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A definition of a field for Euclid's Elements without any set theories

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We introduce a definition of a field for Euclid's Elements [E] without any set theories. Precisely, we define the non-negative part of a ordered field.

Since we never use any notion of set theory, we never say the language is $\{+, \cdot, 0, 1\}$, but we say the symbols of binary operations "+" and "·" and the symbols of constants "0" and "1".

Since we never use any notion of set theory, we never say *infinitely* many variable symbols v_1, v_2, ..., but we say variable symbols a, b, c, ... etc. as many as we need. because the words "infinite" and "finite" are notions of set theories. We introduce a unary predicate $\mathbb{N}(\bullet)$, and we say n is a natural number if $\mathbb{N}(n)$.

The symbol of equality is "=", and the logical connections are "$\wedge$", "$\vee$", "$\Rightarrow$" and "$\neg$" and the quantifiers are "$\forall$" and "$\exists$".

By usual way of BNF, we define *terms*, *equations*, *formulas*. They are not sets but they are on a paper, in our brain, in storages of computers, or etc.. We never say a *set* of formulas against model theory.

By usual way we adopt the axiom of equality — replacing the terms which are connected by "=".

Finally, we work on the classical predicate logic. any proofs or Any deductions are never set.

Here are the definition. Every free variable is bound by universal quantifier.

A0 $\mathbb{N}(0) \wedge \mathbb{N}(1)$.

A1 $(a+b)+c=a+(b+c)$.

A3 $a+b=b=1$

A4 $\Big[P(0,a,b,c,d,e)\wedge\forall n\big[[\mathbb{N}(n)\wedge P(n,a,b,c,d,e)]\Rightarrow P(n+1,a,b,c,d,e)]\big]\Big]$ $\Rightarrow \forall n[\mathbb{N}(n)\Rightarrow P(n,a,b,c,d,e)]$, where $P(x,u,v,w,y,z)$ is a formula on a paper, in our brain, in storages of computers, or etc., and x,u,v,w,y,z are meta-symbols to replace terms.

A5 $a+c=b+c\Rightarrow a=b$.

A6 $\neg[a+1=0]$.

A7 $a\cdot 1=a$.

A8 $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

A9 $a\cdot b=b\cdot a$.

A10 $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.

A11 $\exists c\,[a=0\vee a\cdot c=b]$.

A12 $\exists b\; a=b\cdot b$.

A13 $\exists n\,\exists b\,[\mathbb{N}(n)\wedge a+b=n]$.

We denote $\exists c\; a+c=b$ by $a\leq b$ and denote $\exists n\;[\mathbb{N}(n)\wedge a\cdot n=b]$ by $a\mid b$.

Reference

[E] $E\upsilon\kappa\lambda\epsilon\iota\delta o\upsilon\ \Sigma\tau o\iota\chi\epsilon\iota\alpha$, translation to Japanese by K.Nakamura H.Terasaka S.Ito M.Ikeda, Kyoritsu Shuppan, 1971.

[H] D.Hilbert *Grundlagen der Geometrie. 7 Aufl.* translation to Japanese by K.Nakamura Chikuma Shobou, 2005.