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A global variational principle for nonlinear evolution

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Abstract

This note is devoted to discussing a variational formulation for non-equilibrium systems. We first provide an introductory course on a global variational method for gradient systems based on *Weighted Energy-Dissipation* (WED) functionals. We then review recent results in [2, 3] for doubly nonlinear evolution equations.

This note is based on a joint work with Ulisse Stefanelli (IMATI/CNR).

1 Evolution equations and variational principle

Let X be a configuration space and let u be an X-valued function of time such that $u(t)$ describes a state at time t:

$$u: \begin{array}{ccc} [0,\infty) & \to & X \\ t & \mapsto & u(t) \end{array}$$

Now, an evolution law of state is supposed to be given by an ordinary differential equation in X, which is called *evolution equation*, such as

$$u'(t) = B(u(t)) \ \text{ in } X, \quad 0 < t < \infty,$$

where $u' = \mathrm{d}u/\mathrm{d}t$ and B is an operator in X. In particular, time-evolutionary PDEs (e.g., heat/Navier-Stokes/wave/Schrödinger equations) can be reduced to evolution equations.

Dynamics of non-equilibrium states of irreversible processes (e.g., heat transfer) might be mathematically formulated as *gradient systems*, where B has a gradient structure, i.e., $B = -\nabla\phi$ with some energy/entropy functional $\phi : H \to \mathbb{R}$ on a Hilbert space H,

$$u'(t) = -\nabla\phi(u(t)) \ \text{ in } H, \quad 0 < t < \infty. \tag{GS}$$

Here $\nabla\phi$ denotes a functional derivative of ϕ in a proper sense. Then the energy/entropy $\phi(u(t))$ is decreasing in time (i.e., (GS) enjoys a dissipative structure), and hence, the state $u(t)$ evolves irreversibly.

Morton E. Gurtin [9] proposed a fairly general description for non-equilibrium systems by using the following *generalized gradient system*:

$$\nabla\psi(u'(t)) = -\nabla\phi(u(t)) \ \text{ in } H, \quad 0 < t < \infty \tag{GSS}$$

with two functionals ψ, ϕ in H. In continuum thermomechanics, ψ and ϕ are often called *dissipation functional* and *energy functional*, respectively.

Thus the dynamics of non-equilibrium state is often described as a gradient system. On the other hand, equilibrium states of such irreversible processes are often formulated in a variational fashion. As for (GS), equilibria are critical points of the energy functional ϕ, and hence, the corresponding Euler-Lagrange equation reads,

$$\nabla\phi(u) = 0.$$

state	formulation
non-equilibrium state	(generalized) gradient system
equilibrium state	variational problem

In this note, we shall discuss variational formulations for non-equilibrium systems. We start with the following two examples.

Example 1.1 (Implicit time-discretization of gradient systems)**.** One can incrementally obtain a next step u_n from the previous step u_{n-1} by solving the semi-discretized problem for (GS),

$$\frac{u_n - u_{n-1}}{h} = -\nabla\phi(u_n),$$

which is an Euler-Lagrange equation of the functional

$$I_n(w) := \frac{1}{2}|w|_H^2 + h\phi(w) - (u_{n-1}, w)_H \quad \text{ for } w \in H.$$

This variational formulation seems to be *local in time*. It also requires an approximation (precisely, time-discretization) of the target equation.

Example 1.2 (Brézis-Ekeland's variational principle [6, 7])**.** Let ϕ be a proper lower semicontinuous convex functional on a Hilbert space H. Then Brézis and Ekeland found the following relation,

$$u'(t) + \partial\phi(u(t)) \ni 0, \quad u(0) = u_0 \quad \text{iff} \quad J(u) = \inf J = 0,$$

where J is a functional on $W^{1,2}(0,T;H)$ given by

$$J(u) := \int_0^T \Big(\phi(u(t)) + \phi^*(-u'(t))\Big)\mathrm{d}t + \frac{1}{2}|u(T)|_H^2 - \frac{1}{2}|u_0|_H^2$$

with the domain $D(J) := \{u \in W^{1,2}(0,T;H) \colon u(0) = u_0\}$, where ϕ^* is the convex conjugate of ϕ. Brézis-Ekeland's principle would be *global in time* and it requires no approximation. On the other hand, the original Cauchy problem is not formulated as an Euler-Lagrange equation of J.

The aim of this note is to propose a variational method meeting the following requirements for generalized gradient systems (GGS):

- (GGS) is formulated as a global (in time) minimization problem for an appropriate convex functional (cf. implicit time-discretization method).
- Moreover, (GGS) can be reformulated as an Euler-Lagrange equation of the functional (cf. Brézis-Ekeland's variational principle).

On the other hand, we may allow of approximating (GGS) in compensation.

In this note, we shall propose a variational method using *Weighted Energy-Dissipation* (WED) functionals for (GGS). In Section 2, we give a short guidance on variational methods based on WED functionals for gradient systems. Section 3 is concerned with an extension of the WED functional method to generalized gradient systems. The main part of Sections 4 and 5 is devoted to reviewing recent results of the author and Ulisse Stefanelli in [2, 3]. We shall provide slightly improved results compared to the original ones (see Remark 4.4). In Section 6, we give a couple of remarks on applications to nonlinear PDEs, related variational issues and some perspective of further possibilities of WED functional formalism.

2 A short guidance on WED functional method

The WED functional method has been developed as a new tool in order to possibly reformulate dissipative evolution problems in a variational fashion. In particular, minimizers of WED functionals taking a given initial value are expected to approximate solutions of target systems. This perspective has recently attracted attention and, particularly, the WED formalism has already been matter of consideration. At first, the WED functional approach has been addressed by Mielke and Ortiz [14] in the rate-independent case, namely for a positively 1-homogeneous dissipation ψ.

In this section, we give a short guidance on WED functional formalism to gradient systems. Let us start with the initial-boundary value problem for the heat equation,

$$\begin{cases} \partial_t u - \Delta u = 0 \ \text{ in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0. \end{cases} \tag{Heat}$$

For each $\varepsilon > 0$, let us define the WED functional I_ε for (Heat) as follows:

$$I_\varepsilon(u) := \int_Q e^{-t/\varepsilon} \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2\varepsilon}|\nabla u|^2 \right) \mathrm{d}x\mathrm{d}t$$

for $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ satisfying $u(\cdot,0) = u_0$. Then the Euler-Lagrange equation of I_ε reads,

$$\begin{cases} -\varepsilon\partial_t^2 u + \partial_t u - \Delta u = 0 \ \text{ in } Q, \\ u|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0, \quad \partial_t u(\cdot,T) = 0. \end{cases}$$

Equation above can be regarded as an *elliptic-in-time approximation* of (Heat). Elliptic-in-time regularizations of parabolic problems are classical in the linear case and some results can be found in the monograph by [13]. Then one can prove that $u_\varepsilon \to u$ in $C([0,T];L^2(\Omega))$ as $\varepsilon \to 0$ and the limit u solves (Heat).

