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Existence of eventually positive solutions for a class of fourth order quasilinear differential equations

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1. Introduction

This paper is concerned with the existence of eventually positive solutions of fourth order quasilinear differential equations of the form

$$(p(t)|u''|^{\alpha-1}u'')''+q(t)|u|^{\beta-1}u=0, \tag{1}$$

where α and β are positive constants, $p(t)$ and $q(t)$ are positive continuous functions defined on an infinite interval $[a,\infty)$, $a>0$. Throughout the paper we assume that $p(t)$ satisfies

$$\int_a^\infty\left(\frac{t}{p(t)}\right)^{1/\alpha}dt=\infty, \tag{2}$$

or, more strongly,

$$\int_a^\infty\frac{t}{(p(t))^{1/\alpha}}dt=\infty \quad \text{and} \quad \int_a^\infty\left(\frac{t}{p(t)}\right)^{1/\alpha}dt=\infty. \tag{3}$$

By a solution of (1) we mean a real-valued function $u(t)$ such that $u\in C^2[b,\infty)$ and $p|u''|^{\alpha-1}u''\in C^2[b,\infty)$ and $u(t)$ satisfies (1) at every point of $[b,\infty)$, where $b\geq a$ and b may depend on $u(t)$. Such a solution $u(t)$ of (1) is called nonoscillatory if $u(t)$ is eventually positive or eventually negative. A solution $u(t)$ of (1) is called oscillatory if it has an infinite sequence of zeros clustering at $t=\infty$. Equation (1) itself is called oscillatory if all of its solutions are oscillatory.

If $u(t)$ is a solution of (1), then $-u(t)$ is a solution of (1). Therefore, without loss of generality, we can assume a nonoscillatory solution of (1) is eventually positive. If $u(t)$ is an eventually positive solution of (1), then there is $T\geq a$ such that $u(t)>0$ for $t\geq T$.

The oscillatory and asymptotic behavior of nonoscillatory solutions of (1) has been recently considered by Wu [1] under the condition (2) or (3). The results in [1] are as follows:

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Theorem 1 (Wu [1]). (i) *Suppose* (3) *holds. Then Eq.* (1) *has an eventually positive solution* $u(t)$ *satisfying*

$$\lim_{t\to\infty} u(t) \text{ exists and is a positive finite value} \tag{4}$$

if and only if

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)q(s)ds\right)^{1/\alpha} dt < \infty. \tag{5}$$

(ii) *Suppose* (2) *holds. Then Eq.* (1) *has an eventually positive solution* $u(t)$ *satisfying*

$$\lim_{t\to\infty} \frac{u(t)}{\displaystyle\int_a^t (t-s)\left(\frac{s}{p(s)}\right)^{1/\alpha} ds} \text{ exists and is a positive finite value} \tag{6}$$

if and only if

$$\int_a^\infty q(t)\left(\int_a^t (t-s)\left(\frac{s}{p(s)}\right)^{1/\alpha} ds\right)^\beta dt < \infty. \tag{7}$$

Moreover it is shown [1] that, under the integral condition (3) and the condition $0 < \alpha \le 1 < \beta$ [resp. $0 < \beta < 1 \le \alpha$], Eq. (1) has an eventually positive solution if and only if (5) [resp. (7)] holds.

The purpose of this paper is to show that, in the preceding statements, the conditions $0 < \alpha \le 1 < \beta$ and $0 < \beta < 1 \le \alpha$ can be replaced by the natural conditions $0 < \alpha < \beta$ and $0 < \beta < \alpha$, respectively, provided that $p(t)$ meets additional conditions.

If $p(t) \equiv 1$, then Eq. (1) turns into

$$(|u''|^{\alpha-1}u'')'' + q(t)|u|^{\beta-1}u = 0. \tag{8}$$

The results for (8) in Naito and Wu [2] are as follows:

Theorem 2 (Naito and Wu [2]). (i) *Suppose that* $0 < \alpha < \beta$. *Then Eq.* (8) *has an eventually positive solution if and only if*

$$\int_a^\infty t\left(\int_t^\infty (s-t)q(s)ds\right)^{1/\alpha} dt < \infty. \tag{9}$$

(ii) *Suppose that* $0 < \beta < \alpha$. *Then Eq.* (8) *has an eventually positive solution if and only if*

$$\int_a^\infty t^{(2+(1/\alpha))\beta} q(t)dt < \infty. \tag{10}$$

If $p(t) \equiv 1$, then the conditions (5) and (7) reduce to (9) and (10), respectively. Especially, if $p(t) \equiv 1$ and $\alpha = 1$, the oscillatory and nonoscillatory solutions of (1) were also considered by Ou and Wong [3]. But, this paper does not include the results of [3].

The oscillatory and asymptotic behavior of nonoscillatory solutions of (1) were also considered by Kamo and Usami [4, 5], Manojlović and Milošević [6], Kusano and Tanigawa [7] and Kusano, Manojlović and Tanigawa [8]. In [4] it is asumed that $p(t)$ satisfies

$$\int_a^\infty \left(\frac{t}{p(t)}\right)^{1/\alpha} dt = \infty \quad \text{and} \quad \int_a^\infty \frac{t}{(p(t))^{1/\alpha}} dt < \infty, \tag{11}$$

while in [5, 6] it is asumed that $p(t)$ satisfies

$$\int_a^\infty \left(\frac{t}{p(t)}\right)^{1/\alpha} dt < \infty \quad \text{and} \quad \int_a^\infty \frac{t}{(p(t))^{1/\alpha}} dt < \infty. \tag{12}$$

Kusano, Manojlović and Tanigawa [7, 8] have considered the case

$$\int_a^\infty \left(\frac{t^{\alpha+1}}{p(t)}\right)^{1/\alpha} dt < \infty, \tag{13}$$

which is a stronger condition than (12). Since our condition (3) does not imply (11), (12) and (13), the results in this paper are not included in [4-8].

