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Stationary Phase Method, Feynman Path Integrals and Integration by Parts Formula

By

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Abstract

The primary aim of this paper is a short introductory guide to the following two topics:

1. Stationary phase method for oscillatory integrals over a space of large dimension.
2. Outline of proof of convergence of Feynman path integrals.

In the last part of the paper, the following results of recent research are added.

1. An integration by parts formula for Feynman path integrals:

$$(0.1)\qquad \int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -\int_{\Omega_{x,y}} F(\gamma)\mathrm{Div}\, p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu\int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma),$$

under suitable assumptions. This formula (0.1) is an analogy to Elworthy's integration by parts formula for Wiener integrals (cf. [3]).

2. A semiclassical asymptotic formula which holds in the case of $F(\gamma^*)=0$. Here γ^* is the stationary point of the phase $S(\gamma)$, i.e. $\delta S(\gamma^*)=0$.

Contents

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§ 1. Path Integral Defined by Feynman

For simplicity we restrict ourselves to the case where the configuration space is $\mathbf{R}^1$. In this case Lagrangian function with potential $V(t,x)$ is

$$L(t,\dot{x},x) = \frac{1}{2}\dot{x}^2 - V(t,x).$$

The case where non zero magnetic potential is present is discussed in [13]. Action of path γ is

$$S(\gamma) = \int_a^b L(t,\dot{\gamma}(t),\gamma(t))dt.$$

A classical path is the solution of the variational problem,

$$\delta S(\gamma_0) = 0, \quad \gamma_0(a) = y, \quad \gamma_0(b) = x.$$

A classical path satisfies Euler equation:

$$\begin{aligned} &\frac{d^2}{dt^2}\gamma(t) + \partial_x V(t,\gamma(t)) = 0, \\ &\gamma(b) = x, \quad \gamma(a) = y. \end{aligned}$$

Our assumption for potential $V(t,x)$ is the following (cf. W. Pauli [20]).

Assumption 1.1. 1. $V(t,x)$ is a real continuous function of (t,x). If t is fixed, then it is a function of class C^∞ in x.

2. For any $m \geq 0$ there exists $v_m \geq 0$ such that

$$\max_{|\alpha|=m} \sup_{(t,x)\in[a,b]\times\mathbf{R}^d} \left|\partial_x^\alpha V(t,x)\right| \leq v_m(1+|x|)^{\max\{2-m,0\}}.$$

With this Assumption 1.1 one can prove the following

Proposition 1.2. *Let* $\mu_0 > 0$ *be so small that*

$$\frac{\mu_0^2 d v_2}{8} < 1. \tag{1.1}$$

If $|b-a| \leq \mu_0$, *then for any* $x,\ y \in \mathbf{R}$ *there exists a unique classical path* γ *such that* $\gamma(a) = y$ *and* $\gamma(b) = x$.

Let Δ be an arbitrary division of the interval $[a,b]$ such that

$$\Delta : a = T_0 < T_1 < \cdots < T_J < T_{J+1} = b. \tag{1.2}$$

We set $\tau_j = T_j - T_{j-1}, j = 1,2,\ldots,J+1$ and $|\Delta| = \max_{1\leq j\leq J+1} \tau_j$.

Assume that $|\Delta| \leq \mu_0$. We set $x_0 = y$, $x_{J+1} = x$. For any $x_j \in \mathbf{R}$, $j = 1,2,\ldots,J$, we define a piecewise classical path $\gamma_\Delta(t)$ which is the classical path for $T_{j-1} \leq t \leq T_j$ and satisfies

$$\gamma_\Delta(T_j) = x_j, \quad (j = 0,1,2,\ldots,J+1). \tag{1.3}$$

γ_Δ may have edges at T_j.

Given a functional $F(\gamma)$, we often abbreviate $F(\gamma_\Delta)$ as F_Δ. Once Δ is fixed, it is a function of $(x_{J+1},x_J,\ldots,x_1,x_0)$ and we denote the dependence of $F(\gamma_\Delta)$ on $(x_{J+1},x_J,\ldots,x_1,x_0)$ by writing $F(\gamma_\Delta) = F_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)$.

