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Lyapunov functional techniques on the global stability of equilibria of SIS epidemic models with delays

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1 Introduction

To understand the observed behavior of disease transmission, epidemic models have played a crucial role (see also [1–15] and the references therein). Recently, in order to investigate the spread of vector-borne diseases, Beretta and Takeuchi [1] proposed an SIR (Susceptible-Infected-Recovered) epidemic model with distributed time delays and obtained the global stability of a disease-free equilibrium and local stability of an endemic equilibrium. However, on their global stability analysis of the endemic equilibrium, they required that the delay should be small enough. The global stability of the endemic equilibrium for large delay remained unsolved for a long time. Later, McCluskey [12] introduced a Lyapunov functional and proved that the endemic equilibrium is globally asymptotically stable for any delay whenever it exists. By applying the deformation techniques of time deriavtive of Lyapunov functionals, stability analysis of various kinds of delayed epidemic models have been carried out extensively (see, for example, [4,5,8,9,12–14]).

On the other hand, Brauer and van den Driessche [2] formulated the following SIS (Susceptible-Infected-Susceptible) epidemic model with a bilinear incidence rate:

$$\begin{cases} \dfrac{dS(t)}{dt} = (1-p)A - \mu S(t) - \beta S(t)I(t) + \delta I(t), \\ \dfrac{dI(t)}{dt} = pA + \beta S(t)I(t) - (\mu+\alpha+\delta)I(t), \ t>0 \end{cases} \tag{1.1}$$

with the initial conditions $S(0) > 0$ and $I(0) > 0$.

$S(t)$ and $I(t)$ denote the fractions of susceptible and infective individuals at time t, respectively. It is assumed that there is a constant flow of $A > 0$ into the population in unit time, of which a fraction p $(0 \leq p \leq 1)$ is infective. $\mu > 0$ represents the natural death rate of susceptible and infected individuals. $\alpha \geq 0$ represent the disease-induced death rate and $\delta > 0$ is the recovery rate of infected individuals. $\beta > 0$ is the baseline coefficient which denotes the contact rate between susceptible and infective individuals. By applying the Bendixson-Dulac criterion [6, p.373] and the Poincare-Bendixson theorem [6, p.366], Brauer and van den Driessche [2] showed that the endemic equilibrium of system (1.1) is globally asymptotically stable.

Later, for a wide class of delayed SIS epidemic models with a latency in a vector for the infective, Huang and Takeuchi [8] have fully solved the global asymptotic stability of a disease-free equilibrium and a unique endemic equilibrium by a basic reproduction number of the model. However, their stability analysis is based on a limit system derived from the special property $\lim_{t\to+\infty}(S(t)+I(t)) = 1$. Therefore, how to establish sufficient conditions of the global asymptotic stability for the equilibria of the model with a disease-induced death rate remained an open question. In addition, in modelling the transmission dynamics of communicable diseases, nonlinear incidence rates have also played a vital role in ensuring that the model can give a more reasonable qualitative description for the disease dynamics than a bilinear incidence rate. For instance, Capasso and Serio [3] used a saturated incidence function of the form $\frac{I}{1+kI}$ with $k > 0$ to describe that incidence rates increase more gradually than linear in I and S, and then to prevent the unboundedness of contact rate. Based on the ideas, many authors have investigated the global stability conditions of models with a various type of nonlinear incidence rates which are thought of as appropriate forms when describing each disease dynamics. Moreover, Korobeinikov [10] have constructed suitable Volterra-type Lyapunov function for the classical epidemic models of infectious diseases assuming that the horizontal transmission is governed by an unspecified incidence function.

In this paper, we consider the following delayed SIS epidemic model with a class of nonlinear incidence rates:

$$\begin{cases} \dfrac{dS(t)}{dt} = (1-p)A - \mu S(t) - \beta S(t)G(I(t-\tau)) + \delta I(t), \\ \dfrac{dI(t)}{dt} = pA + \beta S(t)G(I(t-\tau)) - (\mu+\alpha+\delta)I(t),\ t > 0 \end{cases} \tag{1.2}$$

with the initial conditions

$$S(0) = \phi_1(0) > 0,\ I(\theta) = \phi_2(\theta),\ -\tau \le \theta \le 0,\ \phi_2(0) > 0,\ \phi \equiv (\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}^2_+), \tag{1.3}$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$.

Here, $\tau \geq 0$ is the length of an incubation period in the vector population. We assume that the function G is continuously differentiable on $[0, +\infty)$ with $G(0) = 0$ and

(H1) $I/G(I)$ is monotone increasing on $(0, +\infty)$ with $\lim_{I \to +0}(I/G(I)) = 1$,

which implies that G is Lipschitz continuous on $[0, +\infty)$ satisfying $0 < G(I) \leq I$ for all $I > 0$. Furthermore, we assume that

(H2) $G(I)$ is monotone increasing on $[0, +\infty)$.

We note that a linear function $G(I) = I$ and a nonlinear function $G(I) = \frac{I}{1+kI}$ with $k > 0$ satisfy the hypotheses (H1) and (H2).

