigma, \tau]$ , it follows that

$$\|x_t(\sigma, \varphi, g)\| \leq Ce^{-w(t-\sigma)}\|\varphi\| + (w/2) \int_\sigma^t e^{-w(t-s)}\|x_s(\sigma, \varphi, g)\|ds;$$

consequently applying Gronwall's lemma we get  $\|x_t(\sigma, \varphi, g)\| \leq C\|\varphi\|e^{(-w/2)(t-\sigma)} < C\delta = \mu_0$  on  $[\sigma, \tau]$ , which is a contradiction to  $\|x_\tau(\sigma, \varphi, g)\| = \mu_0$ . Thus we must have that  $a = \infty$  and that  $\|x_t(\sigma, \varphi, g)\| < \mu_0$  on  $[\sigma, \infty)$ , as required.

Repeating the above argument and applying Gronwall's lemma again, one can easily deduce the estimate  $\|x_t(\sigma, \varphi, g)\| \leq C\|\varphi\|e^{(-w/2)(t-\sigma)}$ ,  $\forall t \geq \sigma$ ; which implies the desired one with  $M = C$  and  $\nu = w/2$ .  $\square$

**[Proof of Theorem 5]:** In the following, employing almost the same manner as in [10, Theorem 5.1.3], we will establish the theorem by applying Theorem 3.

By the assumption of the theorem, there exists a  $z_0$  such that  $\det \Delta(z_0) = 0$  with  $\operatorname{Re} z_0 > 2c$  for some  $c > 0$ . Without loss of generality, we may assume that  $\det \Delta(z) \neq 0$  for all  $z$  such that  $0 < \operatorname{Re} z \leq 2c$ . Corresponding to the set  $\Sigma_{c/4}^U := \{z \in \sigma(A) \mid \operatorname{Re} z \geq c/4\}$ ,  $X$  is decomposed as  $X = S \oplus U$  with the properties stated in Theorem 1. In particular, since  $\sigma(A|_U) = \{z \in \sigma(A) \mid \operatorname{Re} z > 2c\}$  and  $\sigma(A|_S) = \sigma(A) \setminus \Sigma_{c/4}^U$ , we get the following estimates on the restrictions  $T^S(t)$  and  $T^U(t)$ :

$$\|T^S(t)\| \leq Me^{(c/2)t} \quad (\forall t \geq 0); \quad \|T^U(t)\| \leq Me^{2ct} \quad (\forall t \leq 0),$$

where  $M \geq 1$  is a constant. Now, let us choose a constant  $\mu$ ,  $0 < \mu < \mu_0$ , so that  $k(\mu)(\|\Pi^U\| + 2\|\Pi^S\|) < c/(4M^2)$  ( $< c/(2M)$ ), where  $\Pi^U$  (resp.  $\Pi^S$ ) denotes the projection from  $X$  onto  $U$  (resp.  $S$ ) along the decomposition  $X = S \oplus U$ . Since  $U \neq \{0\}$  and  $2k(\mu)M\|\Pi^S\|/c < 1/(4M)$ , there exists a (nonzero)  $\bar{\psi} \in U$  such that  $2k(\mu)M\|\Pi^S\|/c < \|\bar{\psi}\|/(2\mu) < 1/(4M)$ . Let  $l$  be any positive integer, and consider the set

$$\Omega = \{y \in C((-\infty, l]; X) \mid \Pi^U y(l) = \bar{\psi}, \quad \|y(t)\| \leq \mu e^{c(t-l)} \quad (\forall t \leq l)\}$$

which is equipped with the metric  $d$  defined by

$$d(y, z) := \sup_{-\infty < t \leq l} \|y(t) - z(t)\| e^{c(l-t)}, \quad \forall y, z \in \Omega.$$

Clearly,  $\Omega$  is a complete metric space. For any  $y \in \Omega$  and  $s \in (-\infty, l]$ , it follows that  $\|y(s)\| \leq \mu e^{c(s-l)} \leq \mu < \mu_0$ , and hence  $|g(s, y(s))| \leq k(\mu)\|y(s)\| \leq k(\mu)\mu e^{c(s-l)}$  or  $\|\Gamma^n g(s, y(s))\| \leq |g(s, y(s))| \leq k(\mu)\mu e^{c(s-l)}$ , which yields the estimate

$$\|T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))\| \leq M e^{(c/2)(t-s)} \|\Pi^S\| k(\mu)\mu e^{c(s-l)}, \quad \forall s \leq t \leq l. \quad (21)$$