Let $\nu = 2\pi h^{-1}$, where h is Planck's constant, and Ω_{xy} the space[1] of paths starting y at time a and reaching x at time a. Given a functional $F(\gamma)$ of $\gamma \in \Omega_{xy}$, Feynman [4] considered the following integral on finite dimensional space

$$\begin{aligned} (1.4)\quad & I[F_\Delta](\Delta;\nu,b,a,x,y) \\ &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\mathbf{R}^J} F(\gamma_\Delta)(x_{J+1},x_J,\ldots,x_1,x_0)\,e^{(i\nu S(\gamma_\Delta)(x_{J+1},x_J,\ldots,x_1,x_0))}\prod_{j=1}^{J} dx_j. \end{aligned}$$

Feynman defined his path integral by the formula:

$$\int_{\Omega_{xy}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,b,a,x,y). \tag{1.5}$$

The integral $I[F_\Delta](\Delta;\nu,b,a,x,y)$ of (1.4) is called time slicing approximation of Feynman path integral (1.5).

Does the right hand side of (1.5) give a finite number? Since the integral (1.4) does not converge absolutely, the following questions should be answered.

Q1 Does $I[F_\Delta](\Delta;\nu,b,a,x,y)$ exist for fixed $|\Delta| > 0$?

[1] In these notes Ω is a symbol which expresses vaguely notion of path space.

Q2 Does the $\lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,b,a,x,y)$ exist?

§ 2. Oscillatory Integrals

§ 2.1. What is Oscillatory Integral

First we discuss question Q1 above.

Once the division Δ of the interval is fixed, $I[F_\Delta](\Delta;\nu,b,a,x,y)$ is a special case of the following type of integrals called oscillatory integrals:

$$\int_{\mathbf{R}^n} a(x,y)e^{i\nu\phi(x,y)}dy,$$

where $\phi(x,y)$ is a real valued function of $(x,y)\in\mathbf{R}^m\times\mathbf{R}^n$ and $a(x,y)$ is a function of (x,y). ϕ is called the phase function and a is called the amplitude. Integral (1.4) is the case where $m=2$ and $n=J$.

Precise meaning of oscillatory integral (2.1) is the following (cf. [14]). Consider arbitrary family of fast decreasing C^∞ functions $\{\omega_\epsilon(y)\}_{\epsilon>0}\subset\mathcal{S}(\mathbf{R})$ which converges to 1 in the topology of $\mathcal{E}$. Here $\mathcal{E}$ is the space of C^∞ functions with topology of uniform convergence on every bounded closed intervals together with its all derivatives.

Definition 2.1. Let

$$I(x)=\lim_{\epsilon\to 0}\int_{\mathbf{R}^n}\omega_\epsilon(y)a(x,y)e^{i\nu\phi(x,y)}dy. \tag{2.1}$$

Now we give a sufficient condition for oscillatory integral (2.1) to exist.

Assume $x\in\mathbf{R}^m, y\in\mathbf{R}^n$ and the following conditions.

A1 Phase function $\phi(x,y)\in C^\infty(\mathbf{R}^m\times\mathbf{R}^n)$ is real valued and for any multi-indices α,β with $|\alpha|+|\beta|\geq 2$

$$|\partial_x^\alpha\partial_y^\beta\phi(x,y)|\leq C_{\alpha\beta}.$$

A2 Let $(\partial_{y_j}\partial_{y_k}\phi(x,y))$ be the $n\times n$ square matrix with (j,k) element $\partial_{y_j}\partial_{y_k}\phi(x,y)$. Assume that there exists $C>0$ such that

$$\left|\det(\partial_{y_j}\partial_{y_k}\phi(x,y))\right|\geq C>0$$

for any $(x,y)\in\mathbf{R}^m\times\mathbf{R}^n$. Here det means the determinant.

A3 The amplitude function $a(x,y)$, together with its all derivatives, is uniformly bounded on $\mathbf{R}^m\times\mathbf{R}^n$.

Theorem 2.2 (cf. [1]). *Under the conditions* **A1**, **A2** *and* **A3** *the oscillatory integral $I(x)$ exists. Moreover there exist a positive constant C such that*

$$|I(x)|\leq C\nu^{-n/2}\max_{|\alpha|\leq n+2}\sup_{y\in\mathbf{R}^n}|\partial_x^\alpha a(x,y)|.$$

Under the conditions **A1**, **A2**, **A3** for any fixed $x \in \mathbf{R}^m$ there exists one and only one critical point $y^*(x)$ of $\phi(x,y)$ as a function of y, i.e. $y^*(x)$ is the solution to system of equations

$$\partial_{y_j}\phi(x,y^*(x)) = 0, \qquad (j = 1,2,\ldots,n).$$

Let $H(x,y^*(x))$ be the Hessian matrix of $\phi(x,y)$ with respect to y at $y = y^*(x)$, i.e. $H(x,y^*(x))$ is the $n \times n$ symmetric matrix of which (j,k) element is $\partial_{y_j}\partial_{y_k}\phi(x,y^*(x))$.