If $p = 0$, then system (1.2) always has a disease-free equilibrium $E_0 = (S^0, 0)$, where $S^0 = \frac{A}{\mu}$. We define the basic reproduction number as

$$R_0 = \frac{\beta A}{\mu(\mu+\alpha+\delta)}. \tag{1.4}$$

If either of the conditions

$$\text{(i) } p = 0 \text{ and } R_0 > 1 \quad \text{(ii) } 0 < p \leq 1$$

holds true, then system (1.2) admits a unique endemic equilibrium $E_* = (S^*, I^*)$, where $S^* > 0$ and $I^* > 0$ (see also Lemma 2.2). We remark that the hypothesis (H2) plays an important role to obtain local and global stability of E_*.

By applying functional techniques for a delayed SIR epidemic model in McCluskey [12] and delayed SIRS epidemic models in [5, 14], we establish the global stability of equilibria of system (1.2). In particular, we offer sufficient conditions under which the unique endemic equilibrium E_*

is globally asymptotically stable with respect to the disease-induced death rate α for the case $p = 0$ (see also Corollary 3.1).

The organization of this paper is as follows. In Section 2, we introduce some basic results. In Section 3, we establish the permanence, the local asymptotic stability and the global asymptotic stability of the endemic equilibrium to prove Theorem 3.1 by constructing a Lyapunov functional. In Section 4, similar to the discussion in Section 3, we establish the global stability of the disease-free equilibrium to prove Theorem 4.1. Finally, concluding remarks are offered in Section 5.

2 Basic results

In this section, we offer some definitions and basic lemmas. We denote $Q_H^{E_0}$ (resp. $Q_H^{E_*}$) by a set of the non-negative functions ϕ_i $(i = 1, 2)$ such that $\|\phi - E_0\| < H$ (resp. $\|\phi - E_*\| < H$) with $H > 0$. Here, the norm of ϕ is defined as $\|\phi\| = \sup_{-\tau \le \theta \le 0} |\phi(\theta)|$.

Definition 2.1. The disease-free equilibrium E_0 (resp. the endemic equilibrium E_*) of system (1.2) is uniformly stable if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $|(S(t), I(t)) - E_0| < \varepsilon$ (resp. $|(S(t), I(t)) - E_*| < \varepsilon$) for any $t > 0$ and for any $\phi \in Q_\delta^{E_0}$ (resp. $\phi \in Q_\delta^{E_*}$).

Definition 2.2. The disease-free equilibrium E_0 (resp. the endemic equilibrium E_*) of system (1.2) is globally attractive if and only if $\lim_{t\to+\infty}(S(t), I(t)) = E_0$ (resp. $\lim_{t\to+\infty}(S(t), I(t)) = E_*$) holds for all ϕ.

Definition 2.3. The disease-free equilibrium E_0 (resp. the endemic equilibrium E_*) of system (1.2) is globally asymptotically stable if and only if it is globally attractive and uniformly stable.

Lemma 2.1. *Put $N(t) = S(t) + I(t)$. Under the initial conditions* (1.3), *system (1.2) has a unique solution on* $[0, +\infty)$ *and* $S(t) > 0$, $I(t) > 0$ *hold for all* $t \ge 0$. *Moreover, it holds that*

$$\limsup_{t\to+\infty} N(t) \le \frac{A}{\mu}. \tag{2.1}$$

Proof. We notice that the right-hand side of system (1.2) is completely continuous and locally Lipschitzian on C. Here, C denotes the Banach space $C([-\tau, 0], \mathbb{R}_+^2)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^2$ and designates the norm of an element $\phi \in C$ by $\|\phi\|$. Then, it follows that the solution of system (1.2) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to prove that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from the fact that $\frac{dS(t)}{dt} = (1-p)A > 0$ holds for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us now show that $I(t) > 0$ for all $t \in [0, \alpha)$. Suppose on the contrary that there exists some $t_1 \in (0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. Integrating the second equation of (1.2) from 0 to t_1, we see that

$$I(t_1) = I(0)\mathrm{e}^{-(\mu+\alpha+\delta)t_1} + \int_0^{t_1} (pA + S(u)G(I(u-\tau)))\mathrm{e}^{-(\mu+\alpha+\delta)(t_1-u)}du > 0.$$

This contradicts $I(t_1) = 0$. Furthermore, for $t \in [0, \alpha)$, we obtain

$$\frac{dN(t)}{dt} = A - \mu N(t) - \alpha I(t) \le A - \mu N(t). \tag{2.2}$$

This yields $N(t) \le \max\{N(0), \frac{A}{\mu}\}$, that is, $(S(t), I(t))$ is uniformly bounded on $[0, \alpha)$. By Theorem 3.2 given in Hale [7, Chapter 2], we have $\alpha = +\infty$. It follows that the solution exists and is unique and positive for all $t \ge 0$. From (2.2), we obtain (2.1). Hence, the proof is complete. □

Lemma 2.2. *Let either of the conditions* (i) *or* (ii) *holds true. Then system (1.2) has a unique endemic equilibrium.*

Proof. From the first and second equations of system (1.2), we have

$$S^* = \frac{A - (\mu + \alpha)I^*}{\mu}. \tag{2.3}$$

Substituting (2.3) into the first equation of (1.2), for $I > 0$, we consider the following equation:

$$H(I) \equiv \frac{pA}{I} + \beta \frac{A - (\mu + \alpha)I}{\mu} \frac{G(I)}{I} - (\mu + \alpha + \delta) = 0.$$

By the hypothesis (H1), the function H is strictly monotone decreasing on $(0, +\infty)$ satisfying $\lim_{I \to +0} H(I) = +\infty$ for $0 < p \leq 1$ and

$$\lim_{I \to +0} H(I) = \frac{\beta A}{\mu} - (\mu + \alpha + \delta) = (\mu + \alpha + \delta)(R_0 - 1) > 0$$

for $p = 0$ and $R_0 > 1$. Moreover, $H(I) < 0$ holds for any $I \geq \frac{A}{\mu+\alpha}$. Hence, there exists a unique positive $0 < I^* < \frac{A}{\mu+\alpha}$ such that $H(I^*) = 0$. By (2.3), there exists a unique endemic equilibrium E_*. Hence, the proof is complete. □

3 Global stability of the endemic equilibrium E_*

In this section, we investigate the permanence and local stability of E_* of system (1.2).

