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Macroeconomics and Finance: How Dynamic Macroeconomic Models Treat Finance

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1 Introduction

1.1 Focus of Macroeconomics

- The Great Depression $\Longrightarrow$ Keynesian view that markets may not readily equilibrate.
- The Great Ination $\Longrightarrow$ importance of aggregate supply shocks and spurred real business cycle research.
- The Great Disinflation $\Longrightarrow$ New Keynesianism recognizing the potency of monetary policy.
- The Great Moderation $\Longrightarrow$ dynamic stochastic general equilibrium (DSGE) model as a macroeconomic orthodoxy.
- The Great Panic and Recession of 2008 and 2009 $\Longrightarrow$???

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1.2 Aim of This Presentation

- To show a subjective summary of macroeconomic attempts to explain risk premium, particularly equity premium and term premium in dynamic macroeconomic framework.
- A key feature of recent events has been the close feedback between the real economy and financial conditions.
- Joint analysis of macroeconomy and risk premium, in addition to the financial frictions and macroeconomy, can be a central focus of macroeconomics.

1.3 Structure of Presentation

- Introduction to the Dynamic Macroeconomics
- Equity Premium Puzzle
- Macro–Finance: Joint Explanation of Term Structure and Macroeconomy
- Summary

2 DSGE Model

2.1 History

- Neoclassical growth model, namely Ramsey (1928) – Cass (1965) – Koopmans (1965) model (stochastic extension, Brock and Mirman, 1972), that is Solow (1956) – Swan (1956) model + optimal saving choice, was thought to be very theoretical.
- Kydland and Prescott (1982) use this neoclassical growth model for positive analysis.
- Neoclassical model calibrated to explain the data is then called Real Business Cycle (RBC) models.
- Macroeconomics has been increasingly paying more attention to fitting the data.

- RBC models has developed to explain the data better by incorporating such mechanisms as indivisible labor (Hansen, 1985, Rogerson, 1988), tax distortions (Braun, 1994, McGrattan, 1994), capacity utilization (Greenwood, Hurcowitz and Hoffman, 1988), labor hoarding (Burnside, Eichenbaum and Rebelo, 1996), home production (Benhabib, Rogerson and Wright, 1991), or Labor search and matching (Merz, 1995, Andolfatto, 1996).
- New Keynesian Model, further incorporating nominal rigidities, is now the workhorse model (especially, Christiano, Eichenbaum and Evans, 2005, and Smets and Wouters, 2003) in many policy institutions.

Simplest Example

2.2 Simple RBC Model

- The social planner maximizes

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) - \frac{\chi}{2} h_t^2\right],$$

subject to

$$Y_t = [\exp(z_t) h_t]^{1-\alpha} K_t^{\alpha},$$
$$C_t + I_t = Y_t,$$
$$K_{t+1} = (1-\delta) K_t + I_t,$$

and

$$z_t = \rho_z z_{t-1} + u_t,$$

- Uniqueness of the equilibrium path in this economy can be proved by applying contraction mapping theorem to functional space.
- Note that by specifying functional forms and shocks, the model can fit the data.
- First order necessary conditions are

$$\frac{1}{C_t} = \beta \mathrm{E}_t \frac{1}{C_{t+1}} \left\{(1-\delta) + \alpha [\exp(z_{t+1}) h_{t+1}]^{1-\alpha} K_{t+1}^{a-1}\right\},$$

$$(1-\alpha) [\exp(z_t) h_t]^{-\alpha} \exp(z_t) K_t^{a} = \chi h_t C_t,$$

and

$$C_t + K_{t+1} - (1-\delta) K_t = [\exp(z_t) h_t]^{1-\alpha} K_t^{a}.$$

- Usually, the model is log-linearly approximated as

$$\begin{aligned}
& \mathrm{E}_t\hat{C}_{t+1} \\
= \; & \hat{C}_t \\
& +(1-\alpha)\left[1-\beta(1-\delta)\right]\left(\mathrm{E}_t z_{t+1}+\mathrm{E}_t\hat{h}_{t+1}-\hat{K}_{t+1}\right),
\end{aligned}$$