Now let us move on to an abstract gradient system (GS) in a Hilbert space H,

$$u'(t) = -\nabla\phi(u(t)), \quad 0 < t < T, \qquad u(0) = u_0,$$

where $\nabla\phi$ denotes a functional derivative of ϕ in a proper sense. Define a WED functional I_ε by

$$I_\varepsilon(u) := \int_0^t e^{-t/\varepsilon}\left(\frac{1}{2}|u'(t)|_H^2 + \frac{1}{\varepsilon}\phi(u(t))\right)dt$$

for $u : [0,T] \to H$ satisfying the initial condition $u(0) = u_0$. Then the Euler-Lagrange equation of I_ε is given by

$$\begin{cases} -\varepsilon u''(t) + u'(t) = -\nabla\phi(u(t)), & 0 < t < T, \\ u(0) = u_0, \quad u'(T) = 0. \end{cases}$$

A variational scheme based on the WED functional I_ε for (GS) is stated as follows:

Step 1. Catch global minimizers u_ε of I_ε.

Step 2. Take a limit of u_ε as $\varepsilon \to 0$.

Then the limit of u_ε solves (GS).

The limit as $\varepsilon \to 0$ is clearly the crucial issue for the WED theory and it is usually referred to as the *causal limit.* This name is suggested by the facts that a final condition is imposed on the Euler-Lagrange equation for I_ε (hence (EL) is *non-causal*) and that the causality of the limiting problem (i.e., (GS) here) is restored as $\varepsilon \to 0$.

Causal limit problem

Do the critical points u_ε of I_ε converge to a solution of (GS) as $\varepsilon \to 0$?

Mielke and Stefanelli [15] obtained an affirmative answer to this problem for the gradient system,

$$u'(t) + \partial\phi(u(t)) \ni 0, \quad 0 < t < T, \qquad u(0) = u_0$$

in a Hilbert space H with a subdifferential operator $\partial\phi : H \to H$ of a lower semicontinuous convex functional $\phi : H \to (-\infty, \infty]$.

3 WED approach to generalized gradient systems

Let us go back to a generalized gradient system (GGS), which expresses a balance between the system of *conservative actions* modeled by the gradient $\nabla\phi$ of the *energy* ϕ and that of *dissipative actions* described by the gradient $\nabla\psi$ of the *dissipation* ψ. This in particular motivates the terminology *WED* as the energy ϕ and dissipation ψ

will appear in WED functionals I_ε along with the parameter $1/\varepsilon$ and the exponentially decaying weight $t \mapsto \exp(-t/\varepsilon)$.

The doubly nonlinear dissipative relation (GGS) is extremely general and stands as a paradigm for dissipative evolution. Indeed, let us remark that the formulation (GGS) includes the case of gradient flows, which corresponds to the choice of a quadratic dissipation ψ. Consequently, the interest in providing a variational approach to (GGS) is evident, for it would pave the way to the application of general methods of the Calculus of Variations to a variety of nonlinear dissipative evolution problems.

Let V be a uniformly convex Banach space. Let $\psi : V \to [0,\infty)$ be convex and Gâteaux differentiable and let $\phi : V \to [0,\infty]$ be convex and lower semicontinuous. We are concerned with the following target system,

$$\mathrm{d}_V\psi(u'(t)) + \partial_V\phi(u(t)) \ni 0, \quad 0 < t < T, \qquad u(0) = u_0, \tag{TS}$$

where $\mathrm{d}_V\psi : V \to V^*$ and $\partial_V\phi : V \to V^*$ stand for a gradient operator and a subdifferential operator of ψ and ϕ, respectively.

Here the *subdifferential operator* $\partial_V\phi : V \to V^*$ is defined by

$$\partial_V\phi(u) := \{\xi \in V^* \colon \phi(v) - \phi(u) \geq \langle \xi, v-u\rangle_V \ \forall v \in V\} \quad \text{for } u \in D(\phi),$$

where $D(\phi) := \{u \in V \colon \phi(u) < \infty\}$ is the *effective domain* of ϕ, with the domain $D(\partial_V\phi) := \{u \in D(\phi) \colon \partial_V\phi(u) \neq \emptyset\}$. It is well known that every subdifferential operator is maximal monotone, and moreover, a standard theory was already established in 1970s (see, e.g., [4]). A functional $\psi : V \to \mathbb{R}$ is said to be *Gâteaux differentiable* at u (respectively, in V), if there exists $\xi \in V^*$ such that

$$\lim_{h\to 0} \frac{\psi(u+he) - \psi(u)}{h} = \langle \xi, e\rangle_V \quad \text{for all } e \in V$$

at u (respectively, for all $u \in E$). Then ξ is called the *Gâteaux derivative* of ψ at u and denoted by $\mathrm{d}_V\psi(u)$. Here we note that ψ is Gâteaux differentiable if it is Fréchet differentiable. The *gradient operator* $\mathrm{d}_V\psi : V \to V^*$ of a Gâteaux differentiable functional ψ maps u to $\xi = \mathrm{d}_V\psi(u)$. When ψ is convex and Gâteaux differentiable, the subdifferential operator $\partial_V\phi$ coincides with the gradient operator $\mathrm{d}_V\psi$, and in particular, $\partial_V\psi$ $(= \mathrm{d}_V\psi)$ is single-valued.