In this paper, in addition to (3), we will asume the following condition:

$$\liminf_{t\to\infty} \frac{\displaystyle\int_a^t \left(\frac{s}{p(s)}\right)^{1/\alpha} ds}{t\left(\dfrac{t}{p(t)}\right)^{1/\alpha}} > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\displaystyle\int_a^t \left(\frac{1}{p(s)}\right)^{1/\alpha} ds}{t\left(\dfrac{1}{p(t)}\right)^{1/\alpha}} < \infty. \tag{14}$$

It is easy to see that if $p(t) \equiv 1$, then the conditions (3) and (14) are satisfied. Moreover, for the case where $p(t)$ satisfies

$$0 < \liminf_{t\to\infty} \frac{p(t)}{t^\gamma} \le \limsup_{t\to\infty} \frac{p(t)}{t^\gamma} < \infty \quad \text{for some } \gamma \in \boldsymbol{R},$$

if $\gamma < \alpha$, then the conditions (3) and (14) are satisfied.

2. Results

The main purpose of this paper is to prove the next theorem.

Theorem 3. (i) *Let* $0 < \alpha < \beta$. *Suppose* (3) *and* (14) *hold. Then Eq.* (1) *has an eventually positive solution if and only if* (5) *holds.*

(ii) *Let* $0 < \beta < \alpha$. *Suppose* (3) *and* (14) *hold. Then Eq.* (1) *has an eventually positive solution if and only if* (7) *holds.*

Therefore Theorem 3 gives an extension of Theorem 2. To prove Theorem 3, we give several necessary lemmas.

Lemma 4. *Suppose $x(t) > 0$ and $y(t) > 0$ are continuous functions on $[T, \infty)$. Let $T_0 > T$. If there is a constant $c > 0$ such that*

$$x(t) \int_T^t y(s)ds \geq cy(t) \int_T^t x(s)ds \tag{15}$$

for all $t \geq T_0$. Then there exists a number $0 < \theta_0 < 1$ such that

$$\int_T^t x(s) \int_T^s y(r)drds \geq (1 - \theta_0) \int_T^t x(s)ds \int_T^t y(s)ds \tag{16}$$

for all $t \geq T_0$.

Lemma 5 (Wu [1]). *Suppose* (3) *is satisfied. If $u(t)$ is an eventually positive solution of* (1), *then there is $T \geq a$ such that one of the following cases holds:*

$$u'(t) > 0, \quad u''(t) > 0, \quad (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0 \quad for \ t > T; \tag{17}$$

$$u'(t) > 0, \quad u''(t) < 0, \quad (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0 \quad for \ t > T. \tag{18}$$

Lemma 6. *Suppose* (3) *and* (14) *hold. Let $0 < \alpha < \beta$. If Eq.* (1) *has an eventually positive solution $u(t)$ satisfying* (17), *then, for an arbitrary constant ε with $0 < \varepsilon < \beta - \alpha$, there are $C_0 > 0$ and $T_0 > T$ such that*

$$\int_t^\infty q(s)ds < C_0 t^{\alpha+\varepsilon-\beta} \left(\int_T^t \int_T^s \left(\frac{r-T}{p(r)} \right)^{1/\alpha} drds \right)^{-\alpha}, \quad t > T_0 \tag{19}$$

holds.

Application of Lemma 4 and Lemma 6, we can prove (i) of Theorem 3.

Lemma 7. *Suppose* (3) *and $0 < \beta < \alpha$ hold. If Eq.* (1) *has an eventually positive solution $u(t)$ satisfying* (18), *then*

$$\int_a^\infty t^{\beta/\alpha} \left(\frac{1}{p(t)} \int_t^\infty \int_s^\infty q(r)drds \right)^{1/\alpha} dt < \infty. \tag{20}$$

Application of Lemma 4 and Lemma 7, we can prove (ii) of Theorem 3.

3. Example

We present here an example which illustrates the main results in this paper. Consider Eq. (1) for the special case that $p(t)$ and $q(t)$ satisfy

$$0 < \liminf_{t\to\infty} \frac{p(t)}{t^\gamma} \leq \limsup_{t\to\infty} \frac{p(t)}{t^\gamma} < \infty \quad \text{for some } \gamma \in \boldsymbol{R}, \tag{21}$$

and

$$0 < \liminf_{t\to\infty} \frac{q(t)}{t^\delta} \leq \limsup_{t\to\infty} \frac{q(t)}{t^\delta} < \infty \quad \text{for some } \delta \in \boldsymbol{R}, \tag{22}$$

respectively. Then, both of the conditions (3) and (14) hold if and only if $\gamma < \alpha$. Using Theorem 3, we have the following results for (1): Consider Eq. (1) under the conditions (21) and (22). Then

(i) Let $\gamma < \alpha$ and $0 < \alpha < \beta$. Eq. (1) has an eventually positive solution if and only if $\delta < \gamma - 2(1+\alpha)$.

(ii) Let $\gamma < \alpha$ and $0 < \beta < \alpha$. Eq. (1) has an eventually positive solution if and only if $\delta < -1 - ((1+2\alpha-\gamma)\beta)/\alpha$.

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