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UNPREPARED INDESTRUCTIBILITY

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Abstract. I present a forcing indestructibility theorem for the large cardinal axiom Vopěnka's Principle. It is notable in that there is no preparatory forcing required to make the axiom indestructible, unlike the case for other indestructibility results.

§1. Introduction. This article is based on the talk I gave at the "Aspects of Descriptive Set Theory" RIMS Symposium in October 2011. It is essentially just a survey of the article [3]. I would like to thank the organisers for inviting me to speak at this Symposium.

We shall be concerned with the following axiom schema (which we shall refer to simply as an axiom henceforth).

Vopěnka's Principle: For any first order signature Σ and any proper class A of Σ-structures, there are $\mathcal{M}, \mathcal{N} \in A$ such that there is a non-trivial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

This axiom is at the upper end of the large cardinal hierarchy, lying between supercompact cardinals and huge cardinals in strength.

Vopěnka's Principle has found a number of applications in category theory; indeed, the entire final chapter of Adámek and Rosický's book *Locally presentable and accessible categories* [1] is centred on Vopěnka's Principle, giving many implications of and equivalent statements to Vopěnka's Principle in the context of the book's eponymous categories. Vopěnka's Principle also gained interest from algebraic topologists at the start of this century when Casacuberta, Scevenels and Smith [5] showed that, under the assumption of Vopěnka's Principle, every generalised cohomology theory admits a Bousfield localisation functor. This answered a question that had remained open for 30 years, since Bousfield proved (in ZFC alone) the corresponding result for generalised homology theories (note however that the large cardinal assumption needed has since been reduced by Casacuberta, Bagaria, Mathias and Rosický [2] to a proper class of supercompacts; there is still no known lower bound on the large cardinal strength required).

From a set-theoretic perspective, on the other hand, Vopěnka's Principle has been widely overlooked. A key aim of this research was to show the relative consistency of Vopěnka's Principle with the usual array of statements known to be independent of ZFC. There

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seem to be two main approaches to proving such relative consistency results for large cardinals defined in terms of elementary embeddings. First, one can sometimes show that a preliminary forcing makes the large cardinal indestructible to further forcing satisfying some properties. The best known case of this is the Laver preparation [7], but Hamkins [6] has also proved similar results for other large cardinals. In other situations, one can sometimes apply a *master condition* argument, as pioneered by Silver, in which those generics containing a certain condition give rise to a generic extension in which the large cardinal is preserved. That is, a condition can be found which forces the cardinal in question to retain its large cardinal property.

As described below, for Vopěnka's Principle we find ourselves in a situation that combines the two. In carrying out a master condition argument, we find that in fact master conditions will be dense. Thus, the large cardinal is preserved in all generic extensions for forcings of the given kind, and hence we have an indestructibility theorem without any preparatory forcing required.

§2. **Preliminaries.** As already alluded to, in ZFC Vopěnka's Principle is really an axiom schema, since it refers to proper classes. It is simpler, and probably intuitively clearer for most readers, to work with subsets of V_κ for inaccessible κ than with proper classes. Thus we shall focus here on *Vopěnka cardinals*; only minor technical adjustments are required to translate the proof to the proper class version of Vopěnka's Principle, and these are given in [3].

DEFINITION 1. *A cardinal κ is a* Vopěnka cardinal *if and only if it is inaccessible and V_κ satisfies Vopěnka's Principle where "class" is taken to mean subset of V_κ.*

Note that for Vopěnka cardinals we do not just require that Vopěnka's Principle holds for subsets of V_κ definable in V_κ, but rather for *all* subsets of V_κ. It makes no difference to the proof, though.

Let us begin with a trivial observation.

OBSERVATION 2. *If κ is a Vopěnka cardinal, and $\mathbb{P}$ is a forcing partial order which adds no new subsets of V_κ, then in the generic extension by $\mathbb{P}$, κ remains a Vopěnka cardinal.*

Thus, if we have a forcing iteration such that the tail from some stage onward is κ-distributive, then to prove that κ remains a Vopěnka cardinal in the extension, it suffices to show that it is preserved in the part of the iteration up to that stage.

