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First-order theory of quantum 2-tori T_q^2

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Abstract

We discuss the first-order description of non-commutative quantum 2-tori.

1 Introduction

Quantum tori are geometric objects associated with non-commutative algebras $\mathcal{A}_q$ with q generating multiplicative groups. When q is a root of unity, we have a quantum torus which is a Zariski structure (Zilber's result).

On the other hand, when q generates a cyclic group of countable order we hope to construct new kinds of analytic Zariski structures as non-commutative quantum 2-tori.

In this note we give the first-order description of the quantum 2-tori given in [Z1].

2 Description of the torus $T_q^2(\mathbb{C})$

First we give the description of a quantum 2-torus defined over the complex numbers $\mathbb{C}$. Consider a $\mathbb{C}$-algebra $\mathcal{A}_q^2$ generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV, \quad UU^{-1} = U^{-1}U = VV^{-1} = V^{-1}V = I$$

where $q = e^{2\pi ih}$ with $h \in \mathbb{R}$. Let $\Gamma_q = q^{\mathbb{Z}}$ be a multiplicative subgroup of $\mathbb{C}^*$.

In the literature, the $\mathbb{C}$-algebra $\mathcal{A}_q^2$ itself is called a *quantum torus.*

In this paper, however, the quantum 2-torus $T_q^2(\mathbb{C})$ over $\mathbb{C}$ associated with the algebra $\mathcal{A}_q^2$ and the group Γ_q is the 3-sorted structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{C}^*)$ with the actions U and V satisfying

$$\begin{array}{lcl} U & : & \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v) \\ V & : & \mathbf{u}(\gamma u, v) \mapsto v \mathbf{u}(q^{-1}\gamma u, v) \end{array} \tag{1}$$

and

$$\begin{array}{lcl} U & : & \mathbf{v}(\gamma v, u) \mapsto u \mathbf{v}(q\gamma v, u) \\ V & : & \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u) \end{array} \tag{2}$$

where $\phi : \mathbb{C}^*/\Gamma \to \mathbb{C}^*$ is a (non-definable) "choice function", $\mathbf{u}(\gamma u, v) \in \mathbf{U}_\phi$, and $\mathbf{v}(\gamma v, u) \in \mathbf{V}_\phi$.

We also have

$$\begin{array}{lcl} U^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto \gamma^{-1} u^{-1} \mathbf{u}(\gamma u, v) \\ V^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto v^{-1} \mathbf{u}(q\gamma u, v) \end{array} \tag{3}$$

and

$$\begin{array}{lcl} U^{-1} & : & \mathbf{v}(\gamma v, u) \mapsto u^{-1} \mathbf{v}(q^{-1}\gamma v, u) \\ V^{-1} & : & \mathbf{v}(\gamma v, u) \mapsto \gamma^{-1} v^{-1} \mathbf{v}(\gamma v, u) \end{array} \tag{4}$$

There is also a function $\langle \cdot \mid \cdot \rangle$ called the *pairing function* which plays as an *inner product*

$$\langle \cdot \mid \cdot \rangle : \Big(\mathbf{V}_\phi \times \mathbf{U}_\phi\Big) \cup \Big(\mathbf{U}_\phi \times \mathbf{V}_\phi\Big) \to \Gamma \tag{5}$$

whose properties are given in the next subsection.

2.1 More details

Fix $\phi : \mathbb{C}^*/\Gamma \to \mathbb{C}^*$. Put $\Phi = \mathrm{ran}(\phi)$. We work with Φ^2. Take $\langle u, v \rangle \in \Phi^2$. Let

$$\begin{array}{lcl} \mathbf{U}_{\langle u,v \rangle} & := & \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \gamma_1 . \gamma_2 \in \Gamma\}, \\ \mathbf{V}_{\langle u,v \rangle} & := & \{\gamma_1 \cdot \mathbf{v}(\gamma_2 v, u) : \gamma_1 . \gamma_2 \in \Gamma\}. \end{array} \tag{6}$$

Then let

$$\begin{array}{lcl} \mathbf{U}_\phi & := & \bigcup_{\langle u,v \rangle \in \Phi^2} U_{\langle u,v \rangle}, \\ \mathbf{V}_\phi & := & \bigcup_{\langle u,v \rangle \in \Phi^2} V_{\langle u,v \rangle}. \end{array} \tag{7}$$

