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ALMOST DISJOINT AND INDEPENDENT FAMILIES

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ABSTRACT. I collect a number of proofs of the existence of large almost disjoint and independent families on the natural numbers. This is mostly the outcome of a discussion on math*overflow*.

1. INTRODUCTION

A family $\mathcal{F}\subseteq\mathcal{P}(\omega)$ is an *independent* family (over ω) if for every pair $\mathcal{A}$, $\mathcal{B}$ of disjoint finite subsets of $\mathcal{F}$ the set

$$\bigcap\mathcal{A}\cap\left(\omega\setminus\bigcup\mathcal{B}\right)$$

is infinite. Fichtenholz and Kantorovich showed that there is an independent family on ω of size continuum [3] (also see [6] or [8]). I collect several proofs of this fundamental fact. A typical application of the existence of a large independent family is the result that there are $2^{2^{\aleph_0}}$ ultrafilters on ω due to Pospíšil [11]:

Given an independent family $(A_\alpha)_{\alpha<2^{\aleph_0}}$, for every function $f:2^{\aleph_0}\to 2$ there is an ultrafilter p_f on ω such that for all $\alpha<2^{\aleph_0}$ we have $A_\alpha\in p_f$ iff $f(\alpha)=1$. Now $(p_f)_{f:2^{\aleph_0}\to 2}$ is a family of size $2^{2^{\aleph_0}}$ of pairwise distinct ultrafilters.

Independent families in some sense behave similarly to almost disjoint families. Subsets A and B of ω are *almost disjoint* if $A\cap B$ is finite. A family $\mathcal{F}$ of infinite subsets of $\mathcal{P}(\omega)$ is *almost disjoint* any two distinct elements A, B of $\mathcal{F}$ are almost disjoint.

2. ALMOST DISJOINT FAMILIES

An easy diagonalisation shows that every countably infinite, almost disjoint family can be extended.

Lemma 2.1. *Let $(A_n)_{n\in\omega}$ be a sequence of pairwise almost disjoint, infinite subsets of ω. Then there is an infinite set $A\subseteq\omega$ that is almost disjoint from all A_n, $n\in\omega$.*

Proof. First observe that since the A_n are pairwise almost disjoint, for all $n\in\omega$ the set

$$\omega\setminus\bigcup_{k<n}A_k$$

is infinite. Hence we can choose a strictly increasing sequence $(a_n)_{n\in\omega}$ of natural numbers such that for al $n\in\omega$, $a_n\in\omega\setminus\bigcup_{k<n}A_k$. Clearly, if $k<n$, then $a_n\notin A_k$.

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It follows that for every $k \in \omega$ the infinite set $A = \{a_n : n \in \omega\}$ is almost disjoint from A_k. □

A straight forward application of Zorn's Lemma gives the following:

Lemma 2.2. *Every almost disjoint family of subsets of ω is contained in a maximal almost disjoint family of subsets of ω.*

Corollary 2.3. *Every infinite, maximal almost disjoint family is uncountable. In particular, there is an uncountable almost disjoint family of subsets of ω.*

Proof. The uncountability of an infinite, maximal almost disjoint family follows from Lemma 2.1. To show the existence of such a family, choose a partition $(A_n)_{n\in\omega}$ of ω into pairwise disjoint, infinite sets. By Lemma 2.2, the almost disjoint family $\{A_n : n \in \omega\}$ extends to a maximal almost disjoint family, which has to be uncountable by our previous observation. □

Unfortunately, this corollary only guarantees the existence of an almost disjoint family of size $\aleph_1$, not necessarily of size $2^{\aleph_0}$.

Theorem 2.4. *There is an almost disjoint family of subsets of ω of size $2^{\aleph_0}$.*

All the following proofs of Theorem 2.4 have in common that instead of on ω, the almost disjoint family is constructed as a family of subsets of some other countable set that has a more suitable structure.

First proof. We define the almost disjoint family as a family of subsets of the complete binary tree $2^{<\omega}$ of height ω rather than ω itself. For each $x \in 2^\omega$ let $A_x = \{x \restriction n : n \in \omega\}$.