Then the limit

$$\lim_{\sigma \rightarrow -\infty} \int_{\sigma}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds =: \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds$$

exists in  $X$  (uniformly for  $n$ ). Observe that for any positive integer  $n$  and  $m$ ,

$$\begin{aligned} & \left\| \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds - \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^m g(s, y(s)))ds \right\| \\ & \leq \left\| \int_{-\infty}^{\sigma} T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds \right\| + \left\| \int_{-\infty}^{\sigma} T^S(t-s)\Pi^S(\Gamma^m g(s, y(s)))ds \right\| \\ & \quad + \left\| \int_{\sigma}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds - \int_{\sigma}^t T^S(t-s)\Pi^S(\Gamma^m g(s, y(s)))ds \right\| \\ & \leq (4M/c)\mu k(\mu)\|\Pi^S\| e^{(c/2)(t-\sigma)} e^{c(\sigma-l)} \\ & \quad + \left\| \Pi^S \left( \int_{\sigma}^t T(t-s)(\Gamma^n g(s, y(s)))ds - \int_{\sigma}^t T(t-s)(\Gamma^m g(s, y(s)))ds \right) \right\| \end{aligned}$$

by (21). Since  $\lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n g(s, y(s)))ds = x_t(\sigma, 0, p)$  (here  $p(s) := g(s, y(s))$ ) in  $X$  by Theorem 2, it follows that

$$\begin{aligned} & \limsup_{n, m \rightarrow \infty} \left\| \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds - \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^m g(s, y(s)))ds \right\| \\ & \leq (4M/c)\mu k(\mu)\|\Pi^S\| e^{(c/2)(t-\sigma)} e^{c(\sigma-l)} \rightarrow 0 \quad \text{as } \sigma \rightarrow -\infty, \end{aligned}$$

and hence the limit  $\lim_{n \rightarrow \infty} \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds$  exists in  $X$ .

Let us consider the mapping  $\Phi$  on  $\Omega$  defined by

$$\begin{aligned} (\Phi y)(t) & := T^U(t-l)\bar{\psi} + \lim_{n \rightarrow \infty} \int_l^t T^U(t-s)\Pi^U(\Gamma^n g(s, y(s)))ds \\ & \quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))ds, \quad \forall t \leq l, y \in \Omega. \end{aligned}$$

Then

$$\begin{aligned}
\|(\Phi y)(t)\| &\leq \|T^U(t-l)\bar{\psi}\| + \limsup_{n \rightarrow \infty} \int_t^l \|T^U(t-s)\Pi^U(\Gamma^n g(s, y(s)))\| ds \\
&\quad + \limsup_{n \rightarrow \infty} \int_{-\infty}^t \|T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))\| ds \\
&\leq M e^{2c(t-l)} \|\bar{\psi}\| + \int_t^l M e^{2c(t-s)} \|\Pi^U\| k(\mu) \mu e^{c(s-l)} ds \\
&\quad + \int_{-\infty}^t M e^{(c/2)(t-s)} \|\Pi^S\| k(\mu) \mu e^{c(s-l)} ds \\
&\leq M e^{2c(t-l)} \|\bar{\psi}\| + M k(\mu) \mu e^{c(t-l)} (\|\Pi^U\| + 2\|\Pi^S\|)/c \\
&\leq \mu e^{c(t-l)}, \quad \forall t \leq l.
\end{aligned}$$

Moreover,  $\lim_{n \rightarrow \infty} \int_{-\infty}^t \|T^S(t-s)\Pi^S(\Gamma^n g(s, y(s)))\| ds$  is continuous on  $(-\infty, l]$  as an  $X$ -valued function of  $t$ , because of the inequality