Lemma 3.1. *If* $p = 0$ *and* $R_0 > 1$, *then for any solution of system (1.2) with the initial conditions (1.3), it holds that*

$$\liminf_{t \to +\infty} S(t) \geq v_1 := \frac{A}{\mu + \beta A/\mu}, \ \liminf_{t \to +\infty} I(t) \geq v_2 := qI^* \mathrm{e}^{-(\mu+\delta+\alpha)(\tau+\rho\tau)},$$

where $0 < q < \frac{\beta A - \mu\delta}{\beta(A+\delta I^*)} < 1$ *and* $\rho > 0$ *satisfy* $S^* < S^\triangle := \frac{A}{k}(1 - \mathrm{e}^{-k\rho\tau})$, $k = \mu + \beta q I^*$.

Proof. By Lemma 2.1, we have $\limsup_{t \to +\infty} I(t) \leq \frac{A}{\mu}$, that is, for any $\varepsilon_I > 0$ sufficiently small, there exists a $T_1 = T_1(\varepsilon_I) > 0$ such that $I(t) < \frac{A}{\mu} + \varepsilon_I$ for all $t > T_1$. From the hypothesis (H1), we derive

$$\begin{aligned}\frac{dS(t)}{dt} &\geq A - \left\{\mu + \beta G\left(\frac{A}{\mu} + \varepsilon_I\right)\right\} S(t) \\ &\geq A - \left\{\mu + \beta\left(\frac{A}{\mu} + \varepsilon_I\right)\right\} S(t)\end{aligned}$$

for $t > T_1 + \tau$, which implies that

$$\liminf_{t \to +\infty} S(t) \geq \frac{A}{\mu + \beta(A/\mu + \varepsilon_I)}$$

holds. As the above inequality holds for arbitrary $\varepsilon_I > 0$, it follows that $\liminf_{t \to +\infty} S(t) \geq v_1$.

We now show that $\liminf_{t \to +\infty} I(t) \geq v_2$. First, we prove that it is impossible that $I(t) \leq qI^*$ for all $t \geq \rho\tau$. Suppose on the contrary that $I(t) \leq qI^*$ for all $t \geq \rho\tau$. By the following relation:

$$\beta A - \mu\delta > \beta A - \mu(\mu + \alpha + \delta) = \mu(\mu + \alpha + \delta)(R_0 - 1) > 0,$$

we have

$$S^* = \frac{A+\delta I^*}{\mu+\beta I^*} = \frac{A}{\frac{A(\mu+\beta I^*)}{A+\delta I^*}} = \frac{A}{\mu+\frac{(\beta A-\mu\delta)I^*}{A+\delta I^*}} < \frac{A}{\mu+\beta q I^*},$$

for any $0 < q < \frac{\beta A-\mu\delta}{\beta(A+\delta I^*)}$, one can obtain

$$\frac{dS(t)}{dt} \geq A - (\mu+\beta q I^*)S(t), \text{ for } t \geq \rho\tau+\tau,$$

which yields

$$S(t) \geq e^{-k(t-\rho\tau-\tau)}\left\{S(\rho\tau+\tau) + A\int_{\rho\tau+\tau}^{t} e^{k(\theta-\rho\tau-\tau)}d\theta\right\} > \frac{A}{k}(1-e^{-k(t-\rho\tau-\tau)}) \tag{3.1}$$

for $t \geq \rho\tau+\tau$. Hence, it follows from (3.1) that

$$S(t) > \frac{A}{k}(1-e^{-k\rho\tau}) = S^{\triangle} > S^*, \text{ for } t \geq 2\rho\tau+\tau. \tag{3.2}$$

For $t \geq 0$, we define

$$V(t) = I(t) + \beta S^* \int_{t-\tau}^{t} G(I(u))du. \tag{3.3}$$

Calculating the derivative of V along the solution of system (1.2) gives as

$$\begin{aligned}
\frac{dV(t)}{dt} &= \beta G(I(t-\tau))(S(t)-S^*) + \beta S^* G(I(t)) - (\mu+\alpha+\delta)I(t) \\
&= \beta G(I(t-\tau))(S(t)-S^*) + \left\{\beta S^* \frac{G(I(t))}{I(t)} - (\mu+\alpha+\delta)\right\}I(t) \\
&\geq \beta G(I(t-\tau))(S(t)-S^*) + \left\{\beta S^* \frac{G(I^*)}{I^*} - (\mu+\alpha+\delta)\right\}I(t) \\
&= \beta G(I(t-\tau))(S(t)-S^*) \\
&> \beta G(I(t-\tau))(S^{\triangle}-S^*), \text{ for } t \geq 2\rho\tau+\tau.
\end{aligned} \tag{3.4}$$