If $x, y \in 2^\omega$ are different and $x(n) \neq y(n)$, then $A_x \cap A_y$ contains no sequence of length $> n$. It follows that $\{A_x : x \in 2^\omega\}$ is an almost disjoint family of size continuum. □

Similarly, one can consider for each $x \in [0,1]$ the set B_x of finite initial segments of the decimal expansion of x. $\{B_x : x \in [0,1]\}$ is an almost disjoint family of size $2^{\aleph_0}$ of subsets of a fixed countable set.

Second proof. We again identify ω with another countable set, in this case the set $\mathbb{Q}$ of rational numbers. For each $r \in \mathbb{R}$ choose a sequence $(q_n^r)_{n\in\omega}$ of rational numbers that is not eventually constant and converges to r. Now let $A_r = \{q_n^r : n \in \omega\}$.

For $s, r \in \mathbb{R}$ with $s \neq r$ choose $\varepsilon > 0$ so that

$$(s-\varepsilon, s+\varepsilon) \cap (r-\varepsilon, r+\varepsilon) = \emptyset.$$

Now $A_s \cap (s-\varepsilon, s+\varepsilon)$ and $A_r \cap (r-\varepsilon, r+\varepsilon)$ are both cofinite and hence $A_s \cap A_r$ is finite. It follows that $\{A_r : r \in \mathbb{R}\}$ is an almost disjoint family of size $2^{\aleph_0}$. □

Third proof. We construct an almost disjoint family on the countable set $\mathbb{Z} \times \mathbb{Z}$. For each angle $\alpha \in [0, 2\pi)$ let A_α be the set of all elements of $\mathbb{Z} \times \mathbb{Z}$ that have distance ≤ 1 to the line $L_\alpha = \{(x,y) \in \mathbb{R}^2 : y = \tan(\alpha) \cdot x\}$.

For two distinct angles α and β the set of points in $\mathbb{R}^2$ of distance ≤ 1 to both L_α and L_β is compact. It follows that $A_\alpha \cap A_\beta$ is finite. Hence $\{A_\alpha : \alpha \in [0, 2\pi)\}$ is an almost disjoint family of size continuum. □

Fourth proof. We define a map $e : [0,1] \to \omega^\omega$ as follows: for each $x \in [0,1]$ and $n \in \omega$ let $e(x)(n)$ be the integer part of $n \cdot x$.

For every $x \in [0,1]$ let $A_x = \{(n, e(x)(n)) : n \in \omega\}$. If $x < y$, then for all sufficiently large $n \in \omega$, $e(x)(n) < e(y)(n)$. It follows that $\{A_x : x \in [0,1]\}$ is an almost disjoint family of subsets of $\omega \times \omega$.

Observe that e is an embedding of $([0,1], \leq)$ into $(\omega^\omega, \leq^*)$, where $f \leq^* g$ if for almost all $n \in \omega$, $f(n) \leq g(n)$. □

3. Independent families

Independent families behave similarly to almost disjoint families. The following results are analogs of the corresponding facts for almost disjoint families.

Lemma 3.1. *Let m be an ordinal $\leq \omega$ and let $(A_n)_{n<m}$ be a sequence of infinite subsets of ω such that for all pairs S, T of finite disjoint subsets of m the set*

$$\bigcap_{n\in S} A_n \setminus \left(\bigcup_{n\in T} A_n\right)$$

is infinite. Then there is an infinite set $A \subseteq \omega$ that is independent over the family $\{A_n : n < m\}$ in the sense that for all pairs S, T of finite disjoint subsets of m both

$$\left(A \cap \bigcap_{n\in S} A_n\right) \setminus \left(\bigcup_{n\in T} A_n\right)$$

and

$$\bigcap_{n\in S} A_n \setminus \left(A \cup \bigcup_{n\in T} A_n\right)$$

are infinite.

Proof. Let $(S_n, T_n)_{n\in\omega}$ be an enumeration of all pairs of disjoint finite subsets of m such that every such pair appears infinitely often.