$$\begin{aligned}
&\left\| \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y(s))) ds - \int_{-\infty}^{\bar{t}} T^S(\bar{t}-s)\Pi^S(\Gamma^n g(s, y(s))) ds \right\| \\
&\leq \left\| \int_{-\infty}^{t-\chi} T^S(t-s)\Pi^S(\Gamma^n g(s, y(s))) ds \right\| + \left\| \int_{-\infty}^{\bar{t}-\chi} T^S(\bar{t}-s)\Pi^S(\Gamma^n g(s, y(s))) ds \right\| \\
&\quad + \left\| \int_0^\chi T^S(\tau)\Pi^S(\Gamma^n(g(t-\tau, y(t-\tau)) - g(\bar{t}-\tau, y(\bar{t}-\tau))) d\tau \right\| \\
&\leq (4M/c) \|\Pi^S\| \mu k(\mu) e^{-c\chi/2} \\
&\quad + M \|\Pi^S\| \chi e^{(c\chi/2)} \left( \sup_{0 \leq s \leq \chi} |g(t-s, y(t-s)) - g(\bar{t}-s, y(\bar{t}-s))| \right)
\end{aligned}$$

by (21); here  $\chi$  is any positive number. This observation leads to  $\Phi y \in \Omega$  whenever  $y \in \Omega$ . Furthermore, if  $y_i \in \Omega$  ( $i = 1, 2$ ), then  $|g(s, y_1(s)) - g(s, y_2(s))| \leq k(\mu) \|y_1(s) - y_2(s)\|$  for any  $s \in (-\infty, l]$ ; hence, almost the same calculation as above yields that for any  $t \leq l$ ,

$$\|(\Phi y_1)(t) - (\Phi y_2)(t)\| e^{c(t-l)} \leq d(y_1, y_2) \times k(\mu) M (\|\Pi^U\| + 2\|\Pi^S\|)/c \leq (1/2) d(y_1, y_2),$$

or  $d(\Phi y_1, \Phi y_2) \leq (1/2) d(y_1, y_2)$ . Thus the mapping  $\Phi : \Omega \rightarrow \Omega$  is a contraction. Then there exists a unique  $y \in \Omega$  such that  $\Phi y = y$  by the contraction mapping theorem. Summarizing these facts, we conclude that for any positive integer  $l$ , there is one and only one  $y =: y^{(l)} \in C((-\infty, l]; X)$  with the properties that

$$\|y^{(l)}(t)\| \leq \mu e^{c(t-l)}, \quad \forall t \leq l \quad (22)$$

and

$$\begin{aligned}
y^{(l)}(t) &= T^U(t-l)\bar{\psi} + \lim_{n \rightarrow \infty} \int_l^t T^U(t-s)\Pi^U(\Gamma^n g(s, y^{(l)}(s))) ds \\
&\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^S(t-s)\Pi^S(\Gamma^n g(s, y^{(l)}(s))) ds, \quad \forall t \leq l.
\end{aligned} \quad (23)$$

Define  $\gamma^{(l)} \in C((-\infty, l]; \mathbb{C}^m)$  by  $\gamma^{(l)}(t) := g(t, y^{(l)}(t))$ ,  $\forall t \leq l$ . We assert that for any  $\sigma$  with  $\sigma < l$ ,  $y^{(l)}$  satisfies the relation

$$y^{(l)}(t) = T(t - \sigma)y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s)(\Gamma^n \gamma^{(l)}(s))ds, \quad \forall t \in [\sigma, l]. \quad (24)$$

Indeed, if  $t \in [\sigma, l]$ , then it follows from (23) that

$$\begin{aligned} \Pi^S y^{(l)}(t) &= \lim_{n \rightarrow \infty} \int_{-\infty}^t T^S(t - s)\Pi^S(\Gamma^n \gamma^{(l)}(s))ds \\ &= T^S(t - \sigma)\left(\lim_{n \rightarrow \infty} \int_{-\infty}^{\sigma} T^S(\sigma - s)\Pi^S(\Gamma^n \gamma^{(l)}(s))ds\right) \\ &\quad + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^S(t - s)\Pi^S(\Gamma^n \gamma^{(l)}(s))ds; \end{aligned}$$

hence

$$\Pi^S y^{(l)}(\sigma) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\sigma} T^S(\sigma - s)\Pi^S(\Gamma^n \gamma^{(l)}(s))ds$$

and

$$\Pi^S y^{(l)}(t) = T^S(t - \sigma)\Pi^S y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^S(t - s)\Pi^S(\Gamma^n \gamma^{(l)}(s))ds, \quad \forall t \in [\sigma, l]. \quad (25)$$

In a similar way, one can get that

$$\Pi^U y^{(l)}(\sigma) = T^U(\sigma - l)\bar{\psi} + \lim_{n \rightarrow \infty} \int_l^{\sigma} T^U(\sigma - s)\Pi^U(\Gamma^n \gamma^{(l)}(s))ds$$

and

$$\Pi^U y^{(l)}(t) = T^U(t - \sigma)\Pi^U y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^U(t - s)\Pi^U(\Gamma^n \gamma^{(l)}(s))ds, \quad \forall t \in [\sigma, l]. \quad (26)$$

Then, the assertion (24) immediately follows from (25) and (26).