Remark 3.1 (Model problem). The abstract setting mentioned above is also motivated by the following initial-boundary value problem: Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and consider

$$\begin{cases} \alpha(\partial_t u) - \Delta_m u = 0 \ \text{ in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot,0) = u_0 \ \text{ in } \Omega, \end{cases} \tag{MP}$$

where $\alpha(s) = |s|^{p-2}s$ with $1 < p < \infty$ and Δ_m is a modified Laplacian (m-Laplacian) given by

$$\Delta_m u := \nabla \cdot (|\nabla u|^{m-2}\nabla u), \quad 1 < m < \infty.$$

Set $V = L^p(\Omega)$, $X = W_0^{1,m}(\Omega)$ and define functionals $\psi, \phi : V \to [0,\infty]$ by

$$\psi(u) := \frac{1}{p}\int_\Omega |u(x)|^p \mathrm{d}x, \quad \phi(u) := \begin{cases} \frac{1}{m}\int_\Omega |\nabla u(x)|^m \mathrm{d}x & \text{if } u \in X, \\ \infty & \text{else.} \end{cases}$$

Then one can check that $\mathrm{d}_V\psi(u) = \alpha(u)$ and $\partial_V\phi(u) = -\Delta_m u$ equipped with $u|_{\partial\Omega} = 0$. Moreover, (MP) is reduced to the abstract Cauchy problem,

$$\mathrm{d}_V\psi(u'(t)) + \partial_V\phi(u(t)) = 0, \quad u(0) = u_0.$$

Furthermore, it follows that

- $X \hookrightarrow V$ compactly, provided that $p < m^* := \frac{Nm}{(N-m)_+}$.
- $\psi \in C^1(V)$ and $\phi_X := \phi|_X \in C^1(X)$.
- Since $\psi(u) = (1/p)|u|_V^p$, $\phi(u) = (1/m)|u|_X^m$, they are coercive in V and X, respectively.
- Moreover, $|\mathrm{d}_V\psi(u)|_{V^*}^{p'} = |u|_V^p$ for all $u \in V$ and $|\mathrm{d}_X\phi_X(u)|_{X^*}^{m'} = |u|_X^m$ for all $u \in X$.

We are now in position to state our basic assumptions. Let us first recall that V and V^* are a uniformly convex Banach space and its dual space with norms $|\cdot|_V$ and $|\cdot|_{V^*}$, respectively, and a duality pairing $\langle\cdot,\cdot\rangle_V$. Let X be a reflexive Banach space with a norm $|\cdot|_X$ and a duality pairing $\langle\cdot,\cdot\rangle_X$ such that

$$X \hookrightarrow V \quad \text{and} \quad V^* \hookrightarrow X^*$$

with densely defined compact canonical injections. We also recall that $\psi : V \to [0,\infty)$ is Gâteaux differentiable and convex, and moreover, $\phi : V \to [0,\infty]$ is proper, lower semicontinuous and convex. Let $p \in (1,\infty)$ and $m \in (1,\infty)$ be fixed and introduce our basic assumptions:

(A1) $C_1|u|_V^p \le \psi(u) + C_2 \quad \forall u \in V.$

(A2) $|\mathrm{d}_V\psi(u)|_{V^*}^{p'} \le C_3|u|_V^p + C_4 \quad \forall u \in V.$

(A3) $|u|_X^m \le \ell_1(|u|_V)(\phi(u)+1) \quad \forall u \in D(\phi).$

(A4) $|\eta|_{X^*}^{m'} \le \ell_2(|u|_V)(|u|_X^m + 1) \quad \forall [u,\eta] \in \partial_X\phi,$

where C_i $(i = 1,2,3,4)$ are constants and ℓ_1, ℓ_2 are nondecreasing functions in $\mathbb{R}$.

Remark 3.2. Here we also remark the following for later use.

(i) Condition (A1) implies

(A1)' $C_1|u|_V^p \le \langle \mathrm{d}_V\psi(u), u\rangle_V + C_2' \quad$ for all $u \in V$

with $C_2' := C_2 + \psi(0) \geq 0$.

(ii) In general, Gâteaux differentiable functions might be discontinuous. However, by (A1) and the mean value theorem, ψ is continuous in V.

Note that the existence of global solutions for (TS) was proved by [8] in our functional setting and it is hence out of question here. Instead, we concentrate on the possibility of recovering solutions to (TS) via the minimization of the WED functionals I_ε and the causal limit as $\varepsilon \to 0$.

For (TS), we define a WED functional $I_\varepsilon : L^p(0,T;V) \to [0,\infty]$ by

$$I_\varepsilon(u) := \int_0^T e^{-t/\varepsilon}\left(\psi(u'(t)) + \frac{1}{\varepsilon}\phi(u(t))\right)\mathrm{d}t =: \mathcal{D}_\varepsilon(u) + \mathcal{E}_\varepsilon(u),$$

where $\mathcal{D}_\varepsilon$ and $\mathcal{E}_\varepsilon$ are given by

$$\mathcal{D}_\varepsilon(u) := \int_0^T e^{-t/\varepsilon}\psi(u'(t))\mathrm{d}t, \quad \mathcal{E}_\varepsilon(u) := \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\phi(u(t))\mathrm{d}t.$$

Moreover, the effective domain of I_ε is given by

$$D(I_\varepsilon) := \{u \in W^{1,p}(0,T;V) \colon \psi(u'(\cdot)), \phi(u(\cdot)) \in L^1(0,T),\ u(0) = u_0\}.$$

Then by assumptions (A1)–(A4), one can write

$$D(I_\varepsilon) = \{u \in L^m(0,T;X) \cap W^{1,p}(0,T;V) \colon u(0) = u_0\}.$$

Now our main issues for setting up a WED approach to (TS) are as follows:

- Formulate an Euler-Lagrange equation (EL) for the WED functional I_ε and prove the solvability of (EL).
- Prove that every solution of (EL) minimizes I_ε and every minimizer of I_ε solves (EL).
- Justify the causal limit: solutions u_ε of (EL) converge to a solution u of (TS) as $\varepsilon \to 0$.

By accomplishing these tasks, the following variational scheme will be valid for (TS):

Step 1. Catch global minimizers u_ε of I_ε.

Step 2. Take a limit of u_ε as $\varepsilon \to 0$.

Then the limit of u_ε solves (TS).

4 Euler-Lagrange equations and their solvability

4.1 Euler-Lagrange equations

For the WED functional I_ε, one can immediately obtain an Euler-Lagrange equation,

$$\partial_{\mathcal{V}} I_\varepsilon(u_\varepsilon) \ni 0 \tag{1}$$

with $\mathcal{V} := L^p(0,T;V)$. However, it would be difficult to prove the convergence of u_ε as $\varepsilon \to 0$ for (1). Indeed, I_ε is not Gâteaux differentiable due to the initial constraint $u(0) = u_0$, and the sum rule $\partial(\varphi_1 + \varphi_2) = \partial\varphi_1 + \partial\varphi_2$ is not valid in general for subdifferentials. Then one could not obtain any representation sufficient for establishing specific energy estimates.

Here we propose the following Cauchy problem as an Euler-Lagrange equation for I_ε instead of (1):

$$\begin{cases} -\varepsilon \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}_V\psi(u_\varepsilon'(t))\Big) + \mathrm{d}_V\psi(u_\varepsilon'(t)) + \partial_X\phi_X(u_\varepsilon(t)) \ni 0, \quad 0 < t < T, \\ u_\varepsilon(0) = u_0, \quad \mathrm{d}_V\psi(u_\varepsilon'(T)) = 0, \end{cases} \tag{EL}$$

where ϕ_X stands for the restriction of ϕ on X ($\hookrightarrow V$).