Consider the following sets $\mathbb{C}^*\mathbf{U}_\phi$ and $\mathbb{C}^*\mathbf{V}_\phi$:

$$\begin{array}{lcl} \mathbb{C}^*\mathbf{U}_\phi & := & \{x \cdot \mathbf{u}(\gamma u, v) : \langle u, v\rangle \in \Phi^2, x \in \mathbb{C}^*, \gamma \in \Gamma\} \\ \mathbb{C}^*\mathbf{V}_\phi & := & \{x \cdot \mathbf{v}(\gamma u, v) : \langle u, v\rangle \in \Phi^2, x \in \mathbb{C}^*, \gamma \in \Gamma\} \end{array} \tag{8}$$

The **pairing** function (5) defined above satisfies the following quantitative relations: take $\langle u, v\rangle \in \Phi^2$, then for any $m, k, r, s \in \mathbb{N}$ and $q^s\mathbf{v}(q^m v, u) \in \mathbf{V}_{\langle u,v\rangle}$ and $q^r\mathbf{u}(q^k u, v) \in \mathbf{U}_{\langle u,v\rangle}$ we have

$$\langle q^s\mathbf{v}(q^m v, u) | q^r\mathbf{u}(q^k u, v)\rangle = q^{r-s-km} \tag{9}$$

and

$$\langle q^r\mathbf{u}(q^k u, v) | q^s\mathbf{v}(q^m v, u)\rangle = q^{km+s-r} = \langle q^s\mathbf{v}(q^m v, u) | q^r\mathbf{u}(q^k u, v)\rangle^{-1}. \tag{10}$$

For $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$,

$$\langle q^s\mathbf{v}(v', u) | q^r\mathbf{u}(u', v)\rangle$$

is not defined.

Definition 1 Let ϕ be a choice function of $\mathbb{C}^*/\Gamma$. The pair $(\mathbf{U}_\phi, \mathbf{V}_\phi)$ with operators U, V acting on $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$ satisfying (1) through (4) and the pairing function satisfying (9) and (10) is denoted $(\mathbf{U}, \mathbf{V})_\phi$.

3 The first-order theory of $T_q^2(\mathbb{F})$

Proposition 4.4 of [Z1] claims that once $q = e^{2\pi ih}$ with $h \in \mathbb{R}$ is given any two structures of the form $T_q^2(\mathbb{C})$ are isomorphic. Following Zilber's argument in [Z1], we supply here details of his proof.

Proposition 2 (Proposition 4.4, [Z1]) Given $q = e^{2\pi ih}$ with $h \in \mathbb{R}$ any two structures of the form $T_q^2(\mathbb{C})$ are isomorphic over $\mathbb{C}$. In other words, the isomorphism type of $T_q^2(\mathbb{C})$ does not depend on the system of representative Φ.

Proof: Fix a $q = e^{2\pi ih}$ with $h \in \mathbb{R}$, and set $\Gamma = q^{\mathbb{Z}}$.

Let ϕ, ψ be two choice functions of $\mathbb{C}^*/\Gamma$. Consider two structures $(\mathbf{U}, \mathbf{V})_\phi$ and $(\mathbf{U}, \mathbf{V})_\psi$. We show that these two structures are isomorphic.

Suppose ϕ picks $\langle u_g, v_g\rangle$ from $\mathbb{C}^*/\Gamma$ and ψ picks $\langle u_0, v_0\rangle$ from the same coset of $\langle u_g, v_g\rangle$.

Consider the bases $\{\mathbf{u}(q^k u_g, v_g) : k \in \mathbb{Z}\}$ of $U_{\langle u_g, v_g\rangle}$ and $\{\mathbf{v}(q^k v_g, u_g) : k \in \mathbb{Z}\}$ of $V_{\langle u_g, v_g\rangle}$ in the structure $(\mathbf{U}_\phi, \mathbf{V}_\phi)$.

Since $\langle u_0, v_0\rangle$ and $\langle u_g, v_g\rangle$ are in the same coset of $\mathbb{C}^*/\Gamma$ there are $s, t \in \mathbb{Z}$ such that $u_0 = q^s u_g, v_0 = q^t v_g$.