Next, for each  $l = 1, 2, \dots$  we consider an extension  $\bar{\gamma}^{(l)} \in C(\mathbb{R}; \mathbb{C}^m)$  defined by  $\bar{\gamma}^{(l)}(t) = \gamma^{(l)}(t)$  if  $t \leq l$  and  $\bar{\gamma}^{(l)}(t) = \gamma^{(l)}(l)$  if  $t > l$ . We assert that the function  $\bar{y}^{(l)} : \mathbb{R} \rightarrow X$  defined by

$$\bar{y}^{(l)}(t) = \begin{cases} y^{(l)}(t), & t \leq l \\ T(t - l)y^{(l)}(l) + \lim_{n \rightarrow \infty} \int_l^t T(t - s)(\Gamma^n \bar{\gamma}^{(l)}(s))ds, & t > l \end{cases}$$

satisfies the relation

$$\bar{y}^{(l)}(t) = T(t - \sigma)\bar{y}^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s)(\Gamma^n \bar{\gamma}^{(l)}(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \text{ with } t > \sigma, \quad (27)$$

in  $X$ . Indeed, in the case of  $\sigma \leq t \leq l$  the relation obviously holds (by (24)). If  $\sigma \leq l \leq t$ , then

$$y^{(l)}(l) = T(l-s)y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^l T(l-s)(\Gamma^n \gamma^{(l)}(s))ds$$

by (24), and hence

$$\begin{aligned} & T(t-\sigma)\bar{y}^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n \bar{\gamma}^{(l)}(s))ds \\ &= T(t-\sigma)y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^l T(t-s)(\Gamma^n \gamma^{(l)}(s))ds + \lim_{n \rightarrow \infty} \int_l^t T(t-s)(\Gamma^n \gamma^{(l)}(s))ds \\ &= T(t-l) \left( T(l-s)y^{(l)}(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^l T(l-s)(\Gamma^n \gamma^{(l)}(s))ds \right) \\ &\quad + \lim_{n \rightarrow \infty} \int_l^t T(t-s)(\Gamma^n \gamma^{(l)}(s))ds \\ &= T(t-l)y^{(l)}(l) + \lim_{n \rightarrow \infty} \int_l^t T(t-s)(\Gamma^n \gamma^{(l)}(s))ds \\ &= \bar{y}^{(l)}(t), \end{aligned}$$

as required. Similarly, one can easily check the relation in case of  $l < \sigma \leq t$ .

Now, by virtue of Theorem 3 and Relation (27) we see that the function  $u^{(l)}$  defined by  $u^{(l)}(t) := (\bar{y}^{(l)}(t))[0]$ ,  $\forall t \in \mathbb{R}$ , satisfies  $u^{(l)} \in C(\mathbb{R}; \mathbb{C}^m)$ ,  $u_t^{(l)} = \bar{y}^{(l)}(t)$  in  $X$  and  $u^{(l)}(t) = \int_{-\infty}^t K(t-s)u^{(l)}(s)ds + \bar{\gamma}^{(l)}(t)$  for any  $t \in \mathbb{R}$ ; in particular, if  $t \leq l$ , then  $u_t^{(l)} = y^{(l)}(t)$  in  $X$  and hence  $\bar{\gamma}^{(l)}(t) = \gamma^{(l)}(t) = g(t, y^{(l)}(t)) = g(t, u_t^{(l)})$ ; consequently  $u^{(l)}$  satisfies  $u^{(l)}(t) = \int_{-\infty}^t K(t-s)u^{(l)}(s)ds + g(t, u_t^{(l)})$  for  $t \leq l$ ; that is,  $u^{(l)}$  is a solution of Eq. (18) on  $(-\infty, l]$ . Let us consider a sequence  $\{\varphi^{(l)}\}$  in  $X$  defined by  $\varphi^{(l)} = u_0^{(l)}$ ,  $l = 1, 2, \dots$ . We will verify that the sequence  $\{\varphi^{(l)}\}$  satisfies the desired properties in the theorem. Indeed, from (22) it follows that  $\|\varphi^{(l)}\| = \|y^{(l)}(0)\| \leq \mu e^{-cl} \rightarrow 0$  as  $l \rightarrow \infty$ . Also,  $\sup_{t \geq 0} \|x_t(0, \varphi^{(l)}, g)\| \geq (1/2)\|\bar{\psi}\| =: c_0 > 0$ , because