Remark 4.1 (Euler-Lagrange equation). We emphasize that the functional ϕ of the original WED functional I_ε is replaced in (EL) by its restriction ϕ_X on X, and this replacement will be required to prove the solvability of (EL). Moreover, this formulation of an Euler-Lagrange equation for I_ε is weaker than (EL) with $\partial_X\phi_X$ replaced by $\partial_V\phi$.

We are concerned with *strong solutions* of (EL) defined by

Definition 4.2 (Strong solution of (EL)). *A function $u : [0,T] \to V$ is said to be a* strong solution *of* (EL), *if the following* (i)–(iv) *are all satisfied:*

(i) $u \in L^m(0,T;X) \cap W^{1,p}(0,T;V)$,

(ii) $\xi(\cdot) := \mathrm{d}_V\psi(u'(\cdot)) \in L^{p'}(0,T;V^*)$ *and* $\xi' \in L^{m'}(0,T;X^*) + L^{p'}(0,T;V^*)$,

(iii) *There exists* $\eta \in L^{m'}(0,T;X^*)$ *such that* $\eta(t) \in \partial_X\phi_X(u(t))$ *and* $-\varepsilon\xi'(t)+\xi(t)+\eta(t) = 0$ *for a.a.* $t \in (0,T)$,

(iv) $u(0) = u_0$ *and* $\xi(T) = 0$.

Our main result here is devoted to the solvability of (EL).

Theorem 4.3 (Solvability for (EL)). *Assume that* (A1)–(A4) *hold. Then for every* $u_0 \in D(\phi)$, *the Euler-Lagrange equation* (EL) *admits at least one strong solution* u_ε

satisfying the following energy inequalities:

$$\int_0^T |u_\varepsilon'(t)|_V^p dt \le \frac{1}{C_1}\left(\phi(u_0) + C + \varepsilon\psi(0)\right),$$

$$\int_0^T \phi(u_\varepsilon(t))dt \le \left(\phi(u_0) + C + \varepsilon\psi(0)\right)T + \varepsilon \int_0^T \langle \xi_\varepsilon(t), u_\varepsilon'(t)\rangle_V dt,$$

$$\int_0^T \langle \eta_\varepsilon(t), u_\varepsilon(t)\rangle_X dt \le -\langle \varepsilon\xi_\varepsilon(0), u_0\rangle_X - \int_0^T \langle \varepsilon\xi_\varepsilon(t), u_\varepsilon'(t)\rangle_V dt - \int_0^T \langle \xi_\varepsilon(t), u_\varepsilon(t)\rangle_V dt,$$

$$\int_0^T \langle \xi_\varepsilon(t), u_\varepsilon'(t)\rangle_V dt \le -\phi(u_\varepsilon(T)) + \phi(u_0) + \varepsilon\psi(0),$$

where $\eta_\varepsilon \in \partial_X\phi_X(u_\varepsilon(\cdot))$ *and* $\xi_\varepsilon = \mathrm{d}_V\psi(u_\varepsilon'(\cdot))$, *with some constant* $C \ge 0$.

Remark 4.4 (Improvement of results in [2]). We note that Theorem 4.3 slightly improves the original result in [2], where p was assumed to be not less than 2. In [2], Condition $p \ge 2$ is used only to construct approximate solutions, particularly, to prove regularized WED functionals $I_{\varepsilon,\lambda}$ (see below) are finite in the whole of $L^p(0,T;V)$, since the Moreau-Yosida regularization ϕ_λ of ϕ is bounded by $|\cdot|_V^2$ from above (see the next subsection for more details). Besides, Theorems 5.1 and 5.2 will be also proved without assuming $p \ge 2$ in the next section.

4.2 Sketch of proof for Theorem 4.3

<u>Approximation</u>. We first approximate (EL) as follows:

$$(\mathrm{EL})_\lambda \begin{cases} -\varepsilon\xi_{\varepsilon,\lambda}'(t) + \xi_{\varepsilon,\lambda}(t) + \eta_{\varepsilon,\lambda}(t) = 0, \quad 0 < t < T, \\ \xi_{\varepsilon,\lambda}(t) = \mathrm{d}_V\psi(u_{\varepsilon,\lambda}'(t)), \quad \eta_{\varepsilon,\lambda}(t) = \mathrm{d}_V\phi_\lambda(u_{\varepsilon,\lambda}(t)), \\ u_{\varepsilon,\lambda}(0) = u_0, \quad \xi_{\varepsilon,\lambda}(T) = 0. \end{cases}$$

Here ϕ_λ denotes the *Moreau-Yosida regularization* of ϕ given by

$$\phi_\lambda(u) := \inf_{v\in V}\left(\frac{1}{2\lambda}|u-v|_V^2 + \phi(v)\right) = \frac{1}{2\lambda}|u - J_\lambda u|_V^2 + \phi(J_\lambda u) \quad \text{for } u \in V,$$

where J_λ is the *resolvent* for $\partial_V\phi$ (see [4] for more details). Then ϕ_λ is Gâteaux differentiable and convex in V. For each $\varepsilon > 0$, one can obtain a strong solution $u_{\varepsilon,\lambda}$ of $(\mathrm{EL})_\lambda$ as a minimizer of the regularized WED functional,

$$I_{\varepsilon,\lambda}(u) := \int_0^T e^{-t/\varepsilon}\left(\psi(u'(t)) + \frac{1}{\varepsilon}\phi_\lambda(u(t))\right) \mathrm{d}t =: \mathcal{D}_\varepsilon(u) + \mathcal{E}_{\varepsilon,\lambda}(u)$$

with a domain similar to that of I_ε. Indeed, $I_{\varepsilon,\lambda}$ admits a minimizer $u_{\varepsilon,\lambda}$, since $I_{\varepsilon,\lambda}$ is convex, coercive and lower semicontinuous on $\mathcal{V} := L^\sigma(0,T;V)$ with $\sigma := \max\{2,p\}$. Since $\mathcal{E}_{\varepsilon,\lambda}$ is finite over $\mathcal{V}$, one can apply the sum rule of subdifferentials (see [4]) to get

$$\partial_{\mathcal{V}} I_{\varepsilon,\lambda}(u) = \partial_{\mathcal{V}}\mathcal{D}_\varepsilon(u) + \partial_{\mathcal{V}}\mathcal{E}_{\varepsilon,\lambda}(u).$$

Moreover, one can check $\partial_{\mathcal{V}}\mathcal{E}_{\varepsilon,\lambda}(u)(t) = (e^{-t/\varepsilon}/\varepsilon)\mathrm{d}_V\phi_\lambda(u(t))$.

