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On categoricity of atomic AEC

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Abstract

In recent years, the results about atomic abstract elementary class were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly under the assumption of atomic ω-stability (or *-excellence). I tried the argument around the problem under some weaker conditions.

1. Atomic AEC and splitting

We recall some definitions.

Definition 1 A class of structures $(\mathbf{K}, \prec_{\mathbf{K}})$ (of a language L) is an *abstract elementary class* (*AEC*) if the class $\mathbf{K}$ and class of pairs satisfying the binary relation $\prec_{\mathbf{K}}$ are each closed under isomorphism and satisfy the following conditions ;
A1. If $M \prec_{\mathbf{K}} N$, then $M \subseteq N$.
A2. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
A3. If $\{A_i : i < \delta\}$ is a $\prec_{\mathbf{K}}$-increasing chain :
  (1) $\bigcup_{i<\delta} A_i \in \mathbf{K}$
  (2) for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
  (3) if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$, then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.
A4. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ *and* $A \subseteq B$, then $A \prec_{\mathbf{K}} B$.
A5. There is a Löwenheim-Skolem number LS($\mathbf{K}$) such that if $A \subseteq B \in \mathbf{K}$, there is an $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and $|A'| \leq |A| + LS(\mathbf{K})$.

Definition 2 We say an AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ is *atomic* if $\mathbf{K}$ is the class of atomic models of a countable complete first order theory and $\prec_{\mathbf{K}}$ is first order elementary submodel.

In the following, $\mathbf{K}$ denotes an atomic AEC.

Definition 3 Let T be a countable first order theory.
A set A contained in a model M of T is *atomic* if every finite sequence in

A realizes a principal type over the empty set.

Let A be an atomic set.

$S_{at}(A)$ is the collection of $p \in S(A)$ such that if $a \in \mathcal{M}$ realizes p, Aa is atomic (where $\mathcal{M}$ is the big model).

We refer to a $p \in S_{at}(A)$ as an *atomic type.*

We consider the notion of stability for atomic types.

Definition 4 The atomic class $\mathbf{K}$ is $\lambda -$ *stable* if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{at}(M)| = \lambda$.

Example 5 ([1]) 1. Let $\mathbf{K}_1$ be the class of atomic models of the theory of dense linear order without endpoints. Then $\mathbf{K}_1$ is not $\omega-$stable.

2. Let $\mathbf{K}_2$ be the class of atomic models of the theory of the ordered Abelian group of rationals. Then $\mathbf{K}_2$ is $\omega-$stable.

The notion of independence by splitting is available in this context.

Definition 6 A complete type p over B *splits over* $A \subset B$ if there are b, $c \in B$ which realize the same type over A and a formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.

Let A, B, C be atomic.

We write $A \perp_C B$ and say A is *independent from* B *over* C if for any finite sequence $a \in A$, $\mathrm{tp}_{at}(a/B)$ does not split over some finite subset of C.

Fact 7 ([1]) *Under the atomic $\omega-$stable assumption of $(\mathbf{K}, \prec_{\mathbf{K}})$ (and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.*

Theorem 8 ([1]) *If $\mathbf{K}$ is $\omega-$stable and has a model of power $\aleph_1$, then it has a model of power $\aleph_2$.*

2. Atomic AEC without infinite splitting chain

In Baldwin's book [1] they argue the categoricity of atomic AEC under $\omega-$stability assumption of atomic types. I considered the same problem under some weaker conditions.

Definition 9 Let $\mathbf{K}$ be an atomic AEC and $M \in \mathbf{K}$.

M has no infinite splitting chain if for any nonalgebraic $p \in S_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} (\subset M)$ such that $p \upharpoonright A_{i+1}$ splits over A_i for all $i < \omega$.

We can prove the next facts.

Fact 10 *If* $\mathbf{K}$ *is* $\omega-$*stable, then no model of* $\mathbf{K}$ *has infinite splitting chain.*

Fact 11 *Under the assumption that* $(\mathbf{K}, \prec_{\mathbf{K}})$ *has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms.*

3. Existence of pregeometry

In [1], categoricity of atomic AEC are proved by means of the fact that every model is prime and minimal over a basis of some pregeometry given by a quasi-minimal set. So I tried to define pregeometry in the present context.

At first we prove the next proposition which is some modification of Theorem 8 above.

Proposition 12 *If there are* $N \in \mathbf{K}$ *with* $|N| > \aleph_0$ *and a nonalgebraic type* $p(x) \in S^1_{at}(N)$ *such that* N *has no infinite splitting chain.*

Then there are $M \in \mathbf{K}$ *with* $|M| = \aleph_2$ *and a nonalgebraic type* $q(x) \in S^1_{at}(M)$ *such that* M *has no infinite splitting chain and* q *does not split over some* $b \in M$*, and* $q \restriction b$ *has a Morley sequence* I *in* M *with* $|I| = \aleph_2$.