Another interesting Corollary of this Observation is the following.

COROLLARY 3.

$$\mathrm{Con}(ZFC + \exists\kappa(\kappa \textit{ is a Vopěnka cardinal})) \rightarrow \mathrm{Con}(ZFC + \exists\kappa(\kappa \textit{ is a Vopěnka cardinal } + \square_\kappa))$$

PROOF. The usual (Jensen) partial ordering for forcing $\square_\kappa$ to hold is $< \kappa^+$ strategically closed, and so in particular adds no new subsets of V_κ. $\dashv$

This contrasts with, for example, Solovay's result that $\square_\alpha$ must fail above a supercompact cardinal. Whilst Vopěnka's Principle has greater consistency strength than the existence of a supercompact cardinal, and indeed below any Vopěnka cardinal κ there must be many $< \kappa$-supercompact cardinals, a Vopěnka cardinal need not be supercompact or even weakly compact. The principle $\square_\kappa$ is an example of an incompactness phenomenon, as it directly violates a simple form of reflection, and so it is that it can hold at a Vopěnka cardinal but not large cardinals with more of a "compactness" flavour. See [4] for finer resolution results about the compatibility of square with large cardinals.

§3. The theorem. In this section I will give an outline of the proof of the following main theorem.

THEOREM 4. *Let κ be a Vopěnka cardinal. Suppose $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is the reverse Easton iteration of $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa \rangle$ where*

- *for each $\alpha < \kappa$, $|\dot{\mathbb{Q}}_\alpha| < \kappa$, and*
- *for all $\gamma < \kappa$, there is an η_0 such that for all $\eta \geq \eta_0$,*

$$\mathbb{1}_{\mathbb{P}_\eta} \Vdash \dot{\mathbb{Q}}_\eta \text{ is } \gamma\text{-directed-closed.}$$

Then

$$\mathbb{1}_{\mathbb{P}_\kappa} \Vdash \kappa \text{ is a Vopěnka cardinal.}$$

First let us recall Silver's technique of lifting elementary embeddings. If we have an elementary embedding $j : V \longrightarrow M$ and a partial order $\mathbb{P}$, the idea is to find a V-generic $G \subset \mathbb{P}^V$ and an M-generic $H \subset \mathbb{P}^M$ so that $M[H] \subset V[G]$ and j lifts to an embedding $j' : V[G] \longrightarrow M[H]$. If $j``G \subset H$ we can do this by taking

$$j'(\sigma_G) = j(\sigma)_H$$

for every $\mathbb{P}$-name $\sigma \in V$. Indeed, j' will be well-defined and elementary by the Truth Lemma for forcing, since everything true in the extension model is forced, and $p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ implies $j(p) \Vdash \varphi(j(\sigma_1), \dots, j(\sigma_n))$ by elementarity and the definability of the forcing relation.

If $\mathbb{P}$ is an iteration of increasingly directed-closed forcing partial orders, then it may happen that $j``(G)$ (at least from the critical-point-of-j-th stage onward) is extended by a single condition p — the *master condition.* In this case, choosing G such that $p \in H$ then gives us our lifted embedding j'. On the other hand, in general it does not follow that the embedding will lift for arbitrary choices of G.

Vopěnka's Principle seems to be in a certain sense much more flexible than other "elementary embedding" large cardinal axioms. For each class A there will be many embeddings $j : \mathcal{M} \to \mathcal{N}$ with $\mathcal{M}, \mathcal{N} \in A$ witnessing Vopěnka's Principle for A: for any such j, we can consider Vopěnka's Principle for the class $A \setminus \{\mathcal{M}\}$ to get another. Moreover, the embeddings are not required to respect A at all, merely the elements of A they are between. Yet Vopěnka's Principle is stated by quantifying over classes; to test whether it is true we take names for classes, and see whether we can find embeddings in the generic extension witnessing Vopěnka's Principle for that class. To this end, we can use equivalent names, and in particular, names in which the names for the elements are especially nice. To whit:

LEMMA 5. *Let $\mathbb{P}_\kappa$ be as in the statement of Theorem 4, and let $\dot{A}$ be a $\mathbb{P}_\kappa$-name for a set of Σ-structures with ordinal domains. There is a name $\dot{A}'$ equivalent to $\dot{A}$ such that for for every $\langle \sigma, p \rangle \in \dot{A}$,*

- *σ is the canonical name for the structure $\langle \gamma_\sigma, E^\sigma, R^\sigma \rangle$ using names $\check{\gamma}_\sigma$, $\dot{E}^\sigma$, and $\dot{R}^\sigma$ respectively for the components.*
- *the names $\dot{E}^\sigma$ and $\dot{R}^\sigma$ involve no conditions larger than is necessary:*

  if δ is the least inaccessible cardinal greater than γ_σ such that $|\mathbb{P}_\delta| \leq \delta$ and

  $$\eta \geq \delta \to \Vdash_{\mathbb{P}_\eta} \dot{\mathbb{Q}}_\eta \text{ is } \gamma_\sigma^+\text{-directed-closed}$$

  then $\dot{R}^\sigma$ is a $\mathbb{P}_\delta$-name for a subset of γ_σ, and $\dot{E}^\sigma$ is a $\mathbb{P}_\delta$-name for a subset of γ_σ^2.

The proof of Lemma 5 is a fairly typical case of taking the names for elements, and replacing them with multiple nicer names by extending the corresponding forcing condition. The consideration of structures with ordinals as their underlying sets is simply a convenient way to to get concrete underlying sets, and of course can be achieved by the liberal use of the Axiom of Choice. In the definable proper class form of Vopěnka's Principle, where global choice is tantamount to $V = HOD$, there are other ways around this — see [3].

Whilst the embeddings witnessing Vopěnka's Principle as we have defined it need not respect A, there *is* a reformulation involving large cardinals that do, due to Solovay, Reinhardt and Kanamori:

THEOREM 6 (Solovay, Reinhardt and Kanamori). *An inaccessible cardinal κ is a Vopěnka cardinal if and only if, for every $A \subseteq V_\kappa$, there is an $\alpha < \kappa$ such that for every η strictly between α and κ, there is a λ strictly between η and κ and an elementary embedding*

$$j : \langle V_\eta, \in, A \cap V_\eta \rangle \to \langle V_\lambda, \in, A \cap V_\lambda \rangle$$

with critical point α, such that $j(\alpha) > \eta$.

We call α as in Theorem 6 *extendible below κ for A.*

So now suppose we have a nice name $\dot{A}$ as given by Lemma 5 for a subset of κ of size κ, and suppose that in V, α is extendible for $\dot{A}$ below κ. Let G be $\mathbb{P}_\kappa$-generic over V. Then since A is large, there is some $\langle \sigma, q \rangle$ in $\dot{A}$ with $q \in G$ and γ_σ, the ordinal which is the underlying set of σ_G, greater than α. For each η between α and κ, we have an elementary embedding from $\langle V_\eta, \in, \dot{A} \cap V_\eta \rangle$ to $\langle V_\lambda, \in, \dot{A} \cap V_\lambda \rangle$ with critical point α, for some $\lambda < \kappa$. We shall show that one of these, when restricted to γ_σ, lifts to an elementary embedding from σ_G to another member of A. Of course, this witnesses Vopěnka's Principle for A in the generic extension.

How do we manage this? A master condition argument seems quite possible, and indeed that is the approach we take. Usually though, the generic has to be chosen to contain the specific master condition, which would be a problem for us, since there are many classes for which we want to witness Vopěnka's Principle, each with their own master condition, and no reason why these shouldn't disagree with one another.

The trick we use is to show that there are many possible master conditions for each $\dot{A}$ and σ, corresponding to the many embeddings witnessing the η-extendibility of α for A below κ as η varies. Indeed, there are enough that such master conditions are in fact *dense* in $\mathbb{P}_\kappa$, so any generic must contain one of them.