$$\alpha\hat{K}_t-\hat{C}_t-(1+\alpha)\,\hat{h}_t+(1-\alpha)\,z_t=0,$$

and

$$\begin{aligned}
& \frac{1-\beta(1-\delta)-\alpha\beta\delta}{\alpha\beta}\hat{C}_t+\hat{K}_{t+1}-(1-\delta)\,\hat{K}_t \\
= \; & \frac{1-\beta(1-\delta)}{\alpha\beta}\left[(1-\alpha)\,z_t+(1-\alpha)\,\hat{h}_t+\alpha\hat{K}_t\right],
\end{aligned}$$

where

$$\log\left(\frac{X_t}{X}\right)\approx\frac{X_t-X}{X}\equiv\hat{x}_t.$$

- This system can be expressed in the state space form:

$$\mathbf{A}\begin{pmatrix}\mathrm{E}_t\hat{C}_{t+1}\\ \hat{K}_{t+1}\end{pmatrix}=\mathbf{B}\begin{pmatrix}\hat{C}_t\\ \hat{K}_t\end{pmatrix}+\mathbf{C}\mathrm{E}_t z_{t+1}+\mathbf{D}z_t,$$

or

$$\begin{pmatrix}\mathrm{E}_t\hat{C}_{t+1}\\ \hat{K}_{t+1}\end{pmatrix}=\mathbf{A}^{-1}\mathbf{B}\begin{pmatrix}\hat{C}_t\\ \hat{K}_t\end{pmatrix}+\mathbf{A}^{-1}\mathbf{C}\mathrm{E}_t z_{t+1}+\mathbf{A}^{-1}\mathbf{D}z_t.$$

- With eigendecomposition:

$$\mathbf{A}^{-1}\mathbf{B}=\begin{pmatrix}v_{11}&v_{12}\\ v_{21}&v_{22}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1&0\\ 0&\lambda_2\end{pmatrix}\begin{pmatrix}v_{11}&v_{12}\\ v_{21}&v_{22}\end{pmatrix},$$

we now have

$$\begin{aligned}
& \begin{pmatrix}v_{11}&v_{12}\\ v_{21}&v_{22}\end{pmatrix}\begin{pmatrix}\mathrm{E}_t\hat{C}_{t+1}\\ \hat{K}_{t+1}\end{pmatrix} \\
= & \begin{pmatrix}\lambda_1&0\\ 0&\lambda_2\end{pmatrix}\begin{pmatrix}v_{11}&v_{12}\\ v_{21}&v_{22}\end{pmatrix}\begin{pmatrix}\hat{C}_t\\ \hat{K}_t\end{pmatrix} \\
+ & \begin{pmatrix}c_1\\ c_2\end{pmatrix}\mathrm{E}_t z_{t+1}+\begin{pmatrix}d_1\\ d_2\end{pmatrix}z_t,
\end{aligned}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \mathbf{A}^{-1}\mathbf{C},$$

and

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \mathbf{A}^{-1}\mathbf{D}.$$

- The first row is

$$v_{11}\mathrm{E}_t\hat{C}_{t+1} + v_{12}\hat{K}_{t+1} = \lambda_1 \left(v_{11}\hat{C}_t + v_{12}\hat{K}_t \right) + c_1\mathrm{E}_t z_{t+1}.$$

- If

$$|\lambda_1| > 1, \tag{1}$$

$$\begin{aligned} &\left(v_{11}\hat{C}_t + v_{12}\hat{K}_t \right) \\ &= -\frac{c_1}{\lambda_1}\mathrm{E}_t z_{t+1} - \frac{c_1}{\lambda_1^2}\mathrm{E}_t z_{t+2} - \ldots - \lim_{T\to\infty} \frac{c_1}{\lambda_1^T}\mathrm{E}_t z_{t+T} \\ &= -\frac{\lambda_1 c_1}{\lambda_1 - \rho_z} z_t, \end{aligned}$$

and therefore

$$\hat{C}_t = -\frac{v_{12}}{v_{11}}\hat{K}_t - \frac{\lambda_1 c_1}{v_{11}\left(\lambda_1 - \rho_z\right)} z_t. \tag{2}$$