Moreover if $|N| = \aleph_1$*, then we can take* M *such that* $N \prec M$.

In this note, Morley sequence means the sequence constructed by nonsplitting extensions. Thus Morley sequences are indiscernible.

Lemma 13 *Let* $M \in \mathbf{K}$ *and* $p(x) \in S_{at}(M)$.

Suppose that M *has no infinite splitting chain and* p *does not split over some* $b \in M$.

And let $I = \{a_i : i < \alpha\}$ *be a Morley sequence of* $p \restriction b$ *in* M.

Then I *is totally indiscernible.*

In [8], they characterized generically stable types. We try to modify the notion in this context.

Definition 14 Let $M \in \mathbf{K}$.

A nonalgebraic type $p(x) \in S_{at}(M)$ is *generically stable in* M if for some $A \subset M$, p does not split over A and if $I = \{a_i : i < \alpha\}$ is a Morley sequence of $p \restriction A$ in M, then for any $\phi(x) \in L(M)-$formula, $\{i : M \models \phi(a_i)\}$ is either finite or co-finite.

We can prove the next lemma.

Lemma 15 *Let* $M \in \mathbf{K}$ *and* $q(x) \in S^1_{at}(M)$ *be in Proposition 12.*

Then q *is generically stable in* M.

Moreover if q *does not split over* b*, then* q *is definable over* b *and* $q \restriction b$ *is stationary w.r.t. nonsplitting extension.*

We recall the definition of pregeometry.

Definition 16 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say (X, cl) is *pregeometry*.
(I) $A \subset B \Longrightarrow A \subset \mathrm{cl}(A) \subset \mathrm{cl}(B)$,
(II) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(III) (Finite character) $b \in \mathrm{cl}(A) \Longrightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$,
(IV) (Exchange axiom)
$b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \Longrightarrow c \in \mathrm{cl}(A \cup \{b\})$.

We define big type which is a modified notion in [1].

Definition 17 Let $a \in M$ and $A \subset M \in \mathbf{K}$.
A nonalgebraic atomic type $\mathrm{tp}_{at}(a/A)$ is *big* if there is an atomic model $N \in \mathbf{K}$ such that $A \subset N$ and $\mathrm{tp}_{at}(a/A)$ has a nonalgebraic atomic extension over N.

In the following we argue under the existence of uncountable model $M \in \mathbf{K}$ and a nonalgebraic type $p(x) \in S^1_{at}(M)$. We may assume that p has what is called a minimal U-rank, or U-rank = 1.

Lemma 18 *Let* $\mathbf{K}$ *has no infinite splitting chain and* $M \in \mathbf{K}$. *And let* $p(x) \in S^1_{at}(M)$ *be nonalgebraic and* p *does not split over* b *for some* $b \in M$.
Then $p \restriction b$ *has an extension* $q(x) \in S^1_{at}(c)$ *such that*
$b \in c \in M$ *and* q *is big, but any splitting extension of* q *is not big.*

We may assume that the type q in Proposition 12 above has such property.
We define some closure operator.

Definition 19 Let $M \in \mathbf{K}$ and $p(x) \in S^1_{at}(M)$. And let p does not split over $\emptyset$ (or some finite parameter) and $p \restriction \emptyset$ is stationary.
The operator cl_p is defined by ;
$cl^0_p(X) = X$ and $cl^{n+1}_p(X) = \{a \in (p \restriction \emptyset)(M) \mid a \notin (p \restriction cl^n_p(X))(M)\}$,
and $cl_p(X) = \bigcup_{n<\omega} cl^n_p(X)$ for any $X \subset (p \restriction \emptyset)(M)$.

We can prove the next fact.

Theorem 20 *Let* $\mathbf{K}$ *has no infinite splitting chain and* $M \in \mathbf{K}$ *(with* $|M| > \aleph_0$ *).*
And let $p(x) \in S^1_{at}(M)$ *be a nonalgebraic type such that* p *does not split over* $\emptyset$ *and* $p \restriction \emptyset$ *has no big splitting extension (or* p *has a minimal U-rank among such types).*
Then $((p \restriction \emptyset)(M),\ cl_p)$ *is pregeometry.*

4. Constructible sequence of atomic types

In the argument of categoricity for $*$−excellent AEC, prime models play a crucial role. Now we do not assume the existence of prime models. We try the analogous argument of $F^a_{\kappa(T)}$−prime models in some large atomic model.

First we check the next lemma.