$$y^{(l)}(l) = \bar{\psi} + \lim_{n \rightarrow \infty} \int_{-\infty}^l T^S(l-s)\Pi^S(\Gamma^n g(s, y^{(l)}(s)))ds$$

by (23) and consequently

$$\begin{aligned} \|x_l(0, \varphi^{(l)}, g)\| &= \|u_l^{(l)}\| = \|y^{(l)}(l)\| \\ &\geq \|\bar{\psi}\| - \int_{-\infty}^l M e^{(c/2)(l-s)} \|\Pi^S\| k(\mu) \mu e^{c(s-l)} ds \\ &= \|\bar{\psi}\| - \mu(2M \|\Pi^S\| k(\mu)/c) \\ &\geq (1/2)\|\bar{\psi}\|. \end{aligned}$$

This completes the proof.  $\square$

Let us consider an autonomous abstract equation

$$y(t) = f(t, y_t), \quad (28)$$

where  $f : L_{\rho_1}^1 \rightarrow \mathbb{C}^m$  is continuously Fréchet differentiable and  $\rho_1$  is a constant number such that  $\rho_1 > \rho$ . Assume that  $\bar{y} \in \mathbb{C}^m$  is an equilibrium point for (28); that is,  $\bar{y} = f(\omega_0 \bar{y})$ ; here  $\omega_0$  is a function on  $\mathbb{R}^-$  defined by  $\omega_0(\theta) = 1, \forall \theta \leq 0$ . Denote by  $f'(\omega_0 \bar{y})$  the Fréchet derivative of  $f$  at  $\omega_0 \bar{y}$ . By the Riesz theorem ([15]), there exists a unique function  $K$  belonging to  $L_{\rho_1}^\infty(\mathbb{R}^+; \mathbb{C}^m)$  with the property that  $f'(\omega_0 \bar{y})\varphi = \int_0^\infty K(s)\varphi(-s)ds, \forall \varphi \in L_{\rho_1}^1$ . Notice that  $K$  satisfies the condition  $\|K\|_{1, \rho, +} < \infty$ , together with  $\|K\|_{\infty, \rho, +} < \infty$ . Define a function  $g : \mathbb{R} \times L_\rho^1 \rightarrow \mathbb{C}^m$  by

$$g(t, \varphi) := f(\omega_0 \bar{y} + \varphi) - f(\omega_0 \bar{y}) - f'(\omega_0 \bar{y})\varphi, \quad \varphi \in L_\rho^1.$$

Then, letting  $x(t) := y(t) - \bar{y}$ , one can see that Eq. (28) is transformed to the autonomous equation  $x(t) = \int_{-\infty}^0 K(t-s)x(s)ds + g(x_t)$  (on  $L_\rho^1$ ) which is exactly the same one as Eq. (18) with  $g(t, \varphi) \equiv g(\varphi)$ , and moreover the stability (or instability) property (in  $L_\rho^1$ ) of the equilibrium  $\bar{y}$  for Eq. (28) is reduced to the stability (or instability) property (in  $L_\rho^1$ ) of the zero solution of Eq. (18). Observe that the above  $g$  satisfies the conditions (19) and (20) in Theorem 4 and Theorem 5. Therefore, as a direct consequence of Theorems 4 and 5, we get the following result which is the principle of linearized stability for integral equations ([2, Theorem 3.15]).