- The second row is

$$v_{21}\mathrm{E}_t\hat{C}_{t+1} + v_{22}\hat{K}_{t+1} = \lambda_2 \left(v_{21}\hat{C}_t + v_{22}\hat{K}_t \right) + d_2 z_t.$$

- If

$$|\lambda_2| < 1, \tag{3}$$

$$\hat{K}_{t+1} = \omega_2\hat{K}_t + \frac{v_{11}d_2 - v_{21}\lambda_2 c_1}{v_{11}v_{22} - v_{21}v_{12}} z_t. \tag{4}$$

- Equations (2) and (4) are solutions. Conditions (1) and (3) are usually satisfied thanks to $\beta < 1$.

- Parameters can be estimated by applying Kalman filter for the state space model consisting of equations (2) and (4).

2.3 Implications to Finance

- There exist several exceptions such as Fernández-Villaverde and Rubio-Ramírez (2005), but standard models for positive analysis do not consider higher order terms.
- Therefore, certainty equivalence holds and no risk premium is considered in the model.
- Unconditional expected returns for any asset must be equated.
- Several new challenges are made recently with increasing importance of understanding risk premium.

3 Equity Premium Puzzle

- Mehra and Prescott (1985) argue a particular empirical problem of the RBC model.
- According to Kocherlakota (1996), "Over the last one hundred years, the average real return to stocks in the United States has been about six percent per year higher that on Treasury bills. At the same time, the average real return on treasury bills has been about one percent per year."
- Returns on assets can be different depending on the equity premium, namely the degree to which each asset return covaries with the consumption.
- Yet, the size of the observed equity premium is only justified when risk aversion is set incredibly high.

Risk Free Rate Puzzle

- Weil (1989) shows that this fact also implies the second puzzle.
- According to Kocherlakota (1996), "although Treasury bills offer only a low rate of return, individuals defer consumption at a sufficiently fast rate to generate average per capita consumption growth of around two percent per year... although individual like consumption to be very smooth, and although the risk free rate is very low, they still save enough that per capita consumption grows rapidly."

3.1 CRRA Preference

- Under CRRA preference, which has been intensively used in macroeconomics,

$$u(C_t) = \frac{C_t^{1-\alpha}}{1-\alpha},$$

where α is the coefficient of relative risk aversion.

- Then, we maximize the life-time welfare:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t),$$

subject to the budget constraint:

$$C_t + S_t + B_t \leq R_t^s S_{t-1} + R_t^b B_{t-1} + Y_t.$$

- Then, we can derive the optimality (absence of arbitrage) conditions for bond and stock holdings are expressed

$$\mathrm{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} \left(R_{t+1}^s - R_{t+1}^b \right) \right] = 0, \tag{5}$$

and

$$\mathrm{E}_t \beta \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1}^b \right] = 1, \tag{6}$$

where the pricing kernel or stochastic discount factor is defined as

$$M_{t,t+1} = \mathrm{E}_t \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\alpha}.$$

- Test results on equations (5) and (6) are shown as below.

TABLE 2
THE EQUITY PREMIUM PUZZLE

α	$\bar{e}$	t-stat
0.0	0.0594	3.365
0.5	0.0577	3.280
1.0	0.0559	3.173
1.5	0.0544	3.082
2.0	0.0528	2.987
2.5	0.0512	2.880
3.0	0.0496	2.780
3.5	0.0480	2.685
4.0	0.0464	2.584
4.5	0.0449	2.478
5.0	0.0433	2.370
5.5	0.0418	2.258
6.0	0.0403	2.153
6.5	0.0389	2.044
7.0	0.0372	1.934
7.5	0.0357	1.824
8.0	0.0341	1.715
8.5	0.0326	1.607
9.0	0.0310	1.501
9.5	0.0295	1.395
10.0	0.0279	1.291

In this table, $\bar{e}$ is the sample mean of $e_t = (C_{t+1}/C_t)^{-\alpha}(R^s_{t+1} - R^b_{t+1})$ and α is the coefficient of relative risk aversion. Standard errors are calculated using the implication of the theory that e_t is uncorrelated with e_{t-k} for all k; however, they are little changed by allowing e_t to be MA(1) instead. This latter approach to calculating standard errors allows for the possibility of time aggregation (see Hansen, Heaton, and Yaron 1994).

