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ON VARIANTS OF CM-TRIVIALITY

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ABSTRACT. Following the ideas of Kowalski and Pillay, we introduce a notion weaker than CM-triviality, called CM-linearity, in terms of a fixed invariant collection of types. We show that, under this notion, fields and bad groups are internal to the family of types and we study the internability of groups modulo a nilpotent factor.

1. INTRODUCTION

Recall that an stable theory is called 1-based if the canonical base of a real tuple $\bar{a}$ over an algebraically closed set A is algebraic over $\bar{a}$. Hrushovski and Pillay showed [8] that in a stable 1-based group, definable sets are finite union of cosets of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups. Furthermore, the group is abelian-by-finite. In the finite Morley rank context, the notion of 1-basedness agrees with local modularity and also with k-linearity: the canonical parameter of any uniformly definable family of curves has Morley rank at most k.

Hrushovski, in his *ab initio* strongly minimal set [9], introduced a weaker notion than 1-basedness: CM-triviality, of which Baudisch's renowned non-1-based new uncountably categorical group [2] is a proper example. Pillay [13] showed that neither infinite fields nor bad groups could be interpretable in a CM-trivial stable theory and therefore, in the finite Morley rank context, definable groups are nilpotent-by-finite. In [14] a whole hierarchy of new geometries (n-ample geometries) was exhibited, infinite fields being at the top of the classification, and CM-triviality agrees with non-2-ampleness.

In [15], Pillay and Ziegler reproved the function field case of Mordell-Lang in all characteristics inspired by Hrushovski's original proof but avoiding the use of Zariski Geometries. Instead, motivated on Campana's work [6] on algebraic coreductions, they showed in particular that the collection of types of finite Morley rank in the theory of differentially closed fields in characteristic 0 has the *Canonical Base Property* (in short, CBP) with respect to the field of constants. That is, given a definable set X of bounded differential degree and Morley degree 1 in a saturated differential closed field, the field of definition of the constructible set determined by X is internal over a generic realisation of X in terms of a finite set of elements coming from the constant field.

Clearly, the CBP is a generalisation of 1-basedness, since for the latter, the type of the canonical base of the generic type of X is already algebraic over a realisation.

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For the CBP, we replace algebraicity by internality with respect to a fixed invariant collection Σ of types. Chatizidakis [7] showed that the CBP already implied a strengthening of this notion, called UCBP, introduced by Moosa and Pillay [11] in their study of compact complex spaces.

In [10], following the definability results obtained for 1-based theories, Kowalski and Pillay showed that a group definable in a stable theory having the CBP is then abelian-by-internal.

The goal of this note is to introduce a weaker notion than CM-triviality, called CM-linearity, replacing algebraicity by internality with respect to to a fixed invariant collection Σ of types. Similarly to the generalisation from 1-based to CBP, we show that in a CM-linear stable theory both interpretable fields and bad groups are internal. Furthermore, in the finite Morley rank context, we conclude that every interpretable group is then nilpotent-by-internal. Recent work in [4] shows that the inclusion of CM-trivial theories in CM-linear ones is proper.

2. From trival to linear

The notions that we will present could be generalised to any simple theory, replacing algebraic closure by bounded closure. However, some of the results we will refer to are only true for stable theories. Hence we will only consider this case for the sake of the exposition, All throughout this article, we will be working inside a sufficiently saturated model of a stable theory T. All sets are considered as subsets of the mentioned model.

Definition 2.1. The theory T is called *1-based* if for every eq-algebraically closed set A (i.e. algebraically closed set in T^{eq}) and every real tuple c, we have that $\mathrm{Cb}(c/A)$ is algebraic over c. Equivalently, for every pair of eq-algebraically closed subsets $A \subset B$ and every real tuple c, we have that $\mathrm{Cb}(c/A)$ is algebraic over $\mathrm{Cb}(c/B)$.

Remark 2.2. It is a straightforward generalisation of the results in [13] to show that we may assume that A (and B) are small submodels. If T is 1-based, the same conclusion holds for (possibly imaginary) tuples c.

