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# VARIATIONS OF FODOR REFLECTION PRINCIPLES

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ABSTRACT. Fodor-type Reflection Principles claim the existence of an $\omega_1$-club in $\mathcal{P}_{\omega_1}\omega_2$ such that every element contains a fixed ladder sequence converging to its own supremum. We formulate some variations, all of which follow from MM, e.g. one in which the elements of the $\omega_1$-club have finite intersection with their ladder sequence. Some of the variations given do not involve the reflection of stationary sets of ordinals, but we show that even those variations are not a consequence of PFA.

## 1. THE FODOR-TYPE REFLECTION PRINCIPLE

The following principle has been introduced and studied in [3]. The abbreviation FRP stems from the term *Fodor-type Reflection Principle.*

FRP($\omega_2$) is the statement that for every system $\langle C_\alpha : \alpha \in S\rangle$ where

$$S \subseteq \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\} = S^0_2$$

is stationary there is a $\gamma \in S^1_2$ and a filtration $\langle F_\xi : \xi < \omega_1\rangle$ of $\gamma$ such that

- $\sup(F_\xi) \in S$
- $C_{\sup(F_\xi)} \subseteq F_\xi$

for stationarily many $\xi < \omega_1$.

We need some definitions to understand the above statement. If $\gamma$ is a set of size $\aleph_1$, then a continuous $\subseteq$-chain $\langle F_\xi : \xi < \omega_1\rangle$ is called a *filtration of* $\gamma$ if each $F_\xi$ is countable and $\bigcup_{\xi<\omega_1} F_\xi = \gamma$. $S^1_2$ is the collection of all ordinals of cofinality $\omega_1$ below $\omega_2$.

FRP$^0$($\omega_2$) is the statement that for every ladder system $\langle C_\alpha : \alpha \in S^0_2\rangle$ there is a $\gamma \in S^1_2$ and a filtration $\langle F_\xi : \xi < \omega_1\rangle$ of $\gamma$ such that

- $C_{\sup(F_\xi)} \subseteq F_\xi$

for stationarily many $\xi < \omega_1$.

Now we need the following Lemma:

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**1 Lemma.** *Assume that $\theta$ is large enough. If $S \subseteq S^0_2$ is stationary and $\langle C_\alpha : \alpha \in S\rangle$ a ladder system of $\omega$-sequences, then the sets*

$$\begin{aligned} \mathcal{E}_{\mathrm{in}}(S) &= \{N \in [\omega_2]^{\aleph_0} : \sup(N) \in S \textit{ and } C_{\sup(N)} \subseteq N\} \\ \mathcal{E}_{\mathrm{out}}(S) &= \{N \in [\omega_2]^{\aleph_0} : \sup(N) \in S \textit{ and } C_{\sup(N)} \not\subseteq N\} \\ \mathcal{E}_{\mathrm{fin}}(S) &= \{N \in [\omega_2]^{\aleph_0} : \sup(N) \in S \textit{ and } C_{\sup(N)} \cap N \textit{ is finite}\} \end{aligned}$$

*are all projectively stationary.*

*Proof.* Let $E \subseteq \omega_1$ be stationary, $f : {\omega_2}^{<\omega} \longrightarrow \omega_2$ a function and assume that $M_i$ $(i < \omega)$ is a sequence of models of size $\aleph_1$ such that $M_i \cap \omega_2 = \delta_i$ and $\delta = \sup_{i<\omega} \delta_i \in S$. Also assume that $E, f \in M_0$ and set $M \bigcup_{i<\omega} M_i$. In $M$ we can build a continuous chain $N_\xi$ $(\xi < \omega_1)$ of countable models such that $C_\delta \subseteq N_0$. Then there is $\xi < \omega_1$ such that $N_\xi \cap \omega_1 \in E$, which proves the Lemma for the set $\mathcal{E}_{\mathrm{in}}(S)$. To show the claim for the set $\mathcal{E}_{\mathrm{fin}}(S)$, we use a game from [6, p.272]. This game is as follows:

| | | | | | |
|---|---|---|---|---|---|
| I | $I_0, \xi_0$ | $I_1, \xi_1$ | $I_2, \xi_2$ | $I_3, \xi_3$ | $\dots$ |
| II | $\mu_0$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\dots$ |

where the $I_i$'s are intervals in $\omega_2$ of the form $[\gamma_i, \bar{\gamma}_i]$ and with the property that $\xi_i \in I_i$. The $\mu_i$'s are ordinals below $\omega_2$. We also require that $\mu_i < \gamma_{i+1}$. Player I wins the game if

$$y = \mathrm{cl}_f(\xi_i)_{i<\omega}$$

has the property that

$$y \subseteq \bigcup_{i<\omega} I_i \text{ and } y \cap \omega_1 \in E.$$

[6] shows that Player I has a winning strategy in this game.

