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A genetic method for non-associative algebras (II) (Mendel algebra with mutation)

By

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Abstract. *This is the second part of the paper with the same title. A concept of Mendel algebra with mutation is introduced and it is proved that a certain class of (non-commutative) Jordan algebras and flexible algebras can be found in the algebra and that a classification theory of non-associative algebras based on the Mendel algebras is given from a point of view in genetics.*

Introduction

In the previous paper we have introduced a method of genetics to non-associative algebras and generate them by use of the mathematical formulations of Mendel's law systematically and classify them based on these laws ([6]). There we have not included the concept of mutation in genetics. In this paper we introduce a concept of mutation in the Mendel's laws and find a generation scheme of non-associative algebras including flexible algebra and Jordan algebra by Mendel's laws systematically. Hence we may expect to find a new field of non-associative algebras in genetics.

We introduce a concept of Mendel algebras with mutations following the Mendel's separation law in genetics. We call the linear space M generated by generators $S_1, S_2, ..., S_n$ Mendel algebra, when generators satisfy the following commutation relations and the distributive law:

$$S_i * S_j = \frac{1}{2}\{pS_i + qS_j\} \; (p > 0,\ q > 0, p + q = 1)$$

We notice that in the case where $p = q = 1/2$, the Mendel algebra is called of mutation free. We call the Mendel algebra with mutation Mendel algebra simply.

At first we notice that the Mendel algebra is non-associative and non-commutative when it has mutations. We want to find non-associative algebras including the flexible algebras and Jordan algebras in Mendel algebras. We recall the following definitions:

flexible algebra: $(XY)X = X(YX)$
Jordan algebra: $(((XX)Y)X) = (XX)(YX)$

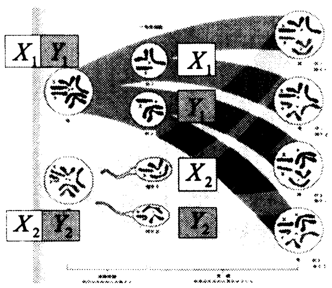
for any pair of elements $^\forall X, ^\forall Y$ of the algebras.

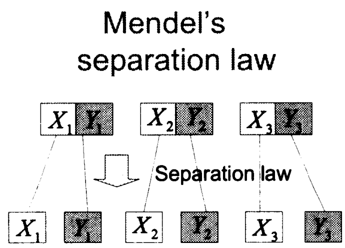
The main results of this paper can be stated as follows:
(1)Mendel algebra is flexible algebra and Jordan algebra(Theorem I and II).
(2)A family of flexible algebras and Jordan algebras can be generated by mathematical formulation of Mendel's laws: Separation law, mating law and independent law and mutation (Theorem III).
(3) We can give a classification of non-associative algebras by use of the shift invariance condition in Mendel algebras. We can discuss these commutation relations in terms of „shift invariant elements"of an algebra. Then we can show that the shift invariant algebras on Mendel algebras automatically derive a family of non-associative algebras including flexible algebras and Jordan algebras.

1. Mendel's laws

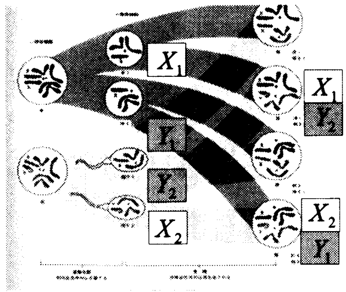
In this section we recall some basic facts on Mendel's law ([4]). In 1860, Mendel has discovered the fundamental laws in genetics, which are called Mendel's laws. They constitute three laws: (1)Separation law, (2) Mating law, (3) Independent law. Later (4) Mutation is discovered. Here we include this law in Mendel's law. We describe the laws by use of figures and we omit its description expect the description on mutation.