We want to transfer the structure of $\mathbf{U}_{\angle u_g, v_g\rangle}$ and $\mathbf{V}_{\langle u_g, v_g\rangle}$ to $\mathbf{U}_{\angle u_0, v_0\rangle}$ and $\mathbf{V}_{\langle u_0, v\rangle}$ as follows. Set

- $\mathbf{u}(u_0, v_0) := q^{st}\mathbf{u}(q^s u_g, v_g)$,
- $\mathbf{u}(q^k u_0, v_0) := v_0^k V^{-k}\mathbf{u}(u_0, v_0)$,
- $\mathbf{v}(v_0, u_0) := \mathbf{v}(q^t v_g, u_g)$,
- $\mathbf{v}(q^k v_0, u_0) := u_0^{-k} U^k \mathbf{v}(v_0, u_0)$,

where $k \in \mathbb{Z}$. First notice that we have

- $$\begin{aligned} \mathbf{u}(q^k u_0, v_0) &= v_0^k V^{-k}\mathbf{u}(u_0, v_0) \\ &= (q^t v_g)^k V^{-k}(q^{st}\mathbf{u}(q^s u_g, v_g)) \\ &= q^{kt} v_g^k q^{st} v_g^{-k}\mathbf{u}(q^{s+k} u_g, v_g) \\ &= q^{kt+st}\mathbf{u}(q^{k+s} u_g, v_g), \end{aligned}$$
- $$\begin{aligned} \mathbf{v}(q^k v_0, u_0) &= u_0^{-k} U^k \mathbf{v}(v_0, u_0) \\ &= q^{-ks} u_g^{-k} U^k \mathbf{v}(g^t v_g, u_g) \\ &= q^{-ks} u_g^{-k} u_g^k \mathbf{v}(g^{t+k} v_g, u_g) \\ &= q^{-sk}\mathbf{v}(q^{k+t} v_g, u_g). \end{aligned}$$

These relations allow us to show that the operator U and V act on the set $\mathbf{U}_{\langle u_0, v_0\rangle} = \{\mathbf{u}(q^k u_0, v_0) : k \in \mathbb{Z}\}$ properly, i.e. U and V obey the rule (1) and (2).

$(\mathbf{u} - 1)$

$$\begin{aligned} U(\mathbf{u}(u_0, v_0)) &= U(q^{st}\mathbf{u}(q^s u_g, v_g)) \\ &= q^{st} U(\mathbf{u}(q^s u_g, v_g)) \\ &= q^{st} q^s u_g \mathbf{u}(q^s u_g, v_g) \\ &= u_0 q^{st}\mathbf{u}(q^s u_g, v_g) \\ &= u_0 \mathbf{u}(u_0, v_0), \end{aligned}$$

$(\mathbf{u} - 2)$

$$\begin{aligned} V(\mathbf{u}(u_0, v_0)) &= V(q^{st}\mathbf{u}(q^s u_g, v_g)) \\ &= q^{st} V(\mathbf{u}(q^s u_g, v_g)) \\ &= q^{st} v_g \mathbf{u}(q^{-1} q^s u_g, v_g) \\ &= q^{(s-1)t} q^t v_g \mathbf{u}(q^{s-1} u_g, v_g) \\ &= q^t v_g q^{(s-1)t}\mathbf{u}(q^{s-1} u_g, v_g) \\ &= v_0 \mathbf{u}(q^{s-1} u_g, q^t v_g) \\ &= v_0 \mathbf{u}(q^{-1} u_0, v_0), \end{aligned}$$

$(\mathbf{u}-3)$

$$\begin{aligned} U(\mathbf{u}(q^k u_0, v_0)) &= U(v_0^k V^{-k}\mathbf{u}(u_0, v_0)) \\ &= U(v_0^k V^{-k} q^{st}\mathbf{u}(q^s u_g, v_g)) \\ &= U(q^{kt+st}\mathbf{u}(q^{k+s}u_g, v_g)) \\ &= q^{kt+st} q^{k+s} u_g \mathbf{u}(q^{k+s}u_g, v_g) \\ &= q^k u_0 q^{kt+st}\mathbf{u}(q^{k+s}u_g, v_g) \\ &= q^k u_0 \mathbf{u}(q^k u_0, v_0), \end{aligned}$$