TABLE 3
THE RISK FREE RATE PUZZLE

α	$\bar{e}$	t-stat
0.0	0.0033	0.0567
0.5	−0.0081	−1.306
1.0	−0.0162	−2.223
1.5	−0.0239	−2.780
2.0	−0.0313	−3.180
2.5	−0.0382	−3.363
3.0	−0.0446	−3.329
3.5	−0.0510	−3.360
4.0	−0.0569	−3.348
4.5	−0.0624	−3.312
5.0	−0.0675	−3.259
5.5	−0.0723	−3.195
6.0	−0.0765	−3.123
6.5	−0.0805	−3.042
7.0	−0.0846	−2.959
7.5	−0.0880	−2.871
8.0	−0.0910	−2.779
8.5	−0.0937	−2.685
9.0	−0.0960	−2.590
9.5	−0.0980	−2.495
10.0	−0.0997	−2.394

In this table, $\bar{e}$ is the sample mean of $e_t = \beta(c_{t+1}/c_t)^{-\alpha}(R^b_{t+1} - 1)$ and α is the coefficient of relative risk aversion. The discount factor β is set equal to 0.99. The standard errors are calculated using the implication of the theory that e_t is uncorrelated with e_{t-k} for all k; however, they are little changed by allowing e_t to be MA(1) instead. This latter approach to calculating standard errors allows for the possibility of time aggregation (see Hansen, Heaton, and Yaron 1994).

- Correlation between consumption growth and equity returns observed from the data is too small for the sizable equity premium.
- There have been several attempts to solve equity premium puzzle.

  1. Preference Modification
  2. Low Frequency Event
  3. Incomplete Market

3.2 Preference Modification

- Habit Formation
- Epstein–Zin Preference

Habit

- What happens if we become more risk averse when the level of consumption decreases in a recession?
- A model of time varying risk aversion, where the risk aversion depends on the level of consumption, may resolve the equity premium and the risk free rate puzzles.

- Abel (1990) and Gali (1994) propose a preference with habit formation as
$$U_t = \frac{c_t^{1-\alpha} C_t^{\gamma} C_{t-1}^{\lambda}}{1-\alpha},$$
where C_t is the aggregate consumption and taken as given when optimizing. We assume that agents tend to be jealous and $\gamma, \lambda < 0$. Therefore, this preference reflects the idea by Duesenberry (1949), "keeping up with Joneses."

- Under this preference, optimality conditions for bond as well as stock holdings in equations (5) and (6) are now rewritten as
$$\mathrm{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{\gamma-\sigma} \left(\frac{C_t}{C_{t-1}} \right)^{\lambda} \left(R_{t+1}^s - R_{t+1}^b \right) \right] = 0,$$
and
$$\mathrm{E}_t \beta \left[\left(\frac{C_{t+1}}{C_t} \right)^{\gamma-\sigma} \left(\frac{C_t}{C_{t-1}} \right)^{\lambda} R_{t+1}^b \right] = 1.$$

- With reasonable parameters, the model solves the equity premium puzzle but not the risk free rate puzzle.

- When γ, which is supposed to be negative, is large in absolute value, marginal utility of own consumption is highly sensitive to fluctuations in per capita consumption and therefore strongly negatively correlated to stock returns.

3.3 Epstein–Zin Preference

- With the standard CRRA preference, the risk aversion is the reciprocal of the IES.

- Epstein and Zin (1989, 1991) propose a preference which disentangles the risk aversion γ and the IES ψ:
$$U_t = \left\{ C_t^{1-\frac{1}{\psi}} + \beta \left[\mathrm{E}_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}.$$

- When the risk aversion γ equals the reciprocal of the IES $\frac{1}{\psi}$, this collapses to the CRRA utility function.