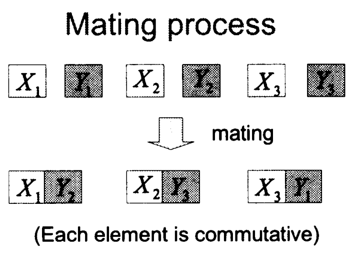
(1) Separation law



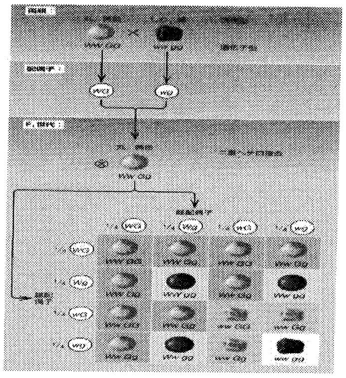


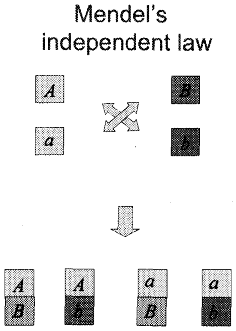
(2) Mating law





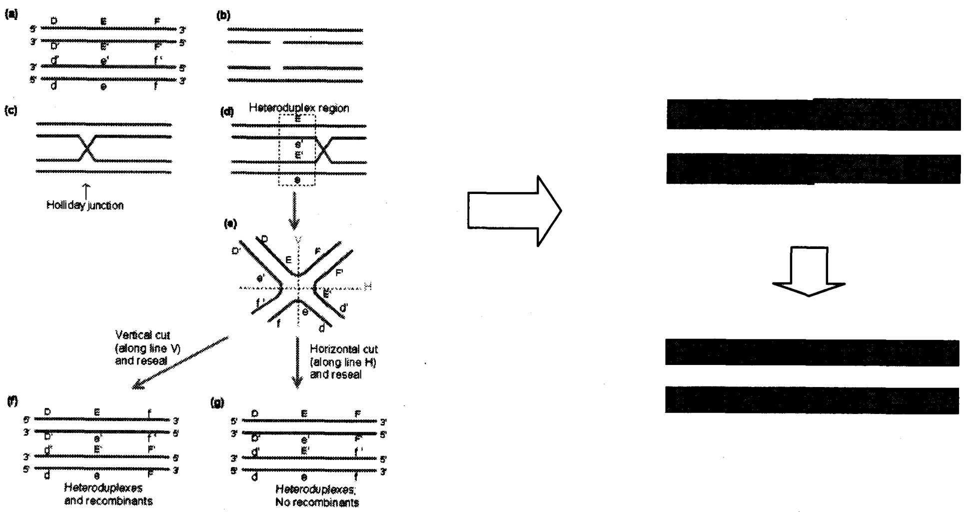
(3) Independent law





(4) Mutation

Here we have to say that our condition of the mutation on the algebra is artificial from the biological view point. Hence we have to make some comments on mutations. In this paper we regard the causes of mutations as the recombinations or the Holiday structures in genetics([4]).



2. Mendel algebra M(p,q)

In this section we introduce a several non-associative algebras which are motivated by Mendel's law ([5]):

(1) Mendel algebra M(p,q)

Let $A(=R[S_1,S_2,\dots,S_n])$ be an algebra. Introducing the product by

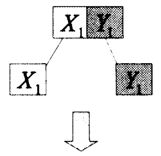
$$\begin{cases} S_i * S_j = \frac{1}{2}\{pS_i + qS_j\} \ (p+q=1(p>0,q>0)) \\ X*Y=\sum_{i,j=1}^{n}\alpha_i\beta_j S_i * S_j (X=\sum_{i=1}^{n}\alpha_i S_i, Y=\sum_{i=1}^{n}\beta_i S_i) \end{cases}$$

we have an algebra $M_{p,q}^{(n)}(R)$ which is called n-dimensional Mendel algebra simply. We see that $M_{p,q}^{(n)}(R)$ is a non-commutative and non-associative algebra in the case of $p \neq q$. Otherwise it is commutative.

We notice a basic property holds on Mendel algebras which might be a mathematical formulation of Hardy-Weinberg's law ([4]):

$$(\sum_{i=1}^{n} \alpha_i S_i)^2 = \sum_{i=1}^{n} \alpha_i S_i \; (\sum_{i=1}^{n} \alpha_i = 1)$$

(2)Original Mendel algebra M(1/2,1/2)
The algebra $M(1/2,1/2)$ is called Mendel algebra mutation free. Putting



$$X_1 * Y_1 = \frac{1}{2}(X_1 + Y_1)$$

$$\begin{cases} S_i * S_j = \frac{1}{2}\{S_i + S_j\} \\ X * Y = \sum_{i,j=1}^{n} \alpha_i \beta_j S_i * S_j (X = \sum_{i=1}^{n} \alpha_i S_i, Y = \sum_{i=1}^{n} \beta_i S_i) \end{cases}$$

we have an algebra $M^{(n)}(1/2,1/2)$ which is called n-dimensional mutation free Mendel algebra .