Theorem 2.3 (Stationary Phase Method). *Suppose that conditions* **A1**, **A2** *and* **A3** *are satisfied. Then*

$$I(x) = \left(\frac{2\pi}{\nu}\right)^{n/2} |\det H(x,y^*(x))|^{-1/2} e^{\frac{\pi i}{4}[n-2\,\mathrm{Ind}(H(x,y^*(x)))]} e^{i\nu\phi(x,y^*(x))}(a(x,y^*(x)) + \nu^{-1}r(\nu,x)).$$

Here $\mathrm{Ind}(H(x,y^*(x)))$ *is the number of negative eigenvalues of matrix* $H(x,y^*(x))$*. For any* $k \geq 0$*, there exist* $K(k) > 0$ *and* $C_k > 0$ *such that for any* α *with* $|\alpha| \leq k$*,*

$$|\partial_x^\alpha r(\nu,x)| \leq C_k \max_{\substack{|\beta_1| \leq K(k) \\ |\beta_2| \leq K(k)}} \sup_{y \in \mathbf{R}^n} \left|\partial_x^{\beta_1}\partial_y^{\beta_2} a(x,y)\right|. \tag{2.2}$$

$a(x,y^*(x))$ is called the amplitude of the main term and $\nu^{-1}r(\nu,x)$ is the remainder (cf. [14] and [1] for more information).

§ 2.2. Time Slicing Approximation is Oscillatory Integral

In order to answer question **Q1**, we prove conditions **A1**, **A2**, **A3** of § 2 hold for (1.4).

From now on we always assume

$$|b-a| \leq \mu_0. \tag{2.3}$$

For any $x, y \in \mathbf{R}$ the classical path γ^* with $\gamma^*(a) = y$, $\gamma^*(b) = x$ is unique. We write

$$S(b,a,x,y) = S(\gamma^*). \tag{2.4}$$

Calculation shows:

Proposition 2.4. *If* $|b-a| \leq \mu_0$*,* $S(b,a,x,y)$ *is of the following form*:

$$S(b,a,x,y) = \frac{|x-y|^2}{2(b-a)} + (b-a)\phi(b,a,x,y).$$

The function $\phi(b,a,x,y)$ *is a function of* (b,a,x,y) *of class* C^1 *and there exists* $C > 0$ *such that*

$$|\phi(b,a,x,y)| \leq C(1+|x|^2+|y|^2).$$

Moreover, $\phi(b,a,x,y)$ *is a* C^∞ *function of* (x,y) *and for any* $m \geq 2$

$$\max_{2 \leq |\alpha|+|\beta| \leq m} \sup_{(x,y) \in \mathbf{R}^2} \left|\partial_x^\alpha \partial_y^\beta \phi(b,a,x,y)\right| = \kappa_m < \infty.$$

In particular,

$$\kappa_2 \leq \frac{v_2}{2}\left(1 - \frac{v_2\mu_0^2}{8}\right)^{-1}.$$

Let Δ be the division of time interval $[a,b]$ such that

$$\Delta : a = T_0 < T_1 < \cdots < T_J < T_{J+1} = b.$$

Assumption 2.5. For any multi-index $\alpha = (\alpha_0, \dots \alpha_{J+1})$ there exists $C_{\alpha,\Delta} > 0$ such that

$$\left|\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)\right| \leq C_{\alpha,\Delta}.$$

We discuss time slicing approximation of path integral.

(2.5) $I[F_\Delta](\Delta; \nu, b, a, x, y)$

$$= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2} \int_{\mathbf{R}^J} F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) e^{i\nu S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)} \prod_{j=1}^{J} dx_j.$$

We show that this is an oscillatory integral which satisfies conditions **A1, A2, A3** of § 2. Condition **A3** is clearly satisfied. We check condition **A1**.

$$S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) = S(\gamma_\Delta)(x_{J+1}, x_J, \dots, x_1, x_0) = \sum_{j=1}^{J+1} S(T_j, T_{j-1}, x_j, x_{j-1})$$
$$= \sum_{j=1}^{J+1}\left(\frac{|x_j - x_{j-1}|^2}{2\tau_j} + \tau_j \phi(T_j, T_{j-1}, x_j, x_{j-1})\right).$$

Note that

(2.6) $\partial_{x_j} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$

$$= \frac{x_j - x_{j-1}}{\tau_j} + \frac{x_j - x_{j+1}}{\tau_{j+1}} + \tau_j \partial_{x_j}\phi_j(x_j, x_{j-1}) + \tau_{j+1}\partial_{x_j}\phi_{j+1}(x_{j+1}, x_j).$$

Here we used abbreviation:

$$\phi_j(x_j, x_{j-1}) = \phi(T_j, T_{j-1}, x_j, x_{j-1}).$$

It follows from (2.6) and Proposition 2.4 that condition **A1** is satisfied.