$(\mathbf{u}-4)$

$$\begin{aligned} V(\mathbf{u}(q^k u_0, v_0)) &= V(q^{kt+st}\mathbf{u}(q^{k+s}u_g, v_g)) \\ &= q^{kt+st} V(\mathbf{u}(q^{k+s}u_g v_g)) \\ &= q^{kt+st} v_g \mathbf{u}(q^{k+s-1}u_g, v_g) \\ &= q^t v_g q^{(k+s-1)t}\mathbf{u}(q^{k+s-1}u_g, v_g) \\ &= v_0 \mathbf{u}(q^{k+s-1}u_g, v_0) \\ &= v_0 \mathbf{u}(q^{k-1}u_0, v_0). \end{aligned}$$

We also see that the operator U and V act on the set $\mathbf{V}_{\langle v_0, u_0\rangle} = \{\mathbf{u}(q^k v_0, u_0) : k \in \mathbb{Z}\}$ properly, i.e. U and V obey the rule (1) and (2) as well.

$(\mathbf{v}-1)$

$$\begin{aligned} U(\mathbf{v}(v_0, u_0)) &= U(\mathbf{v}(q^t v_g, u_g)) \\ &= u_g \mathbf{v}(q^{t+1}v_g, u_g)) \\ &= u_g q^s q^{-s}\mathbf{v}(q^{t+1}v_g, u_g) \\ &= u_0 \mathbf{v}(q v_0, u_0), \end{aligned}$$

$(\mathbf{v}-2)$

$$\begin{aligned} V(\mathbf{v}(v_0, u_0)) &= V(\mathbf{v}(q^t v_g, u_g)) \\ &= q^t v_g \mathbf{v}(q^t v_g, u_g)) \\ &= v_0 \mathbf{v}(v_0, u_0), \end{aligned}$$

$(\mathbf{v}-3)$

$$\begin{aligned} U(\mathbf{v}(q^k v_0, u_0)) &= U(q^{-sk}\mathbf{v}(q^{k+t}v_g, u_g)) \\ &= q^{-sk} U(\mathbf{v}(q^{k+t}v_g, u_g)) \\ &= q^{-sk} u_g \mathbf{v}(q^{k+t+1}v_g, u_g) \\ &= q^s u_g q^{-s(k+1)}\mathbf{v}(q^{k+1+t}v_g, u_g) \\ &= u_0 \mathbf{v}(q^{k+1}v_0, u_0), \end{aligned}$$

$(\mathbf{v}-4)$

$$\begin{aligned} V(\mathbf{v}(q^k v_0, u_0)) &= V(q^{-sk}\mathbf{v}(q^{k+t}v_g, u_g)) \\ &= q^{-sk} V(\mathbf{v}(q^{k+t}v_g u_g)) \\ &= q^{-sk} q^{k+t} v_g \mathbf{v}(q^{k+t}v_g, u_g) \\ &= q^{k+t} v_g q^{-sk}\mathbf{v}(q^{k+t}v_g, u_g) \\ &= q^k v_0 \mathbf{v}(q^k v_0, u_0). \end{aligned}$$

Finally the following relations tell us that we can properly transfer the pairing function from $(\mathbf{U}_{\langle u_g,v_g\rangle}, \mathbf{V}_{\langle v_g,u_g\rangle})$ to $(\mathbf{U}_{\langle u_0,v_0\rangle}, \mathbf{V}_{\langle v_0,u_0\rangle})$:

$$\langle \mathbf{v}(v_0,u_0)|\mathbf{u}(u_0,v_0)\rangle = \langle \mathbf{v}(q^t v_g,u_g)|q^{st}\mathbf{u}(q^s u_g,v_g)\rangle = q^{st-st} = 1$$

and

$$\begin{aligned}\langle \mathbf{v}(q^m v_0,u_0)|\mathbf{u}(q^k u_0,v_0)\rangle &= \langle q^{-sm}\mathbf{v}(q^{m+t}v_g,u_g)|q^{st+kt}\mathbf{u}(q^{k+s}u_g,v_g)\rangle \\ &= q^{st+kt-(-sm)-(m+t)(k+s)} \\ &= q^{-mk}.\end{aligned}$$

We have now shown that the two structures $(\mathbf{U}_{\langle u_g,v_g\rangle}, \mathbf{V}_{\langle v_g,u_g\rangle})$ and $(\mathbf{U}_{\langle u_0,v_0\rangle}, \mathbf{V}_{\langle v_0,u_0\rangle})$ are isomorphic. Therefore so are the two structures $(\mathbf{U},\mathbf{V})_\phi$ and $(\mathbf{U},\mathbf{V})_\psi$. ■

One notice easily that any particular properties of the complex numbers are not used in the proof of Proposition 4.4. Hence we see that Proposition 4.4 can be generalized to quantum 2-tori over any algebraically closed field $\mathbb{F}$ of characteristic zero.