- High risk aversion toghether with high IES is possible. This is a good news for risk free rate puzzle.
- Optimality conditions for bond and stock holdings in equations (5) and (6) are now

$$\mathrm{E}_t\left[U_{t+1}^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(R_{t+1}^s-R_{t+1}^b\right)\right]=0,$$

and

$$\mathrm{E}_t\beta\left[\left(\mathrm{E}_t U_{t+1}^{1-\gamma}\right)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}}U_{t+1}^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}R_{t+1}^b\right]=1.$$

- The existence of the unobservable expected utility may solve both puzzles.
- Kocherlakota (1996), however, claims that if consumption growth is not predictable, equations (5) and (6) hold again under unconditional expectation.

3.4 Low Frequency Event

- Long–Run Risk
- Disaster Shock

Long–Run Risk: Bansal and Yaron (2004)

- Consider again Epstein–Zin Preference

$$U_t=\left\{C_t^{\frac{1-\gamma}{\theta}}+\beta\left[\mathrm{E}_t\left(U_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}},$$

where

$$\theta=\frac{1-\gamma}{1-\frac{1}{\psi}}.$$

- When $\theta=1$, the agent is indifferent to the timing of the resolution of the uncertainty of the consumption path.
- When $\theta<1$, namely risk aversion γ is larger than the reciprocal of the IES $1/\psi$, the agent prefers early resolution.
- When $\theta>1$, namely risk aversion γ is smaller than the reciprocal of the IES $1/\psi$, the agent prefers late resolution.

- The logarithm of the stochastic discount factor for the log utility: $U_t = \log(C_t)$, is
$$m_{t,t+1} = \log(\beta) + \Delta c_{t+1}.$$

- On the other hand, that of Epstein–Zin preference is expressed as

$$\begin{aligned} & m_{t,t+1} \\ = \; & \log(\beta) + \Delta c_t \\ & -(\gamma - 1)\sum_{i=0}^{\infty}(\mathrm{E}_{t+1} - \mathrm{E}_t)\,\Delta c_{t+j+1} \\ & -\frac{1}{2}(\gamma-1)^2 \operatorname{var}_t\left[\sum_{i=0}^{\infty}\beta^j(\mathrm{E}_{t+1} - \mathrm{E}_t)\,\Delta c_{t+j+1}\right]. \end{aligned} \tag{7}$$

- Bansal and Yaron (2004) suppose that the agent prefers early resolution in their model and specify the long–run risks, that is the shocks to the long–run component of consumption, and time varying risk.

- This long–run model can resolve both the equity premium and risk free rate puzzles thanks to the effects from expected growth rate of consumption and time varying risks on the stochastic discount factor in equation (7).

- Question is whether we can statistically distinguish stochastic specification with long–run risk from one without it.

Disaster Shock: Rietz (1988)

- Mehra and Prescott (1985) assume that consumption growth rates are symmetric about their mean and they fall above their mean as often as they fall below.

- Rietz (1988) supposes a situation when consumption fall drastically and equity returns are far below average by incorporating a low-probability, depression-like state as an additional state in the Markov chain.

- Introduction of disaster risk has now become more interested than before due to the recent financial crisis.

- Robert Barro and his coauthors now estimate the disaster risk from historical data.

3.5 Incomplete Market

- When individual cannot write contracts against any contingency, individual consumption growth can be more covary with stock returns than per capita consumption growth. This may solve the equity premium puzzle.
- Under such situation, precautionary saving motive stemming from self-insurance may result in lower natural rate of interest. This may solve the risk free rate puzzle.
- Previous studies, however, report that incomplete market qualitatively alleviate these puzzles but not solve them quantitatively.
- Huggett (1993) and Heaton and Lucas (1995) show that under realistic calibration, the difference between the incomplete markets interest rate and the complete markets interest rate is small. Most individuals are sufficiently far from the debt ceiling.
- Constantinides and Duffie (1995) show analytical solution under incomplete market by assuming that the idiosyncratic shocks are permanent. This could reduce equilibrium real interest rates and increase equity premium.
- Heaton and Lucas (1995), however, find again with the model calibrated to match the data that difference in interest rates between complete and incomplete markets is small.
- Marcet and Singleton (1991), Telmer (1993), Lucas (1994), den Haan (1994), and Heaton and Lucas (1995) show that allocations under incomplete markets are almost the same as those under complete markets.
- Krueger and Lustig (2010) provide various examples of economies in which uninsurable income shocks do not matter for the equity premium.