Setting $\underline{i} = \min_{\theta\in[-\tau,0]} I(\theta+2\rho\tau+2\tau)$, we claim that $I(t) \geq \underline{i}$ for all $t \geq 2\rho\tau+\tau$. Otherwise, if there is a $T \geq 0$ such that $I(t) \geq \underline{i}$ for $2\rho\tau+\tau \leq t \leq 2\rho\tau+2\tau+T$, $I(2\rho\tau+2\tau+T) = \underline{i}$ and $\frac{dI(t)}{dt}|_{t=2\rho\tau+2\tau+T} \leq 0$, then it follows from (3.1) that

$$\begin{aligned}
\left.\frac{dI(t)}{dt}\right|_{t=2\rho\tau+2\tau+T} &= \beta S(t)G(I(t-\tau)) - (\mu+\alpha+\delta)I(t) \\
&\geq \beta S(t)G(I(t)) - (\mu+\alpha+\delta)I(t) \\
&\geq \left\{\beta S(t)\frac{G(I^*)}{I^*} - (\mu+\alpha+\delta)\right\}\underline{i} \\
&> \left\{\beta S^{\triangle}\frac{G(I^*)}{I^*} - (\mu+\alpha+\delta)\right\}\underline{i} \\
&> \left\{\beta S^*\frac{G(I^*)}{I^*} - (\mu+\alpha+\delta)\right\}\underline{i} = 0.
\end{aligned}$$

This is a contradiction. Therefore, $I(t) \geq \underline{i}$ for all $t \geq 2\rho\tau+\tau$. By the hypothesis (H1), it follows from (3.2) that

$$\frac{dV(t)}{dt} > \beta\frac{G(I^*)}{I^*}(S^{\triangle}-S^*)\underline{i} > 0, \text{ for } t \geq 2\rho\tau+2\tau,$$

which implies that $\lim_{t\to+\infty} V(t) = +\infty$. However, from Lemma 2.1, it holds that $\limsup_{t\to+\infty} V(t) \le \frac{A}{\mu} + \beta S^* \frac{A}{\mu} < +\infty$. This leads to a contradiction. Hence the claim is proved.

As the above claim holds, we are left to consider two possibilities:

$$\begin{cases} \text{(i) } I(t) \ge qI^* \text{ for all } t \text{ sufficiently large,} \\ \text{(ii) } I(t) \text{ oscillates about } qI^* \text{ for all } t \text{ sufficiently large.} \end{cases}$$

If the first case holds, then we immediately get the conclusion. If the second case holds, then we show that $I(t) \ge v_2$ for all t sufficiently large. Let $t_1 < t_2$ be sufficiently large such that

$$I(t_1) = I(t_2) = qI^*, \ I(t) < qI^*, \ t_1 < t < t_2.$$

If $t_2 - t_1 \le \tau + \rho\tau$, then we have $\frac{dI(t)}{dt} \ge -(\mu + \alpha + \delta)I(t)$, that is,

$$I(t) \ge I(t_1)\mathrm{e}^{-(\mu+\alpha+\delta)(t-t_1)} = qI^*\mathrm{e}^{-(\mu+\alpha+\delta)(\tau+\rho\tau)} = v_2$$

holds for all $t \ge t_1$. If $t_2 - t_1 \le \tau + \rho\tau$, then we similarly verify that $I(t) \ge v_2$ holds for $t_1 \le t \le t_1 + \tau + \rho\tau$. We now claim that $I(t) \ge v_2$ for all $t_1 + \tau + \rho\tau \le t \le t_2$. Otherwise, there is a $T^* > 0$, such that $I(t) \ge v_2$ for $t_1 \le t \le t_1 + \tau + \rho\tau + T^*$, $I(t_1 + \tau + \rho\tau + T^*) = v_2$ and $\frac{dI(t)}{dt}|_{t=t_1+\tau+\rho\tau+T^*} \le 0$. Then, from (3.2), we get

$$\begin{aligned} \frac{dI(t)}{dt}\Big|_{t=t_1+\tau+\rho\tau+T^*} &= \beta S(t)G(I(t-\tau)) - (\mu+\alpha+\delta)I(t) \\ &\ge \beta S^{\triangle}G(I(t)) - (\mu+\alpha+\delta)I(t) \\ &\ge \left\{\beta S^{\triangle}\frac{G(v_2)}{v_2} - (\mu+\alpha+\delta)\right\}v_2. \end{aligned}$$

However, by the hypothesis (H1), it holds that

$$\frac{dI(t)}{dt}\Big|_{t=t_1+\tau+\rho\tau+T^*} \ge \left\{\beta S^{\triangle}\frac{G(I^*)}{I^*} - (\mu+\alpha+\delta)\right\}v_2 > 0,$$

which is a contradiction. Hence, $I(t) \ge v_2$ for $t_1 \le t \le t_2$. As the interval $[t_1, t_2]$ is arbitrarily chosen, $I(t) \ge v_2$ holds for all t sufficiently large. Thus, we obtain $\liminf_{t\to+\infty} I(t) \ge v_2$. □

Proposition 3.1. *Let either of the conditions* (i) *or* (ii) *holds true. Then the endemic equilibrium E_* is locally asymptotically stable.*