By the assumptions on $(A_n)_{n\in\omega}$, we can choose a strictly increasing sequence $(a_n)_{n\in\omega}$ such that for all $n \in \omega$,

$$a_{2n}, a_{2n+1} \in \bigcap_{k\in S_n} A_k \setminus \left(\bigcup_{k\in T_n} A_k\right).$$

Now the set $A = \{a_{2n} : n \in \omega\}$ is independent over $\{A_n : n < m\}$. Namely, let S, T be disjoint finite subsets of m. Let $n \in \omega$ be such that $S = S_n$ and $T = T_n$. Now by the choice of a_{2n},

$$a_{2n} \in \left(A \cap \bigcap_{k\in S_n} A_k\right) \setminus \left(\bigcup_{k\in T_n} A_k\right).$$

On the other hand,

$$a_{2n+1} \in \bigcap_{k\in S_n} A_k \setminus \left(A \cup \bigcup_{k\in T_n} A_k\right).$$

Since there are infinitely many $n \in \omega$ with $(S,T) = (S_n, T_n)$, it follows that the sets

$$\left(A \cap \bigcap_{k\in S_n} A_k\right) \setminus \left(\bigcup_{k\in T_n} A_k\right)$$

and

$$\bigcap_{k\in S_n} A_k \setminus \left(A \cup \bigcup_{k\in T_n} A_k\right)$$

are both infinite. □

Another straight forward application of Zorn's Lemma yields:

Lemma 3.2. *Every independent family of subsets of ω is contained in a maximal independent family of subsets of ω.*

Corollary 3.3. *Every infinite maximal independent family is uncountable. In particular, there is an uncountable independent family of subsets of ω.*

Proof. By Lemma 3.2, there is a maximal independent family. By Lemma 3.1 such a family cannot be finite or countably infinite. □

As in the case of almost disjoint families, this corollary only guarantees the existence of independent families of size $\aleph_1$. But Fichtenholz and Kantorovich showed that there are independent families on ω of size continuum.

Theorem 3.4. *There is an independent family of subsets of ω of size $2^{\aleph_0}$.*

In the following proofs of this theorem, we will replace the countable set ω by other countable sets with a more suitable structure. Let us start with the original proof by Fichtenholz and Kantorovich [3] that was brought to my attention by Andreas Blass.

First proof. Let C be the countable set of all finite subsets of $\mathbb{Q}$. For each $r \in \mathbb{R}$ let

$$A_r = \{a \in C : a \cap (-\infty, r] \text{ is even}\}.$$

Now the family $\{A_r : r \in \mathbb{R}\}$ is an independent family of subsets of C.

Let S and T be finite disjoint subsets of $\mathbb{R}$. A set $a \in C$ is an element of

$$\bigcap_{r\in S} A_r \setminus \left(C \setminus \bigcup_{r\in T} A_r\right)$$

if for all $r \in S$, $a \cap (-\infty, r]$ is odd and for all $r \in T$, $a \cap (-\infty, r]$ is even. But it is easy to see that there are infinitely many finite sets a of rational numbers that satisfy these requirements. □

The following proof is due to Hausdorff and generalizes to higher cardinals [4]. We will discuss this generalization in Section 4.

Second proof. Let

$$I = \{(n, A) : n \in \omega \wedge A \subseteq \mathcal{P}(n)\}$$

For all $X \subseteq \omega$ let $X' = \{(n, A) \in I : X \cap n \in A\}$. We show that $\{X' : X \in \mathcal{P}(\omega)\}$ is an independent family of subsets of I.