Now we check condition **A2**. Consider $J \times J$ matrix Ψ whose (j,k) element is

$$\Psi_{jk} = \partial_{x_j}\partial_{x_k} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) \qquad (j,k = 1, 2, \dots, J).$$

Then we divide the matrix Ψ into two parts.

$$\Psi = H_\Delta + W_\Delta,$$

where

$$H_\Delta = \begin{pmatrix} \frac{1}{\tau_1}+\frac{1}{\tau_2} & -\frac{1}{\tau_2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{\tau_2} & \frac{1}{\tau_2}+\frac{1}{\tau_3} & -\frac{1}{\tau_3} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\tau_3} & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & -\frac{1}{\tau_J} \\ 0 & 0 & 0 & -\frac{1}{\tau_J} & & \frac{1}{\tau_J}+\frac{1}{\tau_{J+1}} \end{pmatrix}$$

and W_Δ is the matrix whose $(j.k)$ element is

$$w_{jk} = \begin{cases} \partial_{x_j}^2(\tau_j\phi_j + \tau_{j+1}\phi_{j+1}) & \text{if} \quad j=k \\ \partial_{x_k}\partial_{x_j}\tau_j\phi_j & \text{if} \quad k=j-1 \\ \partial_{x_j}\partial_{x_k}\tau_k\phi_k & \text{if} \quad k=j+1 \\ 0 & \text{if} \quad |j-k|\geq 2. \end{cases} \tag{2.7}$$

The matrix H_Δ is a constant matrix with determinant

$$\det H_\Delta = \frac{\tau_1+\tau_2+\cdots+\tau_{J+1}}{\tau_1\tau_2\ldots\tau_{J+1}} = \frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}}.$$

It has its inverse H_Δ^{-1}. Regarding W_Δ as an perturbation, we write

$$\Psi = H_\Delta(I + H_\Delta^{-1}W_\Delta).$$

Proposition 2.6. *Let* $0<\mu_1$ *be so small that* $\mu_1 \leq \mu_0$ *and* $\kappa_2\mu_1^2 < 1$. *Let* $|b-a|\leq \mu_1$. *Then for any* $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{J+2}$

$$(1-\kappa_2\mu_1^2)^J \leq \det(I + H_\Delta^{-1}W_\Delta) \leq (1+\kappa_2\mu_1^2)^J,$$

and

$$(1-\kappa_2\mu_1^2)^J\frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}} \leq \det\Psi = \det(H_\Delta + W_\Delta) \leq (1+\kappa_2\mu_1^2)^J\frac{(b-a)}{\tau_1\tau_2\ldots\tau_{J+1}}. \tag{2.8}$$

Condition **A2** for $I[F_\Delta](\Delta; v, b, a, x, y)$ follows from this proposition if $|b-a|$ is small (cf. [7]).

Consequently, we have proved that conditions **A1**, **A2** and **A3** for § 2 are satisfied if $|b-a| \leq \mu_1$ and Δ is fixed, Hence $I[F_\Delta](\Delta; v, b, a, x, y)$ has a definite value. We answered **Q1** of §1.

§ 2.3. Kumano-go & Taniguchi Theorem

We always assume that $|b-a| \le \mu_1$ in the following.

We apply stationary phase method to $I[F_\Delta](\Delta; \nu, b, a, x, y)$. Let γ^* be the classical path such that $\gamma^*(a) = y$, $\gamma^*(b) = x$.

Theorem 2.7. *If* $|b-a| \le \mu_1$, *then* $\operatorname{Ind} H_\Delta = 0$ *and*

$$I[F_\Delta](\Delta; \nu, b, a, x, y) = \Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2} e^{i\nu S(\gamma^*)} \big(\det(I + H_\Delta^{-1} W_\Delta^*)\big)^{-1/2} p(\Delta, \nu, b, a, x, y)$$

with some function $p(\Delta, \nu, b, a, x, y)$. *Here* W_Δ^* *is* W_Δ *evaluated at* $x_j = \gamma^*(T_j)$.

How does $p(\Delta, \nu, b, a, x, y)$ **behave as** $|\Delta| \to 0$? This is the core of the problem.

The next theorem was known earlier (cf. [15]).

Theorem 2.8 (Kumano-go & Taniguchi). *Assume* $|b-a| \le \mu_0$. *Assume that* F_Δ *satisfies the following property*:

For any $K > 0$, *there exists* A_K *independent of* Δ *such that if* $|\alpha_0| \le K |\alpha_1| \le K, \ldots, |\alpha_{J+1}| \le K$

$$|\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_J}^{\alpha_J} \ldots \partial_0^{\alpha_0} F_\Delta(x_{J+1}, \ldots, x_0)| \le A_K.$$

Then

$$I[F_\Delta](\Delta; \nu, b, a, x, y) = \Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2} e^{i\nu S(\gamma^*)} p(\Delta; \nu, b, a, x, y).$$

Moreover for any $k \ge 0$ *there exist* $K(k) \ge 0$ *and* $C_k > 0$ *such that as long as* $|\alpha_0| \le k, |\alpha_{J+1}| \le k$,

$$|\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} p(\Delta; \nu, b, a, x, y)| \le C_k^J A_{K(k)}. \tag{2.9}$$

Here $K(k)$, C_k *are independent of* Δ *and of* J.

If we let $J \to \infty$ then the bound $C_k^J A_{K(k)}$ obtained by (2.9) may go to ∞. In order to answer **Q2** of § 1 we have to improve Kumano-go & Taniguchi Theorem.

§ 2.4. Stationary Phase Method for Integrals over a Space of Large Dimension — Main Theorem —

Assume $|b-a| \le \mu_1$. Let γ^* be the unique classical path starting from y at time a and reaching x at time b. Let $x_j^* = \gamma^*(T_j)$ for $j = 0, 1, 2, \ldots, J+1$. We set

$$D(\Delta; b, a, x, y) = \det(I + H_\Delta^{-1} W_\Delta^*) = \Big(\frac{\tau_1 \tau_2 \ldots \tau_{J+1}}{(b-a)}\Big) \det(\operatorname{Hess}_{x_J^*, x_{J-1}^*, \ldots x_1^*} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)).$$

Here $\operatorname{Hess}_{x_J^*, x_{J-1}^*, \ldots x_1^*} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ denotes the Hessian matrix at $(x_J^*, x_{J-1}^*, \ldots x_1^*)$ of $S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$. Now we have

Theorem 2.9. *The function $D(\Delta; b, a, x, y)$ is of the following form:*

$$D(\Delta; b, a, x, y) = 1 + (b-a)^2 d(\Delta; b, a, x, y). \tag{2.10}$$

Here for any $K \geq 0$ there exists a positive constant C_K independent of Δ such that if $|\alpha|, |\beta| \leq K$, then

$$|\partial_x^\alpha \partial_y^\beta d(\Delta; b, a, x, y)| \leq C_K. \tag{2.11}$$

Additional assumption is needed for us to improve Kumano-go & Taniguchi Theorem,

Assumption 2.10. The functional $F(\gamma)$ satisfies the following condition:
For any integer $K \geq 0$ there exist constants $A_K > 0$ and $X_K > 0$ such that for any Δ and for $\forall \alpha_j$ satisfying $|\alpha_0| \leq K, |\alpha_1| \leq K, \ldots, |\alpha_{J+1}| \leq K$, the following inequality holds:

$$|\partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{J+1}}^{\alpha_{J+1}} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)| \leq A_K X_K^{J+1}. \tag{2.12}$$

Here A_K, X_K may depend on K but are independent of Δ and of J.

Remark 1. $F(\gamma) \equiv 1$ satisfies **Assumption 2.10** above.

The next theorem states a desired result. We can let $|\Delta| \to 0$.

Theorem 2.11 (cf. [7] [16]). [2] *Suppose that $F(\gamma)$ satisfies Assumption 2.10. Further assume $|b-a| \leq \mu_1$. Then*

$$\begin{aligned} &I[F_\Delta](\Delta; \nu, b, a, x, y) \\ &\quad = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} D(\Delta; b, a, x, y)^{-1/2} \left(F(\gamma^*) + \nu^{-1}(b-a) r(\Delta; \nu, b, a, x, y