Proof. The characteristic equation of system (1.2) at E_* is of the form

$$\lambda^2 + a\lambda + b - \mathrm{e}^{-\lambda\tau}(c\lambda + d) = 0, \tag{3.5}$$

where

$$\begin{cases} a = \mu + \beta G(I^*) + \dfrac{pA}{I^*} + \beta S^*\dfrac{G(I^*)}{I^*}, \\ b = (\mu + \beta G(I^*))\left(\dfrac{pA}{I^*} + \beta S^*\dfrac{G(I^*)}{I^*}\right) - \delta\beta G(I^*), \\ c = \beta S^* G'(I^*), \\ d = \mu\beta S^* G'(I^*). \end{cases}$$

We show that all the roots of (3.5) have negative real part. For the case $\tau = 0$, (3.5) becomes

$$\lambda^2 + (a-c)\lambda + (b-d) = 0. \tag{3.6}$$

Noting from the hypotheses (H1) that $G(I^*) - I^*G'(I^*) \geq 0$, we have

$$a - c = \mu + \beta G(I^*) + \frac{pA}{I^*} + \beta S^* \left(\frac{G(I^*)}{I^*} - G'(I^*) \right) > 0$$

and

$$\begin{aligned} b - d &= \frac{\mu pA}{I^*} + \mu\beta S^* \left(\frac{G(I^*)}{I^*} - G'(I^*) \right) + \beta \frac{G(I^*)}{I^*}(pA + \beta S^* G(I^*) - \delta I^*) \\ &= \frac{\mu pA}{I^*} + \mu\beta S^* \left(\frac{G(I^*)}{I^*} - G'(I^*) \right) + \beta G(I^*)(\mu + \alpha) > 0, \end{aligned}$$

which implies that all the roots of equation (3.6) have negative real part. Hence, all the roots of equation (3.5) have negative real part for sufficiently small τ. Suppose that $\lambda = i\omega$, $\omega > 0$ is a root of (3.5). Substituting $\lambda = i\omega$ into the characteristic equation (3.5) yields equations, which split into its real and imaginary parts as follows:

$$\begin{cases} -\omega^2 + b = d\cos\omega\tau + c\omega\sin\omega\tau, \\ a\omega = c\omega\cos\omega\tau - d\sin\omega\tau. \end{cases} \tag{3.7}$$

Squaring and adding both equations in (3.7), we have

$$\omega^4 + (a^2 - 2b - c^2)\omega^2 + (b + d)(b - d) = 0. \tag{3.8}$$

However, by the hypotheses (H1) and (H2), we obtain

$$\begin{aligned} a^2 - 2b - c^2 &= (\mu + \beta G(I^*))^2 + 2\delta\beta G(I^*) + \left(\frac{pA}{I^*} + \beta S^* \frac{G(I^*)}{I^*} \right)^2 - (\beta S^* G'(I^*))^2 \\ &> (\mu + \beta G(I^*))^2 + 2\delta\beta G(I^*) + (\beta S^*)^2 \left(\frac{G(I^*)}{I^*} + G'(I^*) \right) \left(\frac{G(I^*)}{I^*} - G'(I^*) \right) > 0 \end{aligned}$$

and

$$\begin{aligned} b + d &= (\mu + \beta G(I^*)) \left(\frac{pA}{I^*} + \beta S^* \frac{G(I^*)}{I^*} \right) - \delta\beta G(I^*) + \mu\beta S^* G'(I^*) \\ &= (\mu + \beta G(I^*))(\mu + \alpha) + \mu\delta + \mu\beta S^* G'(I^*) > 0. \end{aligned}$$

This contradicts the fact that the equation (3.8) has a positive root. Hence, all the roots of (3.5) have negative real part for all $\tau \geq 0$, which implies that E_* is locally asymptotically stable. This completes the proof. □

We now investigate the global asymptotic stability of the endemic equilibrium E_* for $R_0 > 1$. If necessary, we hereafter use the following notations:

$$x_t = \frac{S(t)}{S^*}, \; y_t = \frac{I(t)}{I^*}, \; \tilde{y}_t = \frac{G(I(t))}{G(I^*)}.$$

We now apply techniques concerning equation deformation of the time derivative of Lyapunov functional in McCluskey [12].

Theorem 3.1. *Let either of the conditions* (i) *or* (ii) *holds true. If*

$$\mu S^* - \delta I^* \geq 0, \tag{3.9}$$

then the endemic equilibrium E_* of system (1.2) is globally asymptotically stable.

Proof. We consider the following Lyapunov functional:

$$V_*(t) = S^* V_S(t) + I^* V_I(t) + \beta S^* G(I^*) V_+(t) + \frac{\delta}{(2\mu+\alpha)S^*} V_N(t),$$

where

$$\begin{cases} V_S(t) = g\left(\frac{S(t)}{S^*}\right), \; V_I(t) = g\left(\frac{I(t)}{I^*}\right), \; V_+(t) = \int_{t-\tau}^{t} g\left(\frac{G(I(s))}{G(I^*)}\right) ds, \; g(x) = x - 1 - \ln x, \\ V_N(t) = \frac{(N(t)-N^*)^2}{2}, \end{cases}$$

and $N^* = S^* + I^*$. One can see that $g : \mathbb{R}_+ \setminus \{0\} \longrightarrow \mathbb{R}_+$ has a strict global minimum at 1. We now show that $\frac{dV_*(t)}{dt} \leq 0$ holds. First, by the equilibrium condition $(1-p)A = \mu S^* + \beta S^* G(I^*) - \delta I^*$, we have