**Corollary.** *Under the above conditions imposed on the function  $f : L_{\rho_1}^1 \rightarrow \mathbb{C}^m$ , the following statements hold true;*

- (i) *if  $\det(E - \int_0^\infty K(t)e^{-zt}dt) \neq 0$  for all  $z$  with  $\operatorname{Re} z \geq 0$ , then the equilibrium  $\bar{y}$  for Eq. (28) is exponentially stable in  $L_\rho^1$  (with  $0 < \rho < \rho_1$ );*
- (ii) *if  $\det(E - \int_0^\infty K(t)e^{-z_0 t}dt) = 0$  for some  $z_0$  with  $\operatorname{Re} z_0 > 0$ , then the equilibrium  $\bar{y}$  for Eq. (28) is unstable in  $L_\rho^1$  (with  $0 < \rho < \rho_1$ ).*

Before concluding the paper, we give an example to illustrate how our results are effectively applied. Consider the following (scalar) integral equation

$$x(t) = \lambda \int_{-\infty}^t e^{-(t-s)}x(s)ds + C(t) \left( \int_{-\infty}^t e^{-(t-s)}x(s)ds \right)^3, \quad (29)$$

where  $\lambda$  is a real parameter and  $C$  is a continuous function satisfying  $\sup_{t \in \mathbb{R}} |C(t)| \leq c_0$  for some  $c_0 > 0$ . In what follows, let  $\rho$  be a (fixed) constant such that  $0 < \rho < 1$ . It is clear that the function  $K$  on  $\mathbb{R}^+$  defined by  $K(t) = \lambda e^{-t}$  for  $t \geq 0$  satisfies Conditions (9) and (10). Moreover, the function  $g : \mathbb{R} \times L_\rho^1(\mathbb{R}^-; \mathbb{C}) \rightarrow \mathbb{C}$  defined by

$$g(t, \varphi) := C(t) \left( \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta \right)^3, \quad \forall (t, \varphi) \in \mathbb{R} \times L_\rho^1(\mathbb{R}^-; \mathbb{C})$$

is continuous, and it satisfies the condition (5), together with  $g(t, 0) \equiv 0$ . Observe that Eq. (29) is written as Eq. (18) with  $X := L^1_\rho(\mathbb{R}^-; \mathbb{C})$ . Since  $|\int_{-\infty}^0 e^\theta \varphi(\theta) d\theta| \leq \int_{-\infty}^0 |\varphi(\theta)| e^{\rho\theta} d\theta =: \|\varphi\|_{1,\rho}$  for  $\varphi \in X$ , one can see that the above function  $g$  satisfies the conditions (19) and (20) in Theorem 4 and Theorem 5; e.g., for (20) one can take  $k(\mu) := 4c_0\mu^2$  and  $\mu_0 := 1$ . Notice that  $\Delta(z) = 1 - \int_0^\infty K(t)e^{-zt} dt = 1 - \lambda/(1+z)$  for  $\operatorname{Re} z > -\rho$ . Therefore, by virtue of Theorem 4 and Theorem 5 we get:

**Proposition 4.** *Under the above conditions on Eq. (29), the following statements hold true;*

- (i) *if  $\lambda < 1$ , then the zero solution of Eq. (29) is exponentially stable (in  $L^1_\rho$ );*
- (ii) *if  $\lambda > 1$ , then the zero solution of Eq. (29) is unstable (in  $L^1_\rho$ ).*

In case of  $\lambda = 1$ , one can not apply Theorem 4 and Theorem 5 for the stability analysis of the zero solution of Eq. (29). In fact, in the (critical) case of  $\lambda = 1$ , either stable case or unstable case for the zero solution may indeed occur depending on the special choice of  $C$ , as the following result shows:

**Proposition 5.** *Under the above conditions on Eq. (29) with  $\lambda = 1$ , the following statements hold true:*

- (i) *Assume that  $C(t) \leq 0$  on  $\mathbb{R}^+$  with  $\int_0^\infty C(t) dt = -\infty$ . If  $\delta$  is a positive number such that  $\delta \leq 1$ , and if  $\|\varphi\|_{1,\rho} < \delta/\sqrt{c_0}$ , then any solution  $x$  of Eq. (29) (with  $\lambda = 1$ ) through  $(0, \varphi)$  satisfies  $|x(t)| < \delta/\sqrt{c_0}$ ,  $\forall t > 0$ , and  $\lim_{t \rightarrow \infty} x(t) = 0$ ; consequently, the zero solution of Eq. (29) is asymptotically stable:*
- (ii) *Assume that  $C(t) \geq 0$  on  $\mathbb{R}^+$  with  $\int_0^\infty C(t) dt = \infty$ . If  $\int_{-\infty}^0 e^\theta \varphi(\theta) d\theta > c_1 > 0$ , then any solution  $x$  of Eq. (29) (with  $\lambda = 1$ ) through  $(0, \varphi)$  blows up in a finite time; in particular, the zero solution of Eq. (29) is unstable.*