As for the representation of $\partial_{\mathcal{V}}\mathcal{D}_\varepsilon$, we define the operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ by

$$\mathcal{A}(u)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t/\varepsilon}\mathrm{d}_V\psi(u'(t))\right) \quad \text{for } u \in D(\mathcal{A})$$

with the domain

$$D(\mathcal{A}) = \big\{u \in W^{1,p}(0,T;V) \colon \mathcal{A}(u) \in L^{\sigma'}(0,T;V^*), \\ \mathrm{d}_V\psi(u'(T)) = 0 \text{ and } u(0) = u_0\big\}.$$

Then the following proposition can be proved as in [3].

Proposition 4.5 (Representation of $\partial_{\mathcal{V}}D_\varepsilon$). *If ψ is Gâteaux differentiable and convex in V, then $\partial_{\mathcal{V}}\mathcal{D}_\varepsilon = \mathcal{A}$.*

Therefore for each $\varepsilon > 0$ every critical point $u_{\varepsilon,\lambda}$ of $I_{\varepsilon,\lambda}$ (i.e., $\partial_{\mathcal{V}}I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \ni 0$) solves $(\mathrm{EL})_\lambda$. Besides, since $\mathrm{d}_V\phi_\lambda$ is bounded from V into V^* and $u_{\varepsilon,\lambda}$ belongs to $L^\infty(0,T;V)$, it follows that $\eta_{\varepsilon,\lambda} = \mathrm{d}_V\phi_\lambda(u_{\varepsilon,\lambda}(\cdot)) \in L^\infty(0,T;V^*)$, which together with $(\mathrm{EL})_\lambda$ implies that

$$\xi_{\varepsilon,\lambda} \in W^{1,p'}(0,T;V^*).$$

Here we also used the fact that $\xi_{\varepsilon,\lambda} \in L^{p'}(0,T;V^*)$ by (A2) with $u'_{\varepsilon,\lambda} \in L^p(0,T;V)$.

A priori estimates. For simplicity, we omit the subscript ε. Multiplying the approximate equation by $u'_\lambda(t)$, we have

$$-\varepsilon \left\langle \xi'_\lambda(t), u'_\lambda(t)\right\rangle_V + \left\langle \xi_\lambda(t), u'_\lambda(t)\right\rangle_V + \frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(u_\lambda(t)) = 0.$$

By the integration over $(0,T)$,

$$-\varepsilon \int_0^T \left\langle \xi'_\lambda(t), u'_\lambda(t)\right\rangle_V \mathrm{d}t + \int_0^T \left\langle \xi_\lambda(t), u'_\lambda(t)\right\rangle_V \mathrm{d}t + \phi_\lambda(u_\lambda(T)) - \phi_\lambda(u_0) = 0.$$

Recalling $\xi_\lambda(T) = 0$ and $\xi_\lambda(t) = \mathrm{d}_V\psi(u'_\lambda(t))$, one can formally obtain

$$\begin{aligned}\int_0^T \left\langle \xi'_\lambda(t), u'_\lambda(t)\right\rangle_V \mathrm{d}t &= -\left\langle \xi_\lambda(0), u'_\lambda(0)\right\rangle_V - \int_0^T \left\langle \xi_\lambda(t), u''_\lambda(t)\right\rangle_V \mathrm{d}t \\ &\le \psi(0) - \psi(u'_\lambda(0)) - \psi(u'_\lambda(T)) + \psi(u'_\lambda(0)) \\ &= \psi(0) - \psi(u'_\lambda(T)) \le \psi(0)\end{aligned}$$

(see [3] for a rigorous derivation). Therefore it follows that

$$\int_0^T \langle \xi_\lambda(t), u'_\lambda(t)\rangle_V \mathrm{d}t + \phi_\lambda(u_\lambda(T)) \le \phi(u_0) + \varepsilon\psi(0).$$

Here we also used the fact that $\phi_\lambda(u) \le \phi(u)$ for any $u \in V$ (see [4]). Moreover, (A1)$'$ entails

$$C_1 \int_0^T |u'_\lambda(t)|_V^p \mathrm{d}t - C'_2 \le \int_0^T \langle \xi_\lambda(t), u'_\lambda(t)\rangle_V \mathrm{d}t.$$

Hence by (A2), $\mathrm{d}_V\psi(u'_\lambda(\cdot))$ is bounded in $L^{p'}(0,T;V^*)$ for any $\lambda > 0$.
Repeating the same argument with T replaced by t, we have

$$\int_0^t \langle \xi_\lambda(\tau), u'_\lambda(\tau)\rangle_V d\tau + \phi_\lambda(u_\lambda(t)) \le \phi(u_0) + \varepsilon\langle \xi_\lambda(t), u'_\lambda(t)\rangle_V + \varepsilon\psi(0).$$

Integrating this over $(0,T)$ again, we have

$$\int_0^T \phi_\lambda(u_\lambda(t))\mathrm{d}t \le (\phi(u_0) + C + \varepsilon\psi(0))\,T + \varepsilon\int_0^T \langle \xi_\lambda(t), u'_\lambda(t)\rangle_V \mathrm{d}t.$$

By (A3) and (A4) together with the fact that $\phi(J_\lambda u) \le \phi_\lambda(u)$ for all $u \in V$ (see [4]), it holds that

$$\int_0^T |J_\lambda u_\lambda(t)|_X^m \mathrm{d}t \le C, \quad \int_0^T |\eta_\lambda(t)|_{X^*}^{m'} \mathrm{d}t \le C,$$

where J_λ is the resolvent of $\partial_V\phi$ and $\eta_\lambda(t) = \mathrm{d}_V\phi_\lambda(u_\lambda(t))$. By comparison of both sides of the approximate equation, one can deduce that $(\varepsilon\xi'_\lambda)$ is bounded in $L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*)$.

<u>Convergence as $\lambda \to 0$.</u> From the preceding uniform estimates, one can derive

$$\begin{aligned} u_\lambda &\to u && \text{weakly in } W^{1,p}(0,T;V),\\ J_\lambda u_\lambda &\to v && \text{weakly in } L^m(0,T;X),\\ \xi_\lambda &\to \xi && \text{weakly in } L^{p'}(0,T;V^*),\\ \eta_\lambda &\to \eta && \text{weakly in } L^{m'}(0,T;X^*),\\ \xi'_\lambda &\to \xi' && \text{weakly in } L^{m'}(0,T;X^*) + L^{p'}(0,T;V^*). \end{aligned}$$

Then $-\varepsilon\xi' + \xi + \eta = 0$ (it still remains to check $\eta(t) \in \partial_X\phi_X(u(t))$, $\xi(t) = \mathrm{d}_V\psi(u'(t))$ and initial/final conditions).