$$\begin{aligned} \frac{dV_S(t)}{dt} &= \frac{S(t)-S^*}{S^* S(t)} \Big\{ (1-p)A - \mu S(t) - \beta S(t) G(I(t-\tau)) + \delta I(t) \Big\} \\ &= \frac{S(t)-S^*}{S^* S(t)} \Big\{ -\mu(S(t)-S^*) + \beta(S^* G(I^*) - S(t) G(I(t-\tau))) + \delta(I(t) - I^*) \Big\} \\ &= -\frac{\mu S^*}{S(t)} \left(\frac{S(t)}{S^*} - 1\right)^2 + \frac{\delta}{S^*}\left(1 - \frac{S^*}{S(t)}\right)(I(t) - I^*) \\ &\quad + \beta G(I^*) \left(1 - \frac{S^*}{S(t)}\right)\left(1 - \frac{S(t)}{S^*}\frac{G(I(t-\tau))}{G(I^*)}\right) \\ &= -\mu \frac{(x_t - 1)^2}{x_t} + \frac{\delta I^*}{S^*}\left(1 - \frac{1}{x_t}\right)(y_t - 1) + \beta G(I^*)\left(1 - \frac{1}{x_t}\right)(1 - x_t \tilde{y}_{t-\tau}). \end{aligned} \tag{3.10}$$

Second, we calculate $\frac{dV_I(t)}{dt}$. Substituting $\mu + \alpha + \delta = \frac{pA}{I^*} + \beta S^* \frac{G(I^*)}{I^*}$, we obtain

$$\begin{aligned} \frac{dV_I(t)}{dt} &= \frac{I(t)-I^*}{I^* I(t)} \Big\{ pA + \beta S(t) G(I(t-\tau)) - (\mu+\alpha+\delta) I(t) \Big\} \\ &= \frac{I(t)-I^*}{I^* I(t)} \left\{ \beta S(t) G(I(t-\tau)) - \beta S^* \frac{G(I^*)}{I^*} I(t) - pA\left(\frac{I(t)}{I^*} - 1\right) \right\} \\ &= \beta S^* \frac{G(I^*)}{I^*}\left(1 - \frac{I^*}{I(t)}\right)\left(\frac{S(t)}{S^*}\frac{G(I(t-\tau))}{G(I^*)} - \frac{I(t)}{I^*}\right) - \frac{pA}{I(t)}\left(\frac{I(t)}{I^*} - 1\right)^2 \\ &= \beta S^* \frac{G(I^*)}{I^*}\left(1 - \frac{1}{y_t}\right)(x_t \tilde{y}_{t-\tau} - y_t) - \frac{pA}{I^*}\frac{(y_t - 1)^2}{y_t}. \end{aligned} \tag{3.11}$$

We now use the following relation, which plays an important role to cancel the delay term $\tilde{y}_{t-\tau}$ effectively (cf. McCluskey [12]):

$$\begin{aligned} &\left(1 - \frac{1}{x_t}\right)(1 - x_t \tilde{y}_{t-\tau}) + \left(1 - \frac{1}{y_t}\right)(x_t \tilde{y}_{t-\tau} - y_t) + g(\tilde{y}_t) - g(\tilde{y}_{t-\tau}) \\ &= 2 - \frac{1}{x_t} + \tilde{y}_{t-\tau} - \frac{x_t \tilde{y}_{t-\tau}}{y_t} - y_t + g(\tilde{y}_t) - g(\tilde{y}_{t-\tau}) \\ &= -g\left(\frac{1}{x_t}\right) - g\left(\frac{x_t \tilde{y}_{t-\tau}}{y_t}\right) - g(y_t) + g(\tilde{y}_{t-\tau}) + g(\tilde{y}_t) - g(\tilde{y}_{t-\tau}) \\ &= -g\left(\frac{1}{x_t}\right) - g\left(\frac{x_t \tilde{y}_{t-\tau}}{y_t}\right) - (g(y_t) - g(\tilde{y}_t)). \end{aligned}$$

We then obtain

$$\begin{aligned}
&\frac{d}{dt}\left(S^*V_S(t)+I^*V_I(t)+\beta S^*G(I^*)V_+(t)\right)\\
&=-\mu S^*\frac{(x_t-1)^2}{x_t}+\delta I^*\left(1-\frac{1}{x_t}\right)(y_t-1)-\frac{pA}{I^*}\frac{(y_t-1)^2}{y_t}\\
&\quad-\beta S^*G(I^*)\Big(g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t\tilde{y}_{t-\tau}}{y_t}\right)+g(y_t)-g(\tilde{y}_t)\Big).
\end{aligned}$$

Finally, calculating $\frac{dV_N(t)}{dt}$ gives

$$\begin{aligned}
\frac{dV_N(t)}{dt}&=(N(t)-N^*)\{A-\mu S(t)-(\mu+\alpha)I(t)\}\\
&=(N(t)-N^*)\{-\mu(S(t)-S^*)-(\mu+\alpha)(I(t)-I^*)\}\\
&=-\mu(S(t)-S^*)^2-(2\mu+\alpha)(S(t)-S^*)(I(t)-I^*)-(\mu+\alpha)(I(t)-I^*)^2\\
&=-\mu(S^*)^2(x_t-1)^2-(2\mu+\alpha)S^*I^*(x_t-1)(y_t-1)-(\mu+\alpha)(I^*)^2(y_t-1)^2.
\end{aligned}$$