With that idea in mind, it is in fact quite straightforward to show that master conditions for $\dot{A}$ and σ are dense. We factor $\mathbb{P}_\kappa$ as $\mathbb{P}_\xi * \mathbb{P}^\xi$, where P_ξ is big enough to completely determin σ_G. Now, let p be an arbitrary condition in $\mathbb{P}^\xi$. It is bounded below κ, so let η be greater than the support of p, and also large enough that beyond stage η, the forcing iterands $\mathbb{Q}_\nu$ are all $|\mathbb{P}_\xi|^+$ directed closed.

Let $j : \langle V_\eta, \in, \dot{A} \cap V_\eta \rangle \to \langle V_\lambda, \in, \dot{A} \cap V_\lambda \rangle$ in V be an elementary embedding witnessing that α is η-extendible below κ for $\dot{A}$. Crucially, we have that $j(\alpha) > \eta$. So consider what happens to the $\mathbb{P}_\xi$ part G_ξ of our generic when j is applied to it point-wise. For each condition s in P_ξ, the support of s below α is bounded below α (α is inaccessible), and so is unchanged by j. The rest of s, having support starting at α, is sent to something with support starting at $j(\alpha) > \eta$. So the support of $j(s)$ is disjoint from the interval $[\alpha, \eta)$. The "lower parts" must already be in G, and "upper parts" are a directed system of at most $|\mathbb{P}_\xi|$ many conditions in $\mathbb{P}_\eta$, and so are extended by a master condition r in $\mathbb{P}_\eta$, since $\mathbb{P}_\eta$ is $|\mathbb{P}_\xi|^+$ directed closed. Meanwhile, our arbitrary condition p in $\mathbb{P}_\xi$ has support disjoint from the master condition, and so there is a condition extending both p and r, which of course still functions as a master condition.

So, master conditions for $\dot{A}$ and σ are indeed dense, and so our generic G must contain one. Thus, we have that some j witnessing η-extendiblity below κ for $\dot{A}$ lifts to an elementary embedding in the generic extension. We claim that the restriction of this embedding to σ_G witnesses Vopěnka's Principle for A in the generic extension. Since $\langle \sigma, q \rangle \in \dot{A}$, $\langle j(\sigma), j(q) \rangle \in \dot{A}$, by the elementarity of j. We assumed that $q \in G_\xi$, so $j(q) \in j``G_\xi$, and hence the master condition forces that $j(\sigma)_G \in A$. Finally, by the definition of j', $j' \upharpoonright \sigma_G$ is a map from σ_G to $j(\sigma)_G$, and it is elementary since j' is. Thus, we have that $j' \upharpoonright \sigma_G$ is elementary from σ_G to $j(\sigma)_G$, both of which are in A. This completes the proof of Theorem 4.

§4. Corollaries and Optimality. As a taster, here are some immediate corollaries of Theorem 4.

COROLLARY 7. *If the existence of a Vopěnka cardinal is consistent, then the existence of a Vopěnka cardinal is consistent with any of the following.*

- *GCH*
- *A definable well-order on the universe.*
- $\diamondsuit^+_{\kappa^+}$ *for every infinite cardinal* κ.
- *Morasses at every infinite successor cardinal.*

Theorem 4 also allows us to obtain results that may at first be surprising, in light of the reflection properties that other strong large cardinal enjoy. For example, we have the following.

COROLLARY 8. *Suppose* κ *is a Vopěnka cardinal and* $2^\kappa \neq \kappa^+$. *Then there is a generic extension in which* κ *remains Vopěnka and is the least point of failure of the GCH.*

Of course, the proof goes by using the usual κ-length forcing iteration to make the GCH hold up to, but not including, κ, and observing that Theorem 4 applies to this forcing. Corollary 8 contrasts with the result going back to Scott [8] that a measurable cardinal cannot be the least point of failure of the GCH.

To close, let use make a note regarding the optimality of Theorem 4: the assumption that the forcing iterands $\mathbb{Q}_\gamma$ were increasingly *directed* closed was necessary. Indeed, with an iteration of increasingly closed (but not directed closed) partial orders, one can force there to be Kurepa trees at every inaccessible cardinal less than κ. This kills all ineffable cardinals below κ, but for κ to be Vopěnka, there must be many ineffables less than κ (for example every measurable cardinal is ineffable).

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