**[Proof of (i)]:** Noticing that  $-\delta/\sqrt{c_0} < \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta < \delta/\sqrt{c_0}$  because of  $|\int_{-\infty}^0 e^\theta \varphi(\theta) d\theta| \leq \int_{-\infty}^0 e^{\rho\theta} |\varphi(\theta)| d\theta = \|\varphi\|_{1,\rho} < \delta/\sqrt{c_0}$ , we will verify the assertion by dividing into the following three cases:

$$(a) \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta = 0; \quad (b) 0 < \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta < \frac{\delta}{\sqrt{c_0}}; \quad (c) -\frac{\delta}{\sqrt{c_0}} < \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta < 0.$$

In case (a), the assertion holds trivially, because  $x(t) \equiv 0$  on  $(0, \infty)$  by the uniqueness of solutions for the initial value problem. The case (c) is reduced to the case (b) by considering  $-x(t)$  instead of  $x(t)$ . In what follows, we will treat the case (b) to verify the assertion. Observe that  $0 < x(0^+) < \delta/\sqrt{c_0}$  by the nonpositivity of  $C(t)$  because of

$x(0^+) = \int_{-\infty}^0 e^\theta \varphi(\theta) d\theta + C(0) \times (\int_{-\infty}^0 e^\theta \varphi(\theta) d\theta)^3$ . In fact, as long as  $0 \leq x(t) \leq \delta/\sqrt{c_0}$ , the function  $y(t) := \int_{-\infty}^t e^{-(t-s)} x(s) ds$  satisfies  $0 < y(t) < \delta/\sqrt{c_0}$ , and hence  $0 < x(t) = y(t)(1 + C(t)\{y(t)\}^2) < \delta/\sqrt{c_0}$ . This argument shows that  $|x(t)| < \delta/\sqrt{c_0}$  on  $(0, \infty)$ . Notice that the function  $y$  satisfies  $(d/dt)y(t) = -y(t) + x(t) = C(t)\{y(t)\}^3$  on  $(0, \infty)$ . Solving the differential equation for  $y$ , we get  $y(t) = y(0)/\sqrt{1 - 2\{y(0)\}^2 \int_0^t C(s) ds}$ ; hence  $\lim_{t \rightarrow \infty} y(t) = 0$  because of  $\int_0^\infty C(s) ds = -\infty$ , and consequently  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} (y(t) + C(t)\{y(t)\}^3) = 0$ , as required.

**[Proof of (ii)]:** Observe that as long as the solution  $x(t)$  exists and it satisfies  $x(t) \geq c_1$ , the function  $y(t) := \int_{-\infty}^t e^{-(t-s)} x(s) ds$  satisfies  $y(t) = e^{-t} \int_{-\infty}^0 e^s \varphi(s) ds + \int_0^t e^{s-t} x(s) ds > c_1$ , which implies  $x(t) = y(t) + C(t)\{y(t)\}^3 > c_1$  (by the nonnegativity of  $C(t)$ ). Therefore, as long as  $x(t)$  exists, we must have that  $x(t) > c_1$ , together with  $y(t) > c_1$ . Since  $y(t)$  is a solution of the differential equation  $(d/dt)y(t) = C(t)\{y(t)\}^3$ ,  $y(t)$  blows up in a finite time (by the assumption that  $\int_0^\infty C(t) dt = \infty$ ). Thus  $x$  must blow up in a finite time because of  $x(t) \geq y(t)$ , as required.  $\square$

**Open Problem.** *In connection with Theorems 4 and 5, it would be natural to pose the following problem: Can we deduce any stability (or instability) result for Eq. (18) in the critical case? In fact, in Proposition 5 we have treated the critical case for Eq. (29) which is a special one of Eq. (18). But we have not succeeded in solving the above problem for general Eq. (18).*

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