Recall some of the results proved by Hrushovski-Pillay [8]:

Theorem 2.3. *Let G be an interpretable group in a 1-based stable theory. The following hold:*

(1) *Every connected definable subgroup H of G is definable over* $\mathrm{acl}^{eq}(\emptyset)$.
(2) *The connected component of G is abelian.*

Definition 2.4. The theory T is called *CM-trivial* if for every pair of eq-algebraically closed subsets $A \subset B$ and every real tuple c, if $\mathrm{acl}^{eq}(Ac) \cap B = A$, then $\mathrm{Cb}(c/A)$ is algebraic over $\mathrm{Cb}(c/B)$.

Remark 2.5. Remark 2.2 hols for CM-trivial theories as well.
Any 1-based theory is CM-trivial.

Recall some of the results proved by Pillay [13]:

Theorem 2.6. *Let T be a CM-trivial stable theory. The following hold:*

(1) *No infinite field can be interpretable.*
(2) *No bad group can be interpretable.*

(3) An interpretable group of finite Morley rank is nilpotent-by-finite.

Remark 2.7. Recall that a group of finite Morley rank is a *bad group* if it is non-solvable and all of whose proper connected definable subgroups are nilpotent. It is unknown whether bad groups exists. A bad group of minimal rank can be taken simple, i.e. with no proper normal subgroups.

Definition 2.8. A type p over A is *almost internal* to the invariant family of types Σ if for evey realisation a of p there is some superset $B \supset A$ with $a \perp_A B$ and realisations $\beta_1, \ldots, \beta_r$ of types in Σ over B such that a is algebraic over $B, \beta_1, \ldots, \beta_r$. We say that p is almost Σ-internal.

A collection of types $\mathcal{F}$ in a theory T has the *Canonical Base Property* (in short, CBP) with respect to the invariant family of types Σ if for every type p in $\mathcal{F}$ over an eq-algebraically closed A and every realisation $c \models p$, the type $\mathrm{tp}(\mathrm{Cb}(c/A)/c)$ is almost Σ-internal. We say that T has the CBP with respect to the family Σ if the every type in T does.

Remark 2.9. Clearly, any 1-based theory has the CBP with respect to any family.

The theory T has the CBP with respect to the family Σ if and only if for every pair of eq-algebraically closed subsets $A \subset B$ and every real tuple c, we have that $\mathrm{tp}(\mathrm{Cb}(c/A)/\mathrm{Cb}(c/B))$ is almost Σ-internal. So we could have generalised any of the two equivalent notions of 1-basedness.

Proof. One implication is trivial by setting $B = \mathrm{acl}^{\mathrm{eq}}(Ac)$, since then $\mathrm{Cb}(c/B)$ is interalgebraic with c. The other implication follows from the fact that

$$c \underset{\mathrm{Cb}(c/B)}{\perp} B$$

yields

$$c \underset{\mathrm{Cb}(c/B)}{\perp} \mathrm{Cb}(c/A)$$

and use that a non-forking restriction of an almost internal type is again almost internal. □

Conjecture. Does every CM-trivial theory of finite rank have the CBP with respect to the family of all non-1-based types of rank 1?

Taken Theorem 2.3 as a guideline, Kowalski and Pillay proved the following [10]

Theorem 2.10. *Let T be stable and have the CBP with respect to a fixed invariant family Σ and G be a type-definable group. The following hold:*

(1) Given a connected type-definable subgroup $H \leq G$, the type of its canonical parameter $\mathrm{tp}(\ulcorner H \urcorner)$ is almost Σ-internal.

(2) If G is connected, then $G/Z(G)$ is almost Σ-internal.

3. Definable groups in CM-linear theories

In this section, we will introduce a weaker notion than CM-triviality, called CM-linearity, parallel to the generalisation from 1-based to CBP. Furthermore, we will show that interpretable fields and bad groups in a CM-linear theory are almost internal. In the case of a group of finite Morley rank, we show that is is nilpotent-by-internal (cf. Theorem 2.6).