Therefore, by the hypotheses (H1) and (H2), it follows from the relations

$$\begin{aligned}
g(y_t)-g(\tilde{y}_t)&=\frac{1}{\tilde{y}_t}(y_t-\tilde{y}_t)(\tilde{y}_t-1)+g\left(\frac{y_t}{\tilde{y}_t}\right)\\
&\ge\frac{1}{\tilde{y}_t}(y_t-\tilde{y}_t)(\tilde{y}_t-1)\\
&=\frac{1}{I^*}\left(\frac{I(t)}{G(I(t))}-\frac{I^*}{G(I^*)}\right)(G(I(t))-G(I^*))\ge 0,
\end{aligned}\tag{3.12}$$

and

$$\left(1-\frac{1}{x_t}\right)(y_t-1)-(x_t-1)(y_t-1)=-\frac{(x_t-1)^2}{x_t}(y_t-1)\tag{3.13}$$

that

$$\begin{aligned}
\frac{dV_*(t)}{dt}&=-\mu S^*\frac{(x_t-1)^2}{x_t}+\delta I^*\left(1-\frac{1}{x_t}\right)(y_t-1)-\frac{pA}{I^*}\frac{(y_t-1)^2}{y_t}\\
&\quad-\beta S^*G(I^*)\Big(g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t\tilde{y}_{t-\tau}}{y_t}\right)+g(y_t)-g(\tilde{y}_t)\Big)\\
&\quad-\frac{\mu\delta S^*}{2\mu+\alpha}(x_t-1)^2-\delta I^*(x_t-1)(y_t-1)-\frac{(\mu+\alpha)\delta(I^*)^2}{(2\mu+\alpha)S^*}(y_t-1)^2\\
&=-\mu S^*\frac{(x_t-1)^2}{x_t}-\delta I^*\frac{(x_t-1)^2}{x_t}(y_t-1)-\frac{pA}{I^*}\frac{(y_t-1)^2}{y_t}\\
&\quad-\beta S^*G(I^*)\Big(g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t\tilde{y}_{t-\tau}}{y_t}\right)+g(y_t)-g(\tilde{y}_t)\Big)\\
&\quad-\frac{\mu\delta S^*}{2\mu+\alpha}(x_t-1)^2-\frac{(\mu+\alpha)\delta(I^*)^2}{(2\mu+\alpha)S^*}(y_t-1)^2\\
&\le-\frac{(\mu S^*-\delta I^*)(x_t-1)^2}{x_t}-\frac{\mu\delta S^*}{2\mu+\alpha}(x_t-1)^2-\frac{(\mu+\alpha)\delta(I^*)^2}{(2\mu+\alpha)S^*}(y_t-1)^2.
\end{aligned}$$

From the condition (3.9), we see that

$$\frac{dV_*(t)}{dt} \leq -\frac{\mu\delta S^*}{2\mu+\alpha}(x_t-1)^2 - \frac{(\mu+\alpha)\delta(I^*)^2}{(2\mu+\alpha)S^*}(y_t-1)^2 \leq 0.$$

Hence, solutions of system (1.2) limit to M, the largest invariant subset of $\{\frac{dV_*(t)}{dt}=0\}$. Recalling that $\frac{dV_*(t)}{dt}=0$ implies that $x_t=1$ and $y_t=1$, each element of M satisfies $S(t)=S^*$ and $I(t)=I^*$ for all t. Applying LaSalle invariance principle (see Kuang [11, Corollary 5.2]), E_* is globally asymptotically stable. □

Corollary 3.1. *Let the condition* (i) *holds true. Then, the following conditions:*

$$\begin{cases} 0 \leq \alpha < +\infty, & \text{if } \dfrac{\mu(\mu+\delta)(\delta+1)}{\delta\beta A} \geq 1 \\ \alpha \geq \dfrac{-(2\mu+\delta+\mu\delta)+\sqrt{\delta^2(\mu-1)^2+4\delta\beta A}}{2}, & \text{if } \dfrac{\mu(\mu+\delta)(\delta+1)}{\delta\beta A} < 1 \end{cases} \tag{3.14}$$

implies (3.9). In particular, if $G(I)=I$, then (3.9) is equivalent to (3.14).

Proof. From (1.4), I^* satisfies the following equation:

$$\beta(\mu+\alpha)I^* + \mu(\mu+\alpha+\delta)\frac{I^*}{G(I^*)} = \beta A = \mu(\mu+\alpha+\delta)R_0,$$

which yields $I^* \leq \frac{\mu(\mu+\alpha+\delta)(R_0-1)}{\beta(\mu+\alpha)}$. Since

$$\delta^2(\mu-1)^2 + 4\delta\beta A = (2\mu+\delta+\mu\delta)^2 - 4\{\mu(\mu+\delta)(\delta+1) - \delta\beta A\}$$

holds, the condition (3.14) is equivalent to

$$\alpha^2 + (2\mu+\delta+\mu\delta)\alpha + \mu(\mu+\delta)(\delta+1) - \delta\beta A \geq 0,$$

that is,

$$(\mu+\alpha)(\mu+\alpha+\delta) \geq \{\beta A - \mu(\mu+\alpha+\delta)\}\delta,$$