3.6 Rich Households

- According to Aït-Sahalia, Parker and Yogo (2004), "The risk aversion implied by the consumption of luxury goods is more than an order of magnitude less than that implied by national accounts data. For the very rich, the equity premium is much less of a puzzle."

- Parker and Vissing-Jorgensen (2009) document that the consumption of rich households is over five times more volatile than aggregate consumption, which may help to explain average premiums in financial markets.

- Chien, Cole, and Lustig (2009) build a model in which a large fraction of households do not rebalance their portfolios in response to aggregate shocks. As a consequence, households who do rebalance need to sell more stocks in good times and buy more stocks in bad times. This mechanism generates time variation in risk premiums.

4 Macro–Finance

- Macro–finance research has examined the relationship between the term structure of interest rates and the macro–economy in an interdisciplinary fashion.

- Belows are only selective survey of this field:

  1. Structural VAR Models
  2. Affine Term Structure Models
  3. DSGE Models
  4. Nelson–Siegel Models

4.1 Structural VAR Model

- Evans and Marshall (1988) estimate a structural VAR model as

$$\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} Z_t \\ R_{j,t} \end{pmatrix} = \begin{pmatrix} A(L) & 0 \\ C(L) & D(L) \end{pmatrix} \begin{pmatrix} Z_t \\ R_{j,t} \end{pmatrix} + \sigma \begin{pmatrix} \varepsilon_t^Z \\ \varepsilon_t^j \end{pmatrix}$$

- Z_t is a vector of endogenous variables including policy interest rates. $R_{j,t}$ is a bond yield for maturity j. Variables in Z_t are determined independently from $R_{j,t}$ based on the idenfiticaitons imposed by previous studies.

- They assume that yields should be linearly estimated by macroeconomic variables.

4.2 Affine Term Structure Model

- Since Vasicek (1977), various extensions have been made to affine term structure model to better explain the data.
- Multi unobserved factor model proposed by Duffie and Kan (1996) is widely used with high explanatory power of the yield curve.
- Ang and Piazzessi (2003) incorporate macroeconomic variables in addition to unobserved factors and show that fit is improved.
- Ichiue and Ueno (2006) extend Black (1995) and propose a term structure model with explicit considerations to the zero bound constraint.

Duffie and Kan (1996)

- Assume that spot risk free rate is expressed as the affine function of state variables:
$$r_t = \delta_0 + \delta_1' X_t. \tag{8}$$
- Assume that state variables under both P and Q measures follows such Gaussian processes as
$$X_{t+1} = K^{\mathbf{Q}} X_t + \Sigma_X \varepsilon_{t+1}^{\mathbf{Q}},$$
and
$$X_{t+1} = a^{\mathbf{P}} + K^{\mathbf{P}} X_t + \Sigma_X \varepsilon_{t+1}^{\mathbf{P}}. \tag{9}$$
- By modeling the market price of risk also as the affine function:
$$\Lambda_t = \lambda_0 + \Lambda_1 X_t,$$
we have
$$a^{\mathbf{P}} = \Sigma_X \lambda_0, \;\; K^{\mathbf{P}} = K^{\mathbf{Q}} + \Sigma_X \Lambda_1,$$
since the market price of risk should satisfy
$$\varepsilon_t^{\mathbf{P}} = \varepsilon_t^{\mathbf{Q}} - \Lambda_t dt.$$
- The Bond prices are also expressed as the affine function:
$$p_t^{(m)} = \exp\left(A_m + B_m' X_t\right),$$
and therefore, the yields are
$$y_t^{(m)} = -\frac{A_m}{m} - \frac{B_m' X_t}{m}.$$

where

$$B_0 = 0,$$
$$B_m = -\delta_1' + K' B_{m-1},$$
$$A_0 = 0,$$

and

$$A_m = A_{m-1} - \delta_0 + \frac{1}{2} B_m' \Sigma_X \Sigma_X' B_m.$$

- Note that we can compute term premium because we have yields under both P and Q measures.