Definition 3.1. The theory T is called *CM-linear* with respect to the invariant family of types Σ if for every pair of eq-algebraically closed subsets $A \subset B$ and every real tuple c, if $\mathrm{acl}^{\mathrm{eq}}(Ac) \cap B = A$, then $\mathrm{tp}(\mathrm{Cb}(c/A)/\,\mathrm{Cb}(c/B))$ is almost Σ-internal.

Remark 3.2. Every CM-trivial theory is CM-linear with respect to any family Σ.

As in Remark 2.2, we may assume that A and B are small submodels. If T is CM-linear, the same conclusion holds for (possibly imaginary) tuples c.

Inspired by Theorem 2.6, we show the following.

Theorem 3.3. *Let T be a stable theory which is CM-linear with respect to an invariant family of types Σ. The following hold:*

(1) Any interpretable field K is almost Σ-internal.

(2) Any interpretable simple bad group is almost Σ-internal.

(3) Any interpretable group of finite Morley rank is nilpotent-by-(almost Σ-internal).

Proof. (1) We take over the proof of Proposition 3.2 in [13]. Let K be our interpretable field. Consider a generic plane P given by $ax+by+c=z$, where a, b, c are generic independent. Take a line l contained in P given by $y = \lambda x + \mu$ such that λ, μ are generic independent over a, b, c. Take a point $p = (x, y, z)$ in $l \subset P$ generic over a, b, c, λ, μ.

Pillay showed that $\mathrm{acl}^{\mathrm{eq}}(Ap) \cap B = A$, where $A = \mathrm{acl}^{\mathrm{eq}}(\ulcorner P \urcorner)$ is the algebraic closure of the canonical parameter of the plane and $B = \mathrm{acl}^{\mathrm{eq}}(\ulcorner P \urcorner \ulcorner l \urcorner)$ the closure of the parameters of the plane and the line. Furthermore, he also showed that $\mathrm{Cb}(p/A) = \ulcorner P \urcorner$ and $\mathrm{Cb}(p/B) = \ulcorner l \urcorner$. Hence, by CM-linearity, we have that $\mathrm{tp}(\ulcorner P \urcorner / \ulcorner l \urcorner)$ is almost Σ-internal. Note tha the line is given by the equations

$$y = \lambda x + \mu$$
$$z = (a + b\lambda)x + (c + b\mu)$$

Now, since a, b, c, λ, μ are generic independent, so are

$$a\lambda, b\lambda\mu, c\lambda, \lambda, \mu$$

and

$$a\lambda, b\lambda\mu + a\lambda, c\lambda + b\lambda\mu, \lambda, \mu$$

as well. Hence,

$$a\lambda, b\lambda + a, c + b\mu, \lambda, \mu$$

are generic independent and so are

$$a, b\lambda + a, c + b\mu, \lambda, \mu.$$

Since a is in $\ulcorner P \urcorner$, the type $\mathrm{tp}(a/b\lambda + a, c + b\mu, \lambda, \mu)$ is also almost Σ-internal, and hence so is $\mathrm{tp}(a)$.

(2) If G is an interpretable simple bad group, it is the union of the conjugates of a proper nilpotent infinite connected definable subgroup T such that for every g in $G \setminus N_G(T)$, the intersection $T^g \cap T$ is trivial. Furthermore, the group T equals its normaliser $N_G(T)$. With this, given a, b, c, g in G generic independent, Pillay proved in Proposition 3.5 [13] that

$$\mathrm{acl}^{\mathrm{eq}}(b, c, A) \cap B = A,$$

where $A = \mathrm{acl}^{\mathrm{eq}}(\ulcorner(b,c)\cdot G_a\urcorner)$ is the algebraic closure of the canonical parameter of the coset determined by (b,c) of the subgroup

$$G_a = \{(g, g^a) \mid g \in G\}$$

and $B = \mathrm{acl}^{\mathrm{eq}}(\ulcorner P\urcorner\ulcorner l\urcorner)$, where l is the coset determined by (b,c) of the subgroup

$$(T^g)_a = \{(t, t^a) \mid t \in T^g\}.$$

It was shown as well that $\mathrm{Cb}(b,c/A) = \ulcorner(b,c)\cdot G_a\urcorner$ and $\mathrm{Cb}(b,c/B) = \ulcorner(b,c)\cdot(T^g)_a\urcorner$. Note that the latter is contained in $\mathrm{acl}^{\mathrm{eq}}(b,c,\ulcorner T^g\urcorner,\ulcorner a\cdot Z(T^g)\urcorner)$.