which implies that $\mu+\alpha \geq (R_0-1)\delta$ holds. We then have

$$\begin{aligned} \mu S^* - \delta I^* &= \mu\frac{(\mu+\alpha+\delta)I^*}{\beta G(I^*)} - \delta I^* \\ &= \frac{I^*}{\beta G(I^*)}\{\mu(\mu+\alpha+\delta) - \beta\delta G(I^*)\} \\ &\geq \frac{I^*}{\beta G(I^*)}\{\mu(\mu+\alpha+\delta) - \beta\delta I^*\} \\ &\geq \frac{I^*}{\beta G(I^*)}\left\{\mu(\mu+\alpha+\delta) - \beta\delta\frac{\mu(\mu+\alpha+\delta)(R_0-1)}{\beta(\mu+\alpha)}\right\} \\ &= \frac{\mu(\mu+\alpha+\delta)I^*}{\beta G(I^*)}\left\{1 - \frac{\delta(R_0-1)}{\mu+\alpha}\right\} \geq 0, \end{aligned}$$

which implies that (3.9) holds true. Similar to the above discussion, we obtain that (3.9) is equivalent to (3.14) if $G(I)=I$. This completes the proof. □

4 Global stability of the disease-free equilibrium E_0

In this section, we establish the global stability of E_0.

Theorem 4.1. *If $p = 0$ and $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable.*

Proof. We consider the following Lyapunov functional:

$$V_0(t) = S^0 g\left(\frac{S(t)}{S^0}\right) + I(t) + \beta S^0 \int_{t-\tau}^{t} G(I(u))du + \frac{\delta}{(2\mu+\alpha)S^0}\frac{(N(t)-N^0)^2}{2},$$

where $N^0 = S^0$. Similar to the discussion in Section 3, we get

$$\begin{aligned}
\frac{dV_0(t)}{dt} &= -\mu\frac{(S(t)-S^0)^2}{S(t)} - \beta(S(t)-S^0)G(I(t-\tau)) + \delta I(t)\left(1-\frac{S^0}{S(t)}\right)\\
&\quad + \beta S(t)G(I(t-\tau)) - (\mu+\alpha+\delta)I(t)\\
&\quad + \beta S^0(G(I(t)) - G(I(t-\tau)))\\
&\quad - \frac{\mu\delta(S(t)-S^0)^2}{(2\mu+\alpha)S^0} - \delta\left(\frac{S(t)}{S^0}-1\right)I(t) - \frac{(\mu+\alpha)\delta}{(2\mu+\alpha)S^0}I(t)^2\\
&= -\mu\frac{(S(t)-S^0)^2}{S(t)} + \delta I(t)\left\{\left(1-\frac{S^0}{S(t)}\right) - \left(\frac{S(t)}{S^0}-1\right)\right\}\\
&\quad + \beta S^0 G(I(t)) - (\mu+\alpha+\delta)I(t) - \frac{\mu\delta(S(t)-S^0)^2}{(2\mu+\alpha)S^0} - \frac{(\mu+\alpha)\delta}{(2\mu+\alpha)S^0}I(t)^2.
\end{aligned}$$

By the hypothesis (H1), we have

$$\begin{aligned}
\frac{dV_0(t)}{dt} &\leq (\mu+\alpha+\delta)\left(R_0\frac{G(I(t))}{I(t)}-1\right)I(t) - \frac{\mu\delta(S(t)-S^0)^2}{(2\mu+\alpha)S^0} - \frac{(\mu+\alpha)\delta}{(2\mu+\alpha)S^0}I(t)^2\\
&\leq (\mu+\alpha+\delta)(R_0-1)I(t) - \frac{\mu\delta(S(t)-S^0)^2}{(2\mu+\alpha)S^0} - \frac{(\mu+\alpha)\delta}{(2\mu+\alpha)S^0}I(t)^2 \leq 0.
\end{aligned}$$

Therefore, it holds that $\lim_{t\to+\infty}\frac{dV_0(t)}{dt} = 0$, which yields $\lim_{t\to+\infty} S(t) = S^0$ and $\lim_{t\to+\infty} I(t) = 0$. Hence, from Lemma 2.1, applying Lyapunov-LaSalle asymptotic stability theorem [11, Theorem 5.3], E_0 is globally asymptotically stable. □

5 Concluding remarks

In this paper, we investigate the global dynamics of SIS epidemic models with delays. The infection force with a discrete delay is given by a general nonlinear incidence rate of the form $\beta S(t)G(I(t-\tau))$ satisfying monotonicity hypotheses (H1) and (H2).

For the either case (i) or (ii) holds, we obtain sufficient conditions under which the endemic equilibrium E_* of (1.2) is globally asymptotically stable in Theorem 3.1, and for $p = 0$ and $R_0 \leq 1$, we establish the global asymptotic stability of the disease-free equilibrium E_0 of (1.2) in Theorem 4.1. By Proposition 3.1 and Theorem 4.1, when $p = 0$, the basic reproduction number R_0 is a threshold which determines the local stability of the two equilibria E_0 and E_*. In addition, in the proof of Theorem 3.1, we introduced the relations (3.12) and (3.13) to show that the Lyapunov functionals V_* is non-increasing. These techniques are also applicable to construction of suitable

Lyaupnov functionals for the global stability of equilibria of various kinds of delayed epidemic models.

It is also remarkable that Proposition 3.1 shows that the endemic equilibrium E_* is locally asymptotically stable whenever it exists. On the other hand, there is still an open problem whether E_* of system (1.2) is globally asymptotically stable if $\mu S^* - \delta I^* < 0$ when it exists. We leave them as our future work.

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