Let S and T be finite disjoint subsets of $\mathcal{P}(\omega)$. A pair $(n, A) \in I$ is in

$$\bigcap_{X \in S} X' \cap \left(I \setminus \bigcup_{X \in T} X'\right)$$

if for all $X \in S$, $X \cap n \in A$ and for all $X \in T$, $X \cap n \notin A$. Since S and T are finite, there is $n \in \omega$ such that for any two distinct $X, Y \in S \cup T$, $X \cap n \neq Y \cap n$. Let $A = \{X \cap n : X \in S\}$. Now

$$(n, A) \in \bigcap_{X \in S} X' \cap \left(I \setminus \bigcup_{X \in T} X'\right).$$

Since there are infinitely many n such that for any two distinct $X, Y \in S \cup T$, $X \cap n \neq Y \cap n$, this shows that

$$\bigcap_{X \in S} X' \cap \left(I \setminus \bigcup_{X \in T} X'\right)$$

is infinite. □

A combinatorially simple, topological proof of the existence of large independent families can be obtained using the Hewitt-Marczewski-Pondiczery theorem which says that the product space $2^{\mathbb{R}}$ is separable ([5, 9, 10], also see [2]). This is the *first topological proof.*

Third proof. For each $r \in R$ let $B_r = \{f \in 2^{\mathbb{R}} : f(r) = 0\}$. Now whenever S and T are finite disjoint subsets of $\mathbb{R}$,

$$\bigcap_{r \in S} B_r \cap \left(2^{\mathbb{R}} \setminus \bigcup_{r \in T} B_r\right)$$

is a nonempty clopen subset of $2^{\mathbb{R}}$.

The family $(B_r)_{r \in \mathbb{R}}$ is the prototypical example of an independent family of size continuum on any set. A striking fact about the space $2^{\mathbb{R}}$ is that it is separable. Namely, let D denote the collection of all functions $f : \mathbb{R} \to 2$ such that there are rational numbers $q_0 < q_1 < \cdots < q_{2n-1}$ such that for all $x \in \mathbb{R}$,

$$f(x) = 1 \quad \Leftrightarrow \quad x \in \bigcup_{i<n} (q_{2i}, q_{2i+1}).$$

D is a countable dense subset of $2^{\mathbb{R}}$.

For each $r \in \mathbb{R}$ let $A_r = B_r \cap D$. Now for all pairs S, T of finite disjoint subsets of $\mathbb{R}$,

$$\bigcap_{r\in S} A_r \cap \left(D \setminus \bigcup_{r\in T} A_r\right) = D \cap \bigcap_{r\in S} B_r \cap \left(2^{\mathbb{R}} \setminus \bigcup_{r\in T} B_r\right)$$

is infinite, being the intersection of a dense subset with a nonempty open subset of a topological space without isolated points. It follows that $(A_r)_{r\in\mathbb{R}}$ is an independent family of size continuum on the countable set D. □

The *second topological proof* of Theorem 3.4 was pointed out by Ramiro de la Vega.

Fourth proof. Let $\mathcal{B}$ be a countable base for the topology on $\mathbb{R}$ that is closed under finite unions. Now for each $r \in \mathbb{R}$ consider the set $A_r = \{B \in \mathcal{B} : r \in B\}$. Then $(A_r)_{r\in\mathbb{R}}$ is an independent family of subsets of the countable $\mathcal{B}$.

Namely, let S and T be disjoint finite subsets of $\mathbb{R}$. The set $\mathbb{R} \setminus T$ is open and hence there are open sets $U_s \in \mathcal{B}$, $s \in S$, such that each U_s contains s and is disjoint from T. Since $\mathcal{B}$ is closed under finite unions, $U = \bigcup_{s\in S} U_s \in \mathcal{B}$. Clearly, there are actually infinitely many possible choices of a set $U \in \mathcal{B}$ such that $S \subseteq U$ and $T \cap U = \emptyset$. This shows that $\bigcap_{r\in S} A_r \setminus \left(\bigcup_{r\in T} A_r\right)$ is infinite. □

A variant of the Hewitt-Marczewski-Pondiczery argument was mentioned by Martin Goldstern who claims to have heard it from Menachem Kojman.