Since $(u_\lambda(\cdot))$ is equicontinuous in $C([0,T];V)$, from the definition of $\mathrm{d}_V\phi_\lambda$ and the uniform convexity of V (equivalently, locally uniform monotonicity of the duality mapping $F : V \to V^*$), we derive the equicontinuity of $(J_\lambda u_\lambda(\cdot))$ as well. Besides, let us recall that $(J_\lambda u_\lambda(\cdot))$ is bounded in $L^m(0,T;X)$ and X is compactly embedded in V. Hence by the Aubin-Lions compactness lemma (see [16]), we conclude that

$$J_\lambda u_\lambda \to u \quad \text{strongly in } C([0,T];V),$$

which also implies $u(0) = u_0$ and $u_\lambda \to u$ strongly in $L^q(0,T;V)$ for any $q < \infty$.
Furthermore, since $V^* \hookrightarrow X^*$ compactly, we also obtain

$$\xi_\lambda \to \xi \quad \text{strongly in } C([0,T];X^*) \quad \text{and} \quad \xi(T) = 0.$$

By using Fatou's lemma, for a.a. $t \in (0,T)$, one can take a subsequence (non relabelled) $\lambda \to 0$ such that

$$\langle \xi_\lambda(t), u_\lambda(t)\rangle_V \to \langle \xi(t), u(t)\rangle_V$$

(see [3] for more details).
To prove $\eta(t) \in \partial_X\phi_X(u(t))$ and $\xi(t) = \mathrm{d}_V\psi(u'(t))$, we shall employ

Proposition 4.6 (Integration by parts, [3]). *Let $m, p \in (1,\infty)$ and let $u \in L^m(0,T;X) \cap W^{1,p}(0,T;V)$ and $\xi \in L^{p'}(0,T;V^*)$ be such that*

$$\xi' \in L^{m'}(0,T;X^*) + L^{p'}(0,T;V^*).$$

Let $t_1, t_2 \in (0,T)$ be Lebesgue points of the function $t \mapsto \langle \xi(t), u(t)\rangle_V$. Then it holds that

$$\langle\langle \xi', u\rangle\rangle_{L^m_X \cap L^p_V(t_1,t_2)} = \langle \xi(t_2), u(t_2)\rangle_V - \langle \xi(t_1), u(t_1)\rangle_V - \int_{t_1}^{t_2} \langle \xi(t), u'(t)\rangle_V \mathrm{d}t,$$

where $\langle\langle \cdot,\cdot\rangle\rangle_{L^m_X \cap L^p_V(t_1,t_2)}$ denotes a duality pairing between $L^m(t_1,t_2;X) \cap L^p(t_1,t_2;V)$ and its dual space.

Let $0 < t_1 < t_2 < T$ be Lebesgue points of the function $t \mapsto \langle \xi(t), u(t)\rangle_V$ such that $\langle \xi_\lambda(t), u_\lambda(t)\rangle_V$ is convergent at $t = t_1, t_2$. Then by a formal calculation together with Proposition 4.6,

$$\begin{aligned}
\int_{t_1}^{t_2} \langle \eta_\lambda(t), J_\lambda u_\lambda(t)\rangle_X \mathrm{d}t &\le \int_{t_1}^{t_2} \langle \eta_\lambda(t), u_\lambda(t)\rangle_V \mathrm{d}t \\
&= \varepsilon \langle \xi_\lambda(t_2), u_\lambda(t_2)\rangle_V - \varepsilon \langle \xi_\lambda(t_1), u_\lambda(t_1)\rangle_V \\
&\quad - \int_{t_1}^{t_2} \langle \varepsilon \xi_\lambda(t), u_\lambda'(t)\rangle_V \mathrm{d}t - \int_{t_1}^{t_2} \langle \xi_\lambda(t), u_\lambda(t)\rangle_V \mathrm{d}t \\
&\to \varepsilon \langle \xi(t_2), u(t_2)\rangle_V - \varepsilon \langle \xi(t_1), u(t_1)\rangle_V \\
&\quad - \int_{t_1}^{t_2} \langle \varepsilon \xi(t), u'(t)\rangle_V \mathrm{d}t - \int_{t_1}^{t_2} \langle \xi(t), u(t)\rangle_V \mathrm{d}t \\
&= \langle\langle \varepsilon \xi', u\rangle\rangle_{L^m_X \cap L^p_V(t_1,t_2)} - \int_{t_1}^{t_2} \langle \xi(t), u(t)\rangle_V \mathrm{d}t \\
&= \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \mathrm{d}t,
\end{aligned}$$

which implies

$$\limsup_{\lambda\to 0} \int_{t_1}^{t_2} \langle \eta_\lambda(t), J_\lambda u_\lambda(t)\rangle_X \mathrm{d}t \le \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \mathrm{d}t.$$

From the demiclosedness of $\partial_X \phi_X$ (see [4]) and Proposition 1.1 of [11], it follows that

$$\eta(t) \in \partial_X \phi_X(u(t)) \quad \text{for a.a. } t \in (t_1, t_2).$$

Moreover, we have

$$\lim_{\lambda\to 0} \int_{t_1}^{t_2} \langle \eta_\lambda(t), J_\lambda u_\lambda(t)\rangle_X \mathrm{d}t = \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \mathrm{d}t.$$

As for the limit of ξ_λ, it follows that

$$\int_{t_1}^{t_2} \langle \varepsilon\xi_\lambda(t), u_\lambda'(t)\rangle_V \mathrm{d}t \le \varepsilon\langle \xi_\lambda(t_2), u_\lambda(t_2)\rangle_V - \varepsilon\langle \xi_\lambda(t_1), u_\lambda(t_1)\rangle_V - \int_{t_1}^{t_2} \langle \xi_\lambda(t), u_\lambda(t)\rangle_V \mathrm{d}t$$
$$- \int_{t_1}^{t_2} \langle \eta_\lambda(t), J_\lambda u_\lambda(t)\rangle_X \mathrm{d}t =: \mathrm{RHS}.$$