Having such a winning strategy $\sigma \in M_0$, it is straightforward to apply it for our purposes. Player I plays intervals $[\gamma_i, \bar{\gamma}_i]$ such that a final segment of $C_\delta$ is disjoint from $\bigcup_{i<\omega}[\gamma_i, \bar{\gamma}_i]$. This suffices by the definition of the winning condition and note that the responses of Player I to $[\gamma_i, \bar{\gamma}_i]$ will be in the structure $M_i$ as long as $\gamma_i$ and $\bar{\gamma}_i$ are in $M_i$. □

**2 Remark.** *The following holds:*

(1) MM *implies* FRP$(\omega_2)$
(2) FRP$(\omega_2)$ *implies* FRP$^0(\omega_2)$

*Proof.* (2) is clear and for (1): it is well-known that MM implies that every projectively stationary set contains a continuous $\omega_1$-chain (see [2]), so FRP$(\omega_2)$ can be deduced using Lemma 1. The reader will notice that we can even replace "stationarily many $\xi < \omega_1$" with "all $\xi < \omega_1$" in the statement of $\mathrm{FRP}(\omega_2)$ and still deduce this from MM with the same argument. See also Remark 4. $\square$

The natural poset to force the negation of $\mathrm{FRP}^0(\omega_2)$ is the following: conditions of $\mathbb{P}$ are of the form

$$\langle C_\alpha : \alpha \leq \mu, \mathrm{cf}(\alpha) = \omega\rangle, \langle F_\xi^\gamma : \xi < \omega_1, \gamma \in S_2^1\rangle$$

where

(1) $\mu < \omega_2$
(2) for each $\omega$-cofinal $\alpha \leq \mu$, $C_\alpha$ is a cofinal $\omega$-sequence in $\alpha$
(3) for each $\gamma \in S_2^1$, $C_{\sup(F_\xi^\gamma)} \not\subseteq F_\xi^\gamma$ for all $\xi < \omega_1$.

We note that the poset $\mathbb{P}$ is $< \omega_2$-strategically closed. The argument is similar to the argument that the standard forcing to add $\square_{\omega_1}$ is $< \omega_2$-strategically closed (see for example [4, p.255]).

The following theorem shows two things of interest. On the one hand it shows that even though FRP fails after forcing our counterexample to $\mathrm{FRP}^0(\omega_2)$, a strong version of ordinal reflection may still hold. It shows on the other hand that $\mathrm{FRP}^0$ is not a consequence of PFA. It is easy to see that FRP is not a consequence of PFA since PFA is consistent with a non-reflecting subset of $S_2^0$ (see [1]), but the consistency of PFA with $\mathrm{FRP}^0$ requires the following argument. Remember that $\mathrm{Fr}^+(\omega_2)$ is the statement that for every stationary $S \subseteq S_2^0$ there is an $\omega_1$-cofinal ordinal $\gamma < \omega_2$ such that $S \cap \gamma$ is club in $\gamma$. See [5, p.524] for more information on this statement.

**3 Theorem.** *Assume $V \models \mathrm{MM}$. Let $\mathbb{P}$ be as in the previous paragraph. Then*

$$V^{\mathbb{P}} \models \mathrm{PFA} + \mathrm{Fr}^+(\omega_2) + \neg\mathrm{FRP}^0(\omega_2).$$

*Proof.* First notice that the $\mathbb{P}$-generic object is a counterexample to $\mathrm{FRP}^0(\omega_2)$.

**3.1 Claim.** $V^{\mathbb{P}} \models \mathrm{Fr}^+(\omega_2)$

*Proof of Claim 3.1.* Let $\dot{S}$ be a $\mathbb{P}$-name for a stationary subset of $S_2^0$. Now add a continuous $\omega_1$-chain through $\mathcal{E}_{\mathrm{out}}(\dot{S})$. Note that by Lemma 1, this can be done with a forcing $\mathbb{F}_{\mathrm{out}}(\dot{S})$ which preserves stationary subsets of $\omega_1$. We briefly describe that forcing: conditions of $\mathbb{F}_{\mathrm{out}}(\dot{S})$ are continuous chains of the form

$$\langle F_\xi : \xi \leq \zeta\rangle,$$

where

(1) $\zeta < \omega_1$ and for all $\xi \leq \zeta$
(2) $F_\xi$ is a countable subset of $\omega_2^V$
(3) $\sup(F_\xi) \in \dot{S}$
(4) $C_{\sup(F_\xi)} \not\subseteq F_\xi$.

This basically shoots a filtration through $\omega_2^V$ that avoids the ladder system given to us by the poset $\mathbb{P}$ and such that $\sup(F_\xi \cap \omega_2) \in \dot{S}$ for all $\xi < \omega_1$. Now apply MM to the iteration $\mathbb{P} * \mathbb{F}_{\text{out}}(\dot{S})$ and get a sufficiently generic $G \subseteq \mathbb{P} * \mathbb{F}_{\text{out}}(\dot{S})$.