Corollary 3 Suppose $\mathbb{F}$ and $\mathbb{F}'$ are isomorphic algebraically closed fields of characteristic zero. Let $q \in \mathbb{F}$ and $q' \in \mathbb{F}'$ such that both q and q' are transcendental and $\Gamma = q^{\mathbb{Z}}$ and $\Gamma' = q'^{\mathbb{Z}}$ are elementarily equivalent infinite multiplicative subgroups. Then $T^2_q(\mathbb{F})$ and $T^2_{q'}(\mathbb{F}')$ are isomorphic as quantum 2-tori.

This enable us to describe the isomorphism type of the quantum 2-tori using the language $\mathcal{L}_q$ given below.

3.1 The language

From now on we work with $\mathbb{F}$ an algebraically closed field of characteristic zero and $\Gamma = q^{\mathbb{Z}}$ for some $q \in \mathbb{F}$ (say, transcendental).

Let $\mathcal{L}_q = \mathcal{L}_{T^2_q} = \{\mathbf{U},\mathbf{V},\mathbb{F},\Gamma,U,V,q,\mathbf{u}(\cdot,\cdot),\mathbf{v}(\cdot,\cdot),\mathrm{sp},0,1,T_p\}$ where

- $\mathbf{U},\mathbf{V},\mathbb{F},\Gamma$ are unary predicates,
- U, V are 4-ary relations,
- q is a constant symbol,
- sp is a ternary predicate such that $\mathrm{sp}(\gamma,x,y)$ is interpreted as $\gamma \cdot x = y$ for the module operation, (sp stands for the *scalar multiplication*)
- binary function symbols $\mathbf{u}(\cdot,\cdot)$, $\mathbf{v}(\cdot,\cdot)$,
- T_p is a ternary relation symbol corresponding to the pairing function.

3.2 The first-order theory of $T_q^2(\mathbb{F})$

Recall that we treat the quantum torus $T_q^2(\mathbb{F})$ as a 3-sorted structure $(U, V, \mathbb{F})$. Two operators U and V are acting on $\mathbb{F}^*\mathbf{U}$ and $\mathbb{F}^*\mathbf{V}$. We view both $\mathbb{F}^*\mathbf{U}$ and $\mathbb{F}^*\mathbf{V}$ as the following equivalence classes; $\mathbb{F}^*\mathbf{U} \simeq (\mathbb{F} \times \mathbf{U})/E$ where for $(x, y), (x', y') \in \mathbb{F} \times \mathbf{U}$ define

$$(x, y) \sim_E (x', y') \iff \exists\gamma \in \Gamma\,(y' = \gamma y \wedge x' = x\gamma^{-1}) \tag{11}$$

Similarly for $\mathbb{F}^*\mathbf{V}$.

3.2.1 Two actions on U and V

Two operators U and V are acting on $\mathbb{F}^*\mathbf{U}$ and $\mathbb{F}^*\mathbf{V}$ and we treat both U and V as 4-ary relations. These actions have the following properties;

1. $\forall \mathbf{u} \in \mathbf{U}\,\exists u \in \mathbb{F}^*\,(U : \mathbf{u} \mapsto u\mathbf{u})$ and
   $\forall \mathbf{u} \in \mathbf{U}\,\exists v \in \mathbb{F}^{\cdot}\,\exists \mathbf{u}' \in \mathbf{U}\,(V : \mathbf{u} \mapsto v\mathbf{u}' \wedge U : \mathbf{u}' \mapsto q^{-1}u\mathbf{u}')$
2. $\forall \mathbf{v} \in \mathbf{V}\,\exists v \in \mathbb{F}^*\,(V : \mathbf{v} \mapsto v\mathbf{v})$ and
   $\forall \mathbf{v} \in \mathbf{V}\,\exists u \in \mathbb{F}^{\cdot}\,\exists \mathbf{v}' \in \mathbf{V}\,(U : \mathbf{v} \mapsto qu\mathbf{v}' \wedge V : \mathbf{v} \mapsto v\mathbf{v})$

We need to translate the above properties into first-order formulas taking into account the equivalence relation (11); first we express simply that $\mathbf{U}$ and $\mathbf{V}$ are acting on both $\mathbb{F}^*\mathbf{U}$ and $\mathbb{F}^*\mathbf{U}$ as follows.