In particular, since G is simple, its center $Z(G)$ is trivial, so a is in $\ulcorner(b,c)\cdot G_a\urcorner$. By CM-linearity, we have that $\mathrm{tp}(a/b,c,\ulcorner T^g\urcorner,\ulcorner T^{ga}\urcorner,\ulcorner a\cdot Z(T^g)\urcorner)$ is almost Σ-internal. Now, since a, g are independent from b, c, we have that

$$a \underset{a/Z(T^g),g/T}{\perp} b,c$$

and hence $\mathrm{tp}(a/gT, aZ(T^g))$ is almost Σ-internal as well.

Now, for any stable group, a is generic in $aZ(T^g)$ over g. So, the generic type of $aZ(T^g)$ is almost Σ-internal, and so is the generic type of $Z(T^g)$. Since T is nilpotent infinite, its center is infinite (Proposition 1.10 [16]) and by Zilber's indecomposability theorem [17], we conclude that the group generated by $(Z(T)^0)^g$ is normal and almost Σ-internal. Hence, it equals G.

(3) Note first that a simple group G of finite Morley rank in a CM-trivial theory is Σ-internal, by unidimensionality: G is either a simple bad group, whence almost Σ-internal by part (2) or it interprets an infinite field, which is almost Σ-internal by part (1). In both cases, the generic type of G is non-orthogonal to Σ by unidimensionality. Therefore, since G is simple, it is Σ-internal.

Let G be an arbitrary group of finite Morley rank in a CM-linear theory, which we may assume to be connected. By [3], there is a normal series

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{1\}$$

such that each quotient G_i/G_{i+1} for $i < n$ is either finite, abelian, or simple. Furthermore, the series may to be refined to that each quotient is either finite, simple non-abelian, abelian and Σ-internal, or abelian with no almost Σ-internal subgroup.

Let C be the centraliser in G of all almost Σ-internal quotients (which include all simple and all finite quotients). Clearly, if G_i/G_{i+1} is abelian, the derived subgroup of $C \cap G_i$ is contained in $C \cap G_{i+1}$. Otherwise, since C centralises the quotient, then the derived subgroup of $C \cap G_i$ is contained in $C \cap G_{i+1}$ as well. Hence, C has a solvable normal series and G/C is almost Σ-internal (see [12, Remark 6.7]). Suppose that C^0 were not nilpotent. Then there would be an interpretable section of C^0 of the form $K^+ \rtimes T$ for some interpretable field K and an infinite multiplicative subgroup $T \le K^\times$. By part (1) this section is almost Σ-internal and hence centralised by C. Hence, the action of T on K^+ would be trivial, which is a contradiction. Thus C is nilpotent-by-finite.

□

Remark 3.4. [12, Theorem 6.6] implies in particular that if Σ contains all non one-based types of rank 1, then a group G of finite Morley rank has a nilpotent subgroup N such that G/N is almost Σ-internal, without any hypothesis of CM-linearity. Hence part (3) above only provided additionnal information if Σ does not contain all non one-based types of rank 1.

Question. What can we say about groups of infinite rank, or even merely stable ones. In a CM-linear stable group, can one find a nilpotent normal subgroup such that the quotient is almost Σ-internal?

Remark 3.5. The theory of the free pseudospace [3] is neither CM-trivial nor it has the CBP with respect to the family of rank 1 types. However it is CM-linear with respect to the family of rank 1 types [4]. Unfortunately, this theory is trivial and therefore no infinite group is interpretable. Thus it provides little insight about the significance of the above theorem.

Any almost strongly minimal theory has the CBP and is CM-linear with respect to the family of rank 1 types. However, this example is orthogonal to the previous one. We ignore if there is a relevant example of a non-trivial non CM-trivial theory which is CM-linear.

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