Lemma 21 *(* **K** *has no infinite splitting chain.)*
Let $M \in \mathbf{K}$. And let $A \subset B \subset M$ and a be such that $\mathrm{tp}_{at}(a/A)$ has a nonsplitting extension over B (or $A \leq_{TV} B$) and $\mathrm{tp}_{at}(a/A)$ is stationary.
Then the following are equivalent ;
(i) $\mathrm{tp}_{at}(a/A) \vdash \mathrm{tp}_{at}(a/B)$
(ii) For any a' such that $\mathrm{tp}_{at}(a'/A) = \mathrm{tp}_{at}(a/A)$, $\mathrm{tp}_{at}(a'/B)$ does not split over A.

I define some isolation of atomic types.

Definition 22 Let $a \in M \in \mathbf{K}$ and $A \subset M$.
A type $\mathrm{tp}_{at}(a/A)$ is *quasi − isolated* if there is $b \in M$ such that $\mathrm{tp}_{at}(a/b) \vdash \mathrm{tp}_{at}(a/A)$.
A sequence $\{c_i : i < \alpha\} \subset M$ is *quasi − constructible over* A if, for any $\beta < \alpha$, $\mathrm{tp}_{at}(c_\beta / A \cup \{c_i : i < \beta\})$ is quasi-isolated.
M is *quasi − constructible over*A if $M \setminus A$ can be written as a quasi-constructible sequence.

We can prove the next proposition by using Lemma 21 above.

Proposition 23 *Let* **K** *has no infinite splitting chain and $N \in \mathbf{K}$ (with $|N| > \aleph_0$).*
And let a nonalgebraic $p(x) \in S^1_{at}(N)$ be such that p does not split over $\emptyset$ and p has no big splitting extension (or p has a minimal U-rank among such types).
(Suppose that $p \restriction \emptyset$ has a Morley sequence I with $|I| > \aleph_0$ in N.)
Then for any basis J of $((p \restriction \emptyset)(N), cl_p)$, there is a quasi-constructible model over J in N.

5. Categoricity in some large atomic model

At first we recall the definition of Vaughtian triple from [1]. Note that the notion *big* is modified here.

Definition 24 A triple (M, N, ϕ) is called a *Vaughtian triple* if $\phi(M) = \phi(N)$ where $M \prec N \in \mathbf{K}$ with $M \neq N$ and $L(M)$−formula ϕ is big.

In this chapter, we assume that K has no infinite splitting chain where **K** is an atomic AEC. Under this condition we can prove some results about the two cardinal problem.

I tried the argument of categoricity in this context by means of quasi-constructible model. But I do not have the settled result yet. At present I can prove the next theorem by the properties of generically stable types.

If we try to extend the categoricity result to the whole **K**, we need some additional conditions, such as amalgamation property of models, and any atomic set is included in an atomic model, and so on.

In the next Theorem 25, $p \upharpoonright \emptyset$ has a Morley sequence I in N with $|I| = |N|$.

Theorem 25 *Let* **K** *has no infinite splitting chain and* $N \in \mathbf{K}$ *such that* ($|N| > \aleph_0$ *and*) *there is no Vaughtian triple in* N.

And let $p(x) \in S^1_{at}(N)$ *be nonalgebraic such that* p *does not split over* $\emptyset$ *and* $p \upharpoonright \emptyset$ *has no big spitting extension (or* p *has a minimal U-rank among such types).*

Then for $M_i \prec N$ $(i < 2)$ *with* $|M_0| = |M_1|$, $M_0 \cong M_1$.

6. Example of Shelah et al.

Shelah's original work ([4],[5]) showed that categoricity up to $\aleph_\omega$ of a sentence in $L_{\omega_1,\omega}$ implies categoricity in all uncountable cardinalities. Shelah and Hart showed the necessity of the assumption by constructing some example ([6]). This example is adapted by Baldwin and Kolesnikov ([1],[2]).

We can not recall the definition of it and details here.

Theorem 26 ([1],[2]) *For each* $k < \omega$, *there is a* $L_{\omega_1,\omega}$*−sentence* ϕ_{k+2} *such that :*

ϕ_{k+2} *is categorical in* μ *if* $\mu \le \aleph_k$, *and*

ϕ_{k+2} *is not categorical in any* μ *with* $\mu > \aleph_k$.

And they proved the next proposition in [2].

Proposition 27 ([2]) *Let* M *be the standard model of* ϕ_{k+2} *of size* $\aleph_k$. *Then there are* $2^{\aleph_k}$ *Galois types over* M.

This structure is expanded to be an atomic model. And we can check the next fact.

Fact 28 *Let* M *and* ϕ_{k+2} *be the* $L_{\omega_1,\omega}$*−sentence in the Proposition 27 above. Then* M *has an infinite splitting chain (in the expanded language).*

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