- $\forall x_1 \forall u_1 \forall x_2 \forall u_2\, \forall x_1' \forall u_1' \forall x_2' \forall u_2' \forall x_1' \Big(U(x_1, u_1, x_2, u_2) \to (x_1 \in \mathbb{F}^* \wedge u_1 \in \mathbf{U} \wedge x_2 \in \mathbb{F}^* \wedge u_2 \in \mathbf{U})\Big) \bigwedge \Big((U(x_1, u_1, x_2, u_2) \wedge U(x_1', u_1', x_2', u_2' \wedge (x_1, u_1) \sim_E (x_1', u_1')) \to (x_2, u_2) \sim_E (x_2', u_2')\Big)$
  This formula corresponds to $U : \mathbb{F}^*\mathbf{U} \to \mathbb{F}^*\mathbf{U}$. We need three more similar formulas expressing $V : \mathbb{F}^*\mathbf{U} \to \mathbb{F}^*\mathbf{U}$, $U : \mathbb{F}^*\mathbf{V} \to \mathbb{F}^*\mathbf{V}$ and $V : \mathbb{F}^*\mathbf{V} \to \mathbb{F}^*\mathbf{V}$.

Next we write down the above properties 1 and 2;

1. $\forall x \forall \mathbf{u} \exists u \Big((x \in \mathbb{F}^* \wedge \mathbf{u} \in \mathbf{U} \wedge u \in \mathbb{F}^*) \to U(x, \mathbf{u}, x, u\mathbf{u})\Big)$
   This formula corresponds to $U : \mathbf{u} \mapsto u\mathbf{u}$.
   Second part can be translated similarly.

The property 2 is also translated into first-order formulas.

3.2.2 $VU = qUV$

We must write down the equation "$VU = qUV$" using 4-ary relations U and V. This can be done as follows;

$$\forall x_1 \forall u_1 \forall x_2 \forall u_2 \forall x_3 \forall u_3 \forall x_4 \forall u_4 \forall x_5 \forall u_5 \Big(V(x_1, u_1, x_2, u_2) \wedge V(x_2, u_2, x_3, u_3) \wedge V(x_1, u_1, x_4, u_4) \wedge U(x_4, u_4, x_5, u_5) \to x_3 = x_5 \wedge u_3 = qu_5 \Big)$$

3.2.3 The pairing

1. $\forall x \forall y \forall z \Big(T_q(x, y, z) \to ((x \in \mathbf{V} \wedge y \in \mathbf{U} \wedge z \in \Gamma) \vee (x \in \mathbf{U} \wedge y \in \mathbf{V} \wedge z \in \Gamma)) \Big)$
2. $\forall x \forall y \forall z \forall z' \Big((T_q(x, y, z) \wedge T_q(y, x, z')) \to z \cdot z' = 1 \Big)$
3. If $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$ then ternary relation does not hold. This can be translated into
   $\forall x \in \mathbf{V}\, \forall y \in \mathbf{U}\, \forall z\, \forall \gamma \in \Gamma\, \forall x' \in \mathbf{V} (x \neq \gamma x' \to \neg T_q(x, y, z))$
   $\forall x \in \mathbf{V}\, \forall y \in \mathbf{U}\, \forall z\, \forall \gamma \in \Gamma\, \forall y' \in \mathbf{U} (y \neq \gamma y' \to \neg T_q(x, y, z))$

3.2.4 Quantitative properties of pairing functions

Finally we need to take account of quantitative properties of the pairing function (9) and (10) into the theory.

4 Remarks

In this note we only give the first-order description of quantum 2-tori. Stability issue of the theory and $\mathcal{L}_{\omega_1\,\omega}$ description of the tori are discssued in the forthcoming paper.

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