Kalman Filter

- Observation equation is defined as
$$y_t = A + B' X_t + \eta_t.$$

- For simplicity, when we aim at fitting 10 and 20 year zero coupon yields with two unobserved factors, although five factor model is popular,
$$\begin{pmatrix} y_t^{(10)} \\ y_t^{(20)} \end{pmatrix} = -\begin{pmatrix} \frac{1}{120} A_{120} \\ \frac{1}{240} A_{240} \end{pmatrix} - \begin{pmatrix} \frac{1}{120} B_{120}' \\ \frac{1}{240} B_{240}' \end{pmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} + \eta_t.$$

- This altogether with the state equation (9) are jointly estimated via maximum likelihood.

Ang and Piazzessi (2003)

- Short-rate is one of the most important driver of yield curves but reflects macroeconomic variables such as output and inflation via monetary policy rule.

- Equation (8) is now rewritten as
$$r_t = \delta_0 + \delta_1' X_t^o + \delta_2' X_t^u.$$

- The short rate dynamics of the term structure model can be interpreted as a version of the Taylor rule with the errors as unobserved factors. In other words, the pricing kernel is driven by shocks to both observed macro factors and (uncorrelated) unobserved factors.

- "Macro factors primarily explain movements at the short end and middle of the yield curve while unobservable factors still account for most of the movement at the longend of the yield curve."

4.3 DSGE Models for Bond Pricing

- DSGE Models for Bond Pricing is still a very much on-going project.
- Hördahl, Tristani and Vestin (2006) combines the DSGE model with the affine term structure model.
- Rudebusch and Swanson (2008, 2009) and Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2010) aim to explain interrelationship between macroeconomy and term premium in theoretically consistent manner.
- De Graeve, Emiris and Wouters (2009) claim that if the DSGE model is rich enough to explain the data, first order approximated model can explain the yield curve without relying on the term premium.

Hördahl, Tristani and Vestin (2006)

- Hördahl, Tristani and Vestin (2006) combine the affine term structure model with the new Keynesian model.
- They suppose that the macroeconomy is depicted by a standard new Keynesian model with lags:

$$\pi_t = \frac{\mu_\pi}{12}\sum_{i=1}^{12} \mathrm{E}_t \pi_{t+i} + (1-\mu_\pi)\sum_{i=1}^{3} \delta_{\pi i}\pi_{t-i} + \delta_x x_t + \varepsilon_t^\pi,$$

$$\begin{aligned} x_t &= \frac{\mu_x}{12}\sum_{i=1}^{12} \mathrm{E}_t x_{t+i} \\ &\quad + (1-\mu_x)\sum_{i=1}^{3} \zeta_{xi} x_{t-i} + \zeta_r\left(r_t - \mathrm{E}_t\pi_{t+11}\right) + \varepsilon_t^x, \end{aligned}$$

and

$$r_t = (1-\rho)\left[\beta \mathrm{E}_t \pi_{t+11} - \pi_t^* + \gamma x_t\right] + \rho r_{t-1} + \eta_t.$$

- It is also assumed unobserved inflation target:

$$\pi_t^* = \phi_\pi \pi_{t-1}^* + u_{\pi,t}.$$

- This system can be solved as in equations (2) and (4). Therefore, the short-rate can be expressed by the state variables obtained from the new Keynesian model:

$$r_t = \delta_0 + \delta_1' Z_t. \tag{10}$$

- Hördahl, Tristani and Vestin (2006) jointly estimate the new Keynesian model and the affine term structure model based on equation (10) together.