Fifth proof. Let P be the set of all polynomials with rational coefficients. For each $r \in \mathbb{R}$ let $A_r = \{p \in P : p(r) > 0\}$. If $S, T \subseteq \mathbb{R}$ are finite and disjoint, then there is a polynomial in P such that $p(r) > 0$ for all $r \in A$ and $p(r) \leq 0$ for all $r \in T$. All positive multiples of p satisfy the same inequalities. It follows that $(A_r)_{r\in\mathbb{R}}$ is an independent family of size $2^{\aleph_0}$ over the countable set P. □

The next proof was pointed out by Tim Gowers. This is the *dynamical proof.*

Sixth proof. Let X be a set of irrationals that is linearly independent over $\mathbb{Q}$. Kronecker's theorem states that for every finite set $\{r_1, \dots, r_k\} \subseteq X$ with pairwise distinct r_i, the closure of the set $\{(nr_1, \dots, nr_k) : n \in \mathbb{Z}\}$ is all of the k-dimensional torus $\mathbb{R}^k/\mathbb{Z}^k$ ([7], also see [1]).

For each $r \in X$ let A_r be the set of all $n \in \mathbb{Z}$ such that the integer part of $n \cdot r$ is even. Then $\{A_r : r \in X\}$ is an independent family of size continuum. To see this, let $S, T \subseteq X$ be finite and disjoint. By Kronecker's theorem there are infinitely many $n \in \mathbb{Z}$ such that for all $r \in S$, the integer part of $n \cdot r$ is even and for all $r \in T$, the integer part of $n \cdot r$ is odd. For all such n,

$$n \in \bigcap_{r\in S} A_r \cap \bigcap_{r\in T} \mathbb{Z} \setminus A_r.$$

□

The following proof was mentioned by KP Hart. Let us call it the *almost disjoint proof.*

Seventh proof. Let $\mathcal{F}$ be an almost disjoint family on ω of size continuum. To each $A \in \mathcal{F}$ we assign the collection A' of all finite subsets of ω that intersect A. Now $\{A' : A \in \mathcal{F}\}$ is an independent family of size continuum.

Given disjoint finite sets $S, T \subseteq \mathcal{F}$, by the almost disjointness of $\mathcal{F}$, each $A \in S$ is almost disjoint from $\bigcup T$. It follows that there are infinitely many finite subsets of ω that intersect all $A \in S$ but do not intersect any $A \in T$. Hence

$$\bigcap_{A \in S} A' \cap \left(\omega \setminus \bigcup_{A \in T} A'\right)$$

is infinite. □

The last proof was communicated by Peter Komjáth. This is the *proof by finite approximation.*

Eighth proof. First observe that for all $n \in \omega$ there is a family $(X_k)_{k<n}$ of subsets of 2^n such that for any two disjoint sets $S, T \subseteq n$,

$$\bigcap_{k \in S} X_k^n \cap \left(2^n \setminus \bigcup_{k \in T} X_k\right)$$

is nonempty. Namely, let $X_k = \{f \in 2^n : f(k) = 0\}$.

Now choose, for every $n \in \omega$, a family $(X_s^n)_{s \in 2^n}$ of subsets of a finite set Y_n such that for disjoint sets $S, T \subseteq 2^n$,

$$\bigcap_{s \in S} X_s^n \cap \left(2^n \setminus \bigcup_{s \in T} X_s^n\right)$$

is nonempty. We may assume that the Y_n, $n \in \omega$, are pairwise disjoint.

For each $\sigma \in 2^\omega$ let $X_\sigma = \bigcup_{n \in \omega} X^n_{\sigma \restriction n}$. Now $\{X_\sigma : \sigma \in 2^\omega\}$ is an independent family of size $2^{\aleph_0}$ on the countable set $\bigcup_{n \in \omega} Y_n$. □

4. Independent families on larger sets

We briefly point out that for every cardinal κ there is an independent family of size 2^κ of subsets of κ. We start with a corollary of the Hewitt-Marczewski-Pondiczery Theorem higher cardinalities.

Lemma 4.1. *Let κ be an infinite cardinal. Then 2^{2^κ} has a dense subset D such that for every nonempty clopen subset A of 2^{2^κ}, $D \cap A$ is of size κ. In particular, 2^{2^κ} has a dense subset of size κ.*

Proof. For each finite partial function s from κ to 2 let $[s]$ denote the set $\{f \in 2^\kappa : s \subseteq f\}$. The product topology on 2^κ is generated by all sets of the form $[s]$. Every clopen subset of 2^κ is compact and therefore the union of finitely many sets of the form $[s]$. It follows that 2^κ has exactly κ clopen subsets. The continuous functions from 2^κ to 2 are just the characteristic functions of clopen sets. Hence there are only κ continous functions from 2^κ to 2. Let D denote the set of all continuous functions from 2^κ to 2.