(3) Alternative Mendel algebra
Let $A(= R[S_1, S_2, ..., S_n])$ be an algebra. Introducing the product by

$$\begin{cases} S_i * S_j = \frac{1}{2}\{S_i - S_j\} \\ X * Y = \sum_{i,j=1}^{n} \alpha_i \beta_j S_i * S_j (X = \sum_{i=1}^{n} \alpha_i S_i, Y = \sum_{i=1}^{n} \beta_i S_i) \end{cases}$$

we have an algebra $M_{(-)}^{(n)}(R)$ which is called n-dimensional alternative Mendel algebra. Then we see that $M_{(-)}^{(n)}(R)$ is a non-commutative and non-associative algebra.

3. Mendel algebra is flexible algebra

In this section we treat flexible algebras from our point of view. We begin with the definition ([6]): An algebra A is called flexible algebra, if the following commutation relation is satisfied:

$$\forall X, \forall Y \in A \Rightarrow (XY)X = X(YX).$$

Next we proceed to flexible algebras generated by Mendel algebras.

Theorem I
(1) A Mendel algebra $M(p,q)$($n \geq 2$) is a non-commutative, non-associative flexible algebra if $p \neq q$. Especially it is commutative when $p = q = 1/2$.
(2) $M_{(-)}^{(n)}(n \geq 2)$ is a non-commutative, non-associative flexible algebra.
Proof : Putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i$, we see $((XY)X) = \sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k$, and $(X(YX)) = \sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k)$. Hence to prove the assertion, it is enough to prove the following equality:

$$\sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k = \sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k).$$

At first we notice the following equalities:

$$(*)\begin{cases}(S_i * S_j) * S_k = p^2 S_i + pqS_j + qS_k \\ S_i * (S_j * S_k) = pS_i + pqS_j + q^2 S_k\end{cases}$$

Hence we have

$$\begin{aligned}&\sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k - \sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k) \\ &= \sum \alpha_i \beta_j \alpha_k (pq(S_i - S_k)) = 0\end{aligned}$$

Hence we have proved the assertion.

The proof for alternative Mendel algebra is almost same and may be omitted.

4. Mendel algebra is Jordan algebra

In this section we make a Jordan algebra by a genetic method ([3], [7]): An algebra J is called Jordan algebra if the commutation relation holds for $\forall X, \forall Y \in J$:

$$(((XX)Y)X) = ((XX)(YX)).$$

When it is commutative, it is called Jordan algebra simply. Otherwise it is called non-commutative Jordan algebra. We can prove the following theorem:

Theorem II

(1) Mendel algebra $M^{(n)}(p,q)\,(n \geq 2)$ is a non-commutative Jordan algebra, when $p \neq q$ .Otherwise it is commutative Jordan algebra.

(2) $M^{(n)}_{(-)}(n \geq 2)$ is a Jordan algebra .

Proof of (1): At first we notice the following identities:

$$(**)\begin{cases}(((S_i * S_j) * S_k) * S_l) = p^3 S_i + p^2 qS_j + pqS_k + qS \\ ((S_i * S_j) * (S_k * S_l)) = p^2 S_i + pqS_j + pqS_k + q^2 S_l.\end{cases}$$

Putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i$, we have

$$(((XX)Y)X) = \sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l,$$

$$((XX)(YX)) = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l),$$

Hence to prove the assertion, it is enough to prove the following equality:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l).$$

For this we decompose the both sides in the following manner:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum_{i=j=l} \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l)$$

$$\sum \alpha_i \alpha_j \beta_k \alpha_k (S_i * (S_j * S_k)) * S_l = \sum_{i=k=l} \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l),$$

where the second sum is remained sum. Since $((S_i * S_i) * S_k) * S_i = ((S_i * S_i) * (S_k * S_i))$,

the first term of the both sides are identical. Next we decompose the remained sum into two parts: $\Sigma' = \Sigma'_1 + \Sigma'_2$: The first sum is taken for the case of two of the indices (i, j, l) are identical and the remained sum is taken for three different indices. The second terms of the both sides can be written as follows:

$$\sum\nolimits_2{}'\alpha_i\alpha_j\beta_k\alpha_l((S_i*S_j)*S_k)*S_l=\sum\nolimits_\sigma\alpha_{\sigma(i)}\alpha_{\sigma(j)}\beta_k\alpha_{\sigma(l)}\{(S_{\sigma(i)}*S_{\sigma(j)})*S_k*S_{\sigma(l)}\}$$

$$\sum\nolimits_2{}'\alpha_i\alpha_j\beta_k\alpha_l(S_i*S_j)*(S_k*S_l)=\sum\nolimits_\sigma\alpha_{\sigma(i)}\alpha_{\sigma(j)}\beta_k\alpha_{\sigma(l)}\{(S_{\sigma(i)}*S_{\sigma(j)})*(S_k*S_{\sigma(l)})\},$$

where the sum is taken through the permutations of three words. By use of the identities $_{(**)}$, we can obtain the assertion.