4.4 Rudebusch and Swanson (2008)

- Models explained above and below separate the determination of macroeconomic variables and the term premium. Macroeconomic variables are determined irrespective of term premium.
- Only consumption based model approach can theoretically explain the joint determination of premium and macroeconomic variables.
- Rudebusch and Swanson (2008) simulate a new Keynesian model with internal habit formation with higher order approximation based on perturbation method.
- They conclude that "standard DSGE models, even with nominal rigidities, labor market frictions, and consumption habits, appear to fall short of being able to price nominal bonds."
- Yet, Rudebusch and Swanson (2009) report that if Epstein–Zin preference is employed, "the DSGE model is able to fit the asset pricing facts without compromising its ability to fit the macroeconomic data.
- Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2010) extends this analysis with endogenous capital etc.

De Graeve, Emiris and Wouters (2009)

- De Graeve, Emiris and Wouters (2009) estimate the standard DSGE model based on Christiano, Eichenbaum and Evans (2005) or Smets and Wouters (2003), which incorporates nominal price and wage rigidities, consumption habit, investment growth adjustment cost, endogenous capacity utilization and structurally decompose the US yield curves.
- They conclude that contrary to the previous studies, the US yield curve is consistent with the expectations hypothesis under the rational expectation of the standard DSGE models.

4.5 Nelson–Siegel Model

- Yield curves based on Nelson and Siegel (1987) model show remarkable fit.
- Recent paper by Christensen, Diebold and Rudebusch (2009) construct the Arbitrage-Free Nelson–Siegel model. With this model, we can understand the underlying factor in the context of "level," "slope," and "curvature."

Litterman and Sheinkman (1991)

- Litterman and Sheinkman (1991) show with the principle component analysis that most of the variations in the Treasury yields are explained by only three factors.
- They are interpreted as "level," "slope," and "curvature."

4.6 Nelson and Siegel (1987)

- Nelson and Siegel (1987) propose an yield curve model as

$$y(m) = L + S\left(\frac{1-e^{-m\lambda}}{m\lambda}\right) + C\left(\frac{1-e^{-m\lambda}}{m\lambda} - e^{-m\lambda}\right).$$

- The factor loading for the first term is constant, that for the second term starts at 1 and decays monotonically to 0 and that for the third term starts at 0, increases and decreases to 0. Therefore, these terms are considered to represent "level," "slope," and "curvature."
- We can estimate four parameters, L, S, C and λ for each period.

Diebold and Li (2006)

- Diebold and Li (2006) propose a dynamic Nelson–Siegel model:

$$y_t(m) = L_t + S_t\left(\frac{1-e^{-m\lambda}}{m\lambda}\right) + C_t\left(\frac{1-e^{-m\lambda}}{m\lambda} - e^{-m\lambda}\right),$$

  where factors, L_t, S_t and C_t, are assumed to follow autoregressive processes.

4.7 Christensen, Diebold and Rudebusch (2009)

- Above models based on Nelson and Siegel (1987) are not from theory just from mere fitting.
- Christensen, Diebold and Rudebusch (2009) combine Nelson–Siegel model with theory, namely no arbitrage.
- We can understand how such factors as "level," "slope," and "curvature" affect yields in a theoretically consistent manner.
- Christensen, Diebold and Rudebusch (2009) assume that yields are explained by three factors, L_t, S_t and C_t.
- Instantaneous risk-free rate is given by

$$r_t = L_t + S_t,$$

while state equations are defined under Q measure as

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt - \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & \sigma_S & 0 \\ 0 & 0 & \sigma_C \end{pmatrix} \begin{pmatrix} dW_t^{\mathbf{Q},L} \\ dW_t^{\mathbf{Q},S} \\ dW_t^{\mathbf{Q},C} \end{pmatrix}.$$

- Under these settings, yields can be expressed as Nelson–Siegel model:

$$\begin{aligned} y_t(m) &= L_t + S_t \left(\frac{1 - e^{-m\lambda}}{m\lambda} \right) \\ &\quad + C_t \left(\frac{1 - e^{-m\lambda}}{m\lambda} - e^{-m\lambda} \right) \\ &\quad + \frac{A(m)}{m}, \end{aligned}$$

where $A(m)$ is a yield adjustment factor, which is time-invariant but depends only on the maturity.

5 Summary

- Puzzles in the financial markets have not been solved completely.

- A model with a preference disentangling risk aversion and intertemporal substitution together with the low frequency events seem to be promising in solving puzzles in financial markets.