Besides, we find that

$$\begin{aligned}
\mathrm{RHS} &\to \varepsilon\langle \xi(t_2), u(t_2)\rangle_V - \varepsilon\langle \xi(t_1), u(t_1)\rangle_V - \int_{t_1}^{t_2} \langle \xi(t), u(t)\rangle_V \mathrm{d}t - \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \mathrm{d}t \\
&= \langle\langle \varepsilon\xi', u\rangle\rangle_{L^m_X \cap L^p_V(t_1,t_2)} + \int_{t_1}^{t_2} \langle \varepsilon\xi(t), u'(t)\rangle_V \mathrm{d}t \\
&\quad - \int_{t_1}^{t_2} \langle \xi(t), u(t)\rangle_V \mathrm{d}t - \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \mathrm{d}t = \int_{t_1}^{t_2} \langle \varepsilon\xi(t), u'(t)\rangle_V \mathrm{d}t,
\end{aligned}$$

which yields

$$\limsup_{\lambda\to 0} \int_{t_1}^{t_2} \langle \varepsilon\xi_\lambda(t), u_\lambda'(t)\rangle_V \mathrm{d}t \le \int_{t_1}^{t_2} \langle \varepsilon\xi(t), u'(t)\rangle_V \mathrm{d}t.$$

Consequently, we obtain $\xi(t) = \mathrm{d}_V\psi(u'(t))$ for a.a. $t \in (0,T)$ from the arbitrariness of $0 < t_1 < t_2 < T$ (see [3] for more details). Furthermore, energy inequalities of Theorem 4.3 also follows from energy inequalities and limiting procedures for approximate solutions. Thus we have proved Theorem 4.3. □

5 Minimization of WED functionals and causal limit

Now, we are ready to solve the causal limit problem for (TS). The next theorem is concerned with a relation between solutions for (EL) and minimizers of the WED functional I_ε.

Theorem 5.1 (Existence and uniqueness of minimizers of I_ε). *Assume* (A1)–(A4) *and let $u_0 \in D(\phi)$. Then the strong solution u_ε of* (EL) *obtained in Theorem* 4.3 *minimizes the WED functional I_ε. In addition, if either ϕ or ψ is strictly convex, then the minimizer of I_ε is unique.*

As for the causal limit, we have:

Theorem 5.2 (Convergence of minimizers of I_ε). *Assume that* (A1)–(A4) *hold and either ϕ or ψ is strictly convex. Let $u_0 \in D(\phi)$. For each $\varepsilon > 0$, let u_ε denote the unique minimizer of the WED functional I_ε. Then for any sequence $\varepsilon_n \searrow 0$ there exist a subsequence $(\varepsilon_{n'})$ and the limit u such that*

$$u_{\varepsilon_{n'}} \to u \quad \text{strongly in } C([0,T];V), \quad \text{weakly in } W^{1,p}(0,T;V) \cap L^m(0,T;X).$$

Moreover, the limit u is a strong solution of the target system (TS).

5.1 Sketch of proof for Theorem 5.1

Let $v \in D(I_\varepsilon)$ and let $u_{\varepsilon,\lambda}$ be a minimizer of $I_{\varepsilon,\lambda}$. Then it follows that

$$I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \leq I_{\varepsilon,\lambda}(v),$$

since $D(I_\varepsilon) \subset D(I_{\varepsilon,\lambda})$. By using Lebesgue's dominated convergence theorem, we deduce that

$$I_{\varepsilon,\lambda}(v) \to I_\varepsilon(v) \quad \text{as} \quad \lambda \to 0.$$

Moreover, we also have

$$\begin{aligned}\liminf_{\lambda\to 0} I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) &= \liminf_{\lambda\to 0} \Big(\mathcal{D}_\varepsilon(u_{\varepsilon,\lambda}) + \mathcal{E}_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})\Big) \\ &\geq \liminf_{\lambda\to 0} \Big(\mathcal{D}_\varepsilon(u_{\varepsilon,\lambda}) + \mathcal{E}_\varepsilon(J_\lambda u_{\varepsilon,\lambda})\Big) \\ &\geq \mathcal{D}_\varepsilon(u_\varepsilon) + \mathcal{E}_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon).\end{aligned}$$

Here we also employed the fact that $\phi(J_\lambda u) \leq \phi_\lambda(u)$ for all $u \in V$. Thus we conclude that $I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(v)$ for all $v \in D(I_\varepsilon)$. The uniqueness of minimizer follows from the strict convexity of I_ε □

5.2 Sketch of proof for Theorem 5.2

From the energy inequalities established in Theorem 4.3, one can obtain

$$\int_0^T |u_\varepsilon'(t)|_V^p \mathrm{d}t \leq C, \quad \int_0^T \phi(u_\varepsilon(t))\mathrm{d}t \leq C.$$

Here we remark that it is no longer valid for strong solutions of (EL) to directly test equation by $u_\varepsilon'(t)$ (see Definition 4.2). So we also established energy estimates in the construction of strong solutions for (EL) in §4. Furthermore, by assumptions (A2)–(A4),

$$\int_0^T |\xi_\varepsilon(t)|_{V^*}^{p'} \mathrm{d}t \leq C, \quad \int_0^T |u_\varepsilon(t)|_X^m \mathrm{d}t \leq C, \quad \int_0^T |\eta_\varepsilon(t)|_{X^*}^{m'} \mathrm{d}t \leq C.$$

Hence one can take a sequence (non relabelled) $\varepsilon \to 0$ such that

$$\begin{aligned} u_\varepsilon \to u \quad & \text{weakly in } W^{1,p}(0,T;V) \cap L^m(0,T;X), \\ & \text{strongly in } C([0,T];V), \\ \xi_\varepsilon \to \xi \quad & \text{weakly in } L^{p'}(0,T;V^*), \\ \eta_\varepsilon \to \eta \quad & \text{weakly in } L^{m'}(0,T;X^*). \end{aligned}$$

By comparison of equation, $(\varepsilon\xi_\varepsilon')$ is bounded in $L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*)$. Then it follows that

$$\begin{aligned} \varepsilon\xi_\varepsilon' \to 0 \quad & \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*), \\ \varepsilon\xi_\varepsilon(t) \to 0 \quad & \text{weakly in } X^* \text{ for all } t > 0. \end{aligned}$$

Thus $\xi + \eta = 0$ and $u(0) = u_0$.

As in the proof of Theorem 4.3, one can prove

$$\eta(t) \in \partial_X \phi_X(u(t)), \quad \xi(t) = \mathrm{d}_V \psi(u'(t))$$

for a.e. $t \in (0,T)$. Since $\eta = -\xi \in L^{p'}(0,T;V^*)$, we finally conclude that

$$\eta(t) \in \partial_V \phi(u(t)) \quad \text{for a.a. } t \in (0,T)$$

by using the following proposition:

Proposition 5.3 (Coincidence between $\partial_X \phi_X$ and $\partial_V \phi$, [3]). *Let V and X be normed spaces such that $X \hookrightarrow V$ continuously. Let $\phi : V \to (-\infty, \infty]$ be a proper lower semicontinuous and convex. Moreover, let ϕ_X be the restriction of ϕ onto X. If $D(\phi) \subset X$, then*

$$D(\partial_V \phi) = \{w \in D(\partial_X \phi_X) \colon \partial_X \phi_X(w) \cap V^* \neq \emptyset\},$$

and moreover, $\partial_V \phi(u) = \partial_X \phi_X(u) \cap V^*$ for all $u \in D(\partial_V \phi)$.