Proof of (2): The proof can be performed in a completely similar manner and may be omitted.

4. Tensor product of Mendel algebras

We can define the tensor product $M_1\otimes M_2$ of two Mendel algebras M_1 and M_2 as follows: Putting $M_1=R[S_1,S_2,..,S_n]$, $M_2=R[S'_1,S'_2,..,S'_m]$, we define

$$M_1\otimes M_2=R[S_i\otimes S'_j : i=1,2,..,n, j=1,2,..,m].$$

We define the product by

$$(S_i\otimes S'_j)\hat{*}(S_k\otimes S'_l)=(S_i*S_k)\otimes(S'_j*'S'_l).$$

Then we have the following formula:

(1) $(S_i\otimes S'_j)\hat{*}(S_k\otimes S'_l)=1/2^2(S_i\otimes S'_j+S_i\otimes S'_l+S_k\otimes S'_j+S_k\otimes S'_l)$

(2) Putting $X=\sum_{i=1}^{n}\alpha_iS_i, Y=\sum_{j=1}^{m}\beta_jS'_j$ and $U=\sum_{i=1}^{n}\alpha'_iS_i, V=\sum_{j=1}^{m}\beta'_jS'_j$, we have $X\otimes Y=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_jS_i\otimes S'_j$, $U\otimes V=\sum_{k=1}^{n}\sum_{l=1}^{m}\alpha'_k\beta'_lS_k\otimes S'_l$. Then we have $(X\otimes Y)*(U\otimes V)=\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha'_k\beta_j\beta'_l(S_i\otimes S'_j+S_i\otimes S'_l+S_k\otimes S'_j+S_k\otimes S')$. We can prove the following theorem:

Theorem III

(1) The tensor product of Mendel algebras $M^{(n)}(p,q)\,(n\geq 2)$ is a flexible algebra .

(2) The tensor product of Mendel algebras $M^{(n)}(p,q)\,(n\geq 2)$ is a Jordan algebra .

5. Genetic generations of non-associative algebras

In the previous paper we have generated non-associative algebras by use of (1) separation law, (2) mating law, (3) independent law in genetics ([6]). Here we shall generate non-associative algebras four laws adding (4) mutations. By these generation scheme, we can generate a wider class of non-commutative, non-associative algebras including flexible and Jordan algebras systematically. We make a comment only on generations by mutations and will not repeat other things.

Generation by mutation

We choose an algebra A which is generated by elements $\{a_1,a_2,...,a_n\}$:We can make a new Mendel algebra introducing the following product:

$$\begin{cases}\Omega_{i_1i_2..i_n}*\Omega_{j_1j_2...j_n}=p_{i_1i_2..i_n}\Omega_{i_1i_2..i_n}+q_{j_1j_2...j_n}\Omega_{j_1j_2...j_n}\\ \qquad p_{i_1i_2..i_n}+q_{j_1j_2...j_n}=1\end{cases},$$

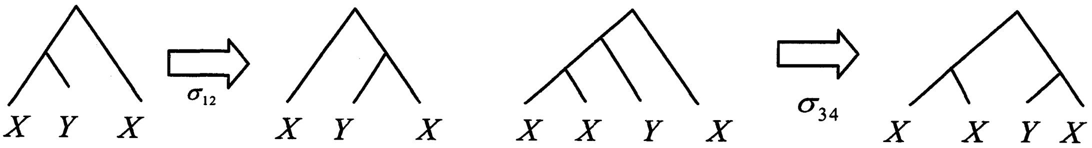
where $\Omega_{i_1i_2...i_n}$ is the product of elements $a_{i_n},a_{i_{n-1}},...,a_{i_1}$. Following the discussions in the previous paper, we can prove the following theorem:

Theorem IV
(1) We can generate non-commutative Mendel algebras from a given Mendel algebra by the genetic generations.
(2) We can obtain commutative and non-commutative flexible algebra and Jordan algebra by each genetic generation systematically.