**3.1.1 Subclaim.** *$G \restriction \mathbb{P}$ extends to a condition $p_G \in \mathbb{P}$.*

*Proof of Subclaim 3.1.1.* This is because we have forced a good filtration for $G \restriction \mathbb{P}$, so it can be extended to a condition. □

**3.1.2 Subclaim.** *$p_G \Vdash \{\sup(F_\xi) : \xi < \omega_1\}$ is an $\omega_1$-club in $\dot{S}$.*

*Proof of Subclaim 3.1.2.* Clear because the filtrations given by filters for $\mathbb{F}_{\text{out}}(\dot{S})$ are continuous chains. □

This last Subclaim finishes the proof of $\mathrm{Fr}^+(\omega_2)$ in $V^{\mathbb{P}}$. □

**3.2 Claim.** $V^{\mathbb{P}} \models$ PFA.

*Proof of Claim 3.2.* Assume that $\mathbb{P} \Vdash \mathbb{Q}$ is proper. Then look at the iteration $\mathbb{P} * \mathbb{Q} * \mathbb{F}$ where $\mathbb{F} = \mathbb{F}_{\text{out}}(S_2^0)$.

**3.2.1 Subclaim.** *$\mathbb{P} * \mathbb{Q} * \mathbb{F}$ is proper.*

*Proof of Subclaim 3.2.1.* Let $N \prec H_\theta$ containing everything in sight and set $\gamma = N \cap \omega_1$, $\delta = \sup(N \cap \omega_2)$. Given an N-generic sequence for the iteration, we make sure that the $\mathbb{P}$-entries of that sequence are extended with a ladder $C_\delta \not\subseteq N$. This is easily possible and makes sure that the $\mathbb{F}$-entries of our $N$-generic sequence of conditions will be extendable since the requirement for that will be

$$C_\delta \not\subseteq F_\gamma = N \cap \omega_2.$$

□

This subclaim basically suffices, the rest of the argument is similar to Claim 3.1, i.e. reprove Subclaims 3.1.1 and 3.1.2. □

□

## 2. A DUAL TO FRP

In this section we turn our attention to a statement that is dual to FRP. This dual statement asks for a filtration whose countable members meet each ladder sequence only on a finite set and we denote it by dFRP.

dFRP($\omega_2$) says that for every ladder system $\langle C_\alpha : \alpha \in S\rangle$ where $S \subseteq S^0_2$ is stationary there is a $\gamma \in S^1_2$ and a filtration $\langle F_\xi : \xi < \omega_1\rangle$ of $\gamma$ such that

- $\sup(F_\xi) \in S$
- $C_{\sup(F_\xi)} \cap F_\xi$ is finite

for stationarily many $\xi < \omega_1$.

We mention two variations of dFRP. dFRP$^+(\omega_2)$ is the same as dFRP($\omega_2$) except that the last line in the definition is replaced by "... for all $\xi < \omega_1$".

dFRP$^0(\omega_2)$ says that for every ladder system $\langle C_\alpha : \alpha \in S^0_2\rangle$ there is a $\gamma \in S^1_2$ and a filtration $\langle F_\xi : \xi < \omega_1\rangle$ of $\gamma$ such that

- $C_{\sup(F_\xi)} \cap F_\xi$ is finite

for stationarily many $\xi < \omega_1$.

**4 Remark.** *The following holds:*

(1) MM *implies* dFRP$^+(\omega_2)$
(2) dFRP$^+(\omega_2)$ *implies* dFRP($\omega_2$)
(3) dFRP($\omega_2$) *implies* dFRP$^0(\omega_2)$

*Proof.* Similar to Remark 2, Lemma 1 for $\mathcal{E}_{\text{fin}}(S)$ shows (1). The rest is fairly clear. □

Similar to (1) in Remark 2, Lemma 1 for $\mathcal{E}_{\text{fin}}(S)$ shows that MM implies the statement dFRP$^+(\omega_2)$.

It is interesting to note that a statement analogous to dFRP$^+$ for $\omega_1$ would say the following: for every ladder system on $\omega_1$ there is a club $C \subseteq \omega_1$ such that $C$ intersects each ladder only on a finite set. This statement is known to follow from PFA (see e.g. [5, p.133]) and is sometimes referred to as "negation of $\clubsuit_{\omega_1}$(club)".

We can use the techniques described earlier to get an interesting result: though $\clubsuit_{\omega_1}$(club) fails under PFA, even the weakest form of dFRP($\omega_2$) is independent of PFA.

**5 Theorem.** PFA *is consistent with the negation of* dFRP$^0(\omega_2)$.

*Proof.* This is using the exact same arguments as in the proof of Theorem 3, except that we need to modify the definition of $\mathbb{P}$ in the obvious way: conditions have the property that for each $\gamma \in S^1_2$, $C_{\sup(F^\gamma_\xi)} \cap F^\gamma_\xi$ is unbounded in $\sup(F^\gamma_\xi)$ for all $\xi < \omega_1$. □

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