This completes our proof. □

6 Final remarks

(i) Theorems 4.3, 5.1 and 5.2 can be applied to the initial-boundary value problem (MP) (see Remark 3.1). The WED functional for (MP) is the following:

$$I_\varepsilon(u) := \int_Q e^{-t/\varepsilon} \left(\frac{1}{p} |\partial_t u|^p + \frac{1}{m\varepsilon} |\nabla u|^m \right) \mathrm{d}x\mathrm{d}t$$

for $u \in W^{1,p}(0,T;L^p(\Omega)) \cap L^m(0,T;W_0^{1,m}(\Omega))$ satisfying $u(0) = u_0$. Then the Euler-Lagrange equation of I_ε reads,

$$\begin{cases} -\varepsilon \partial_t \alpha(\partial_t u) + \alpha(\partial_t u) - \Delta_m u = 0 & \text{in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0, \quad \alpha(\partial_t u)(\cdot, T) = 0. \end{cases} \tag{2}$$

Assume that $p < m^*$. Then by Theorem 4.3, the Euler-Lagrange equation (2) has at least one solution u_ε. Moreover, by Theorem 5.1, the solution u_ε minimizes the WED functional I_ε and it is a unique minimizer. Finally, by Theorem 5.2, we have

$$u_\varepsilon \to u \quad \begin{array}{l} \text{weakly in } L^m(0,T;W_0^{1,m}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)), \\ \text{strongly in } C([0,T];L^p(\Omega)), \quad \text{as } \varepsilon \to 0, \end{array}$$

and moreover, the limit u solves (MP).

(ii) The convergence of a sequence of WED functionals might be one of related issues in view of variational analysis. Indeed, Γ-convergence of functionals generally ensures that the limit of their minimizers also minimizes the limiting functional. More precisely, set $I_{\varepsilon,h}$ as follows:

$$I_{\varepsilon,h}(u) = \int_0^T e^{-t/\varepsilon}\Big(\psi_h(u'(t)) + \frac{1}{\varepsilon}\phi_h(u(t))\Big)\mathrm{d}t$$

with two sequences of convex functionals $\psi_h, \phi_h : V \to (-\infty, \infty]$ involving an additional parameter $h > 0$ and initial constraints $u(0) = u_{0,h} \in D(\phi_h)$. In [3], some sufficient condition is provided for the Mosco convergence $I_{\varepsilon,h} \to I_\varepsilon$ as $h \to 0$. More precisely, it consists of *separate* Γ*-liminf conditions* for ψ_h and ϕ_h as well as a suitable *joint recovery sequence condition*.

(iii) Another noteworthy point of the variational approach using WED functionals is in *minimizations of convex functionals*. As discussed so far, (generalized) gradient systems are always reduced to minimizations of convex functionals via the WED functional formalism. It would not so peculiar for gradient systems with convex energies (see two examples in Section 1). Moreover, the WED functional approach proposed here can be also applied to reformulate other variational problems with possibly non-convex functionals to minimizing problems of convex functionals. One can find such attempts for a nonlinear wave equation and Lagrange systems in [17] and [12], respectively.

Furthermore, the WED functional method seems to be applicable to numerical analysis of nonlinear PDEs. In fact, one can numerically solve doubly nonlinear parabolic problems such as (MP) by using various techniques accumulated so far for minimization of convex functionals. This observation will be fully discussed in a forthcoming report.

(iv) In [1], the WED functional formalism is extended to another type of doubly nonlinear evolution equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial\psi(u(t)) + \partial\phi(u(t)) \ni 0, \quad 0 < t < T,$$

which is arising from porous medium equation, enthalpy formulation of Stefan problem and so on. Moreover, (TS) is also treated in a more general setting there.

References

[1] Akagi, G. and Stefanelli. S., in preparation.

[2] Akagi, G. and Stefanelli. S., A variational principle for doubly nonlinear evolution, Applied Mathematics Letters **23** (2010), 1120–1124.

[3] Akagi, G. and Stefanelli. S., Weighted energy-dissipation functionals for doubly nonlinear evolution, Journal of Functional Analysis **260** (2011), 2541–2578.

[4] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach spaces*, Noordhoff, Leiden, 1976.

[5] Brézis, H., Crandall, M.G. and Pazy, A., Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure. Appl. Math. **23** (1970), 123–144.

[6] Brézis, H. and Ekeland, I., Un principe variationnel associé à certaines équations paraboliques, Le cas dépendant du temps, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), Ai, A1197–A1198.

[7] Brézis, H. and Ekeland, I., Un principe variationnel associé à certaines équations paraboliques, Le cas indépendant du temps, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), Ai, A971–A974.

[8] Colli, P., On some doubly nonlinear evolution equations in Banach spaces, Japan J. Indust. Appl. Math. **9** (1992), 181–203.

[9] Gurtin, M.E., Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, Physica D **92** (1996), 178–192.

[10] Ilmanen, T., Elliptic regularization and partial regularity for motion by mean curvature, Mem. Amer. Math. Soc. **108** (1994), x+90 pp.

[11] Kenmochi, N., Some nonlinear parabolic variational inequalities, Israel J. Math. **22** (1975), 304–331.

[12] Liero, M. and Stefanelli, U., A new minimum principle for Lagrangian mechanics, Preprint IMATI-CNR, 1PV12/1/0, 2012.

[13] Lions, J.-L. and Magenes, E., *Problèmes aux limites non homogènes et applications. Vol. 1*, Dunod, Paris, 1968.

[14] Mielke, A. and Ortiz, M., A class of minimum principles for characterizing the trajectories of dissipative systems, ESAIM Control Optim. Calc. Var. **14** (2008), 494–516.

[15] Mielke, A. and Stefanelli, U., Weighted energy-dissipation functionals for gradient flows, ESAIM Control Optim. Calc. Var. **17** (2011), 52–85.

[16] Simon, J., Compact sets in the space $L^p(0,T;B)$, Ann. Math. Pura. Appl. (4) **146** (1987), 65–96.

[17] Stefanelli, U., The De Giorgi Conjecture on elliptic regularization, Math. Models Meth. Appl. Sci. **21** (2011), 1377–1394.