6. Classifications of non-associative algebras based on Mendel algebras

We have obtained flexible algebra and Jordan algebra from the shift invariant conditions on Mendel algebras without mutations ([6]). We recall basic facts in the previous paper and state the analogous results for Mendel algebras with mutations. Details are omitted. We can describe any algebra in terms of brackets in the formal languages. Shift implies that the change of the neighboring brackets in an acceptable manner in the sense of formal language and shift invariance implies the elements give the same elements by the shifts of brackets.

Examples
We give two examples of shift invariant elements which are connected to non-associative algebras([3]):



flexible algbera $(XY)X = X(YX)$ Jordan algbera $(((XX)Y)X) = ((XX)(YX))$.

Based on this fact, we can get a group of non-associative algebras which are related to Mendel algebras.

Proposition(Shift invariance of flexible algebra)
We assume the following shift invariant elements: $X*(Y*Z)=(X*Y)*Z$ for $\forall X, \forall Y, \forall Z \in M(A)$. Then we have $X* = Z*$. Hence we have a flexible algebra.

Proof: Putting $X=\sum\alpha_i S_i, Y=\sum\beta_i S_i, Z=\sum\gamma_i S_i$ we consider the shift invariant condition: $X*(Y*Z)=(X*Y)*Z$. Restricting special element, we consider $((S_i*S_j)*S_k)=((S_i*(S_j*S_k))$. Then from (*), we see that $S_i = S_k$. Hence we obtain $X*(Y*X)=(X*Y)*X$.

Proposition(Shift invariance of Jordan algebra)
We assume that $((X*Y)*Z)*W=(X*Y)*(Z*W)$. Then we have $X=Y=W$. Hence we have a Jordan algebra.

Proof :From (**),we have $S_i = S_l = S_l$ from $(((S_i*S_j)*S_k)*S_l)=((S_i*S_j)*(S_k*S_l))$.

Hence putting $X=\sum\alpha_i S_i, Y=\sum\beta_i S_i$, we have the commutation relation of a Jordan algebra.

Hence we see that the shift invariance condition chooses a class of non-associative algebras in Mendel algebras. Therefore we may expect to list up non-associative algebras connected to Mendel algebras using the shift invariance of elements in the following table:

(The table of possible commutation relations)

(1)The terms of shift invariant conditions of degree 3

$((XY)Z), (X(YZ))$

(2) The terms of shift invariant conditions of degree 4

$(((XY)Z)W)$, $((X(YZ))W)$, $((XY)(ZW))$, $(X((Y(ZW))$, $(X((YZ)W))$

(3) The terms of shift invariant conditions of degree 5

$(((XY)Z)W)U$, $(X(YZ))W)U$, $(X((YZ)W))U$, $(X(Y(ZW)))U$, $(X(Y(Z(WU)))$, $X(Y((ZW)U))$
$X((Y(ZW))U)$, $X(((YZ))W)U)$, $((XY)(ZW))U$, $((XY)Z)(WU)$, $(XY)(Z(WU))$, $X((YZ)(WU))$

Examples of calculations of shift invariant elements tell us that the commutation relations of flexible algebra and Jordan algebra are basic and that we can get the algebras with commutation relations which are generated by those of flexible algebras and Jordan algebras.

References

[1] Y.Asou,H.Kouriyama, M.Matsuo, K.Nouno and O.Suzuki Construction of non-associative algebras in algebras generated by Chomsky sentences (To appear in the RIMS Proceeding Lecture Note

[2] N. Chomsky:Context-free grammar and pushdown storge,Quaterly Prog. Rep. No.65,187-194,CambridgeMass(1952)

[3]N.Jackobson:General representation theory of Jordan algebras,Trans. Amer. Math. Soc. 70(1951)

[4] D.L.Hartl: Essential genetics:A genomics perspective,John and Bartlett Publishers, Inc.M.A.U.S.A(2002)p.519

[5] A.Micali and Ph. Revoy:Sur les algebre gametique,Proc. Edinburgh Math. Soc. 29(1986),197-206

[6] A.Micali and O.Suzuki: A genetics method for non-associative algebras. Report on Mathematical studies in RIMS , Kyoto University 29(2010),197-206

[7] T.A.Springer and F.D.Veldkamp:Octonions, Jordan algebra and exceptional group, Springer Monograph in Mathematics, Springer 2000