Since finitely many points in 2^κ can be separated simultaneously by pairwise disjoint clopen sets, every finite partial function from 2^κ to 2 extends to a continuous functions defined on all of 2^κ. It follows that D is a dense subset of 2^{2^κ} of size κ.

Now, if A is a nonempty clopen subset of 2^{2^κ}, then there is a finite partial function s from 2^κ to 2 such that $[s] \subseteq A$. Cleary, the number of continuous extensions of s to all of 2^κ is κ. Hence $D \cap A$ is of size κ. □

As in the case of independent families on ω, from the previous lemma we can derive the existence of large independent families of subsets of κ.

Theorem 4.2. *For every infinite cardinal cardinal κ, there is a family $\mathcal{F}$ of size 2^κ such that for all disjoint finite sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$, the set*

$$\left(\bigcap \mathcal{A}\right) \setminus \bigcup \mathcal{B}$$

is of size κ.

First proof. Let $D \subseteq 2^{2^\kappa}$ be as in Lemma 4.1. For each $x \in 2^\kappa$ let $B_x = \{f \in 2^{2^\kappa} : f(x) = 0\}$ and $A_x = D \cap B_x$. Whenever S and T are disjoint finite subsets of 2^κ, then

$$\left(\bigcap_{x \in S} B_x\right) \setminus \bigcup_{x \in T} B_x$$

is a nonempty clopen subset of 2^{2^κ}. It follows that

$$\left(\bigcap_{x \in S} A_x\right) \setminus \bigcup_{x \in T} A_x = D \cap \left(\left(\bigcap_{x \in S} B_x\right) \setminus \bigcup_{x \in T} B_x\right)$$

is of size κ. It follows that $\mathcal{F} = \{A_x : x \in 2^\kappa\}$ is as desired. □

We can translate this topological proof into combinatorics as follows:

The continuous functions from 2^κ to 2 are just characteristic functions of clopen sets. The basic clopen sets are of the form $[s]$, where s is a finite partial function from κ to 2. All clopen sets are finite unions of sets of the form $[s]$. Hence we can code clopen subsets of 2^κ in a natural way by finite sets of finite partial functions from κ to 2. We formulate the previous proof in this combinatorial setting. The following proof is just a generalization of our second proof of Theorem 3.4. This is essentially Hausdorff's proof of the existence large independent families in higher cardinalities.

Second proof. Let D be the collection of all finite sets of finite partial functions from κ to 2. For each $f : 2^\kappa \to 2$ let A_f be the collection of all $a \in D$ such that for all $s \in a$ and all $x : \kappa \to 2$ with $s \subseteq x$ we have $f(x) = 1$.

Claim 4.3. For any two disjoint finite sets $S, T \subseteq 2^\kappa$ the set

$$\left(\bigcap_{x \in S} A_x\right) \setminus \bigcup_{x \in T} A_x$$

is of size κ.

For all $x \in S$ and all $y \in T$ there is $\alpha \in \kappa$ such that $x(\alpha) \neq y(\alpha)$. It follows that for every $x \in S$ there is a finite partial function s from κ to 2 such that $s \subseteq x$ and for all $y \in T$, $s \not\subseteq T$. Hence there is a finite set a of finite partial functions from κ to 2 such that all $x \in S$ are extensions of some $s \in a$ and no $y \in T$ extends any $s \in a$. Now $a \in (\bigcap_{x\in S} A_x) \setminus \bigcup_{x\in T} A_x$. But for every $\alpha < \kappa$ we can build the set a in such a way that α is in the domain of some $s \in a$. It follows that there are in fact κ many distinct sets $a \in (\bigcap_{x\in S} A_x) \setminus \bigcup_{x\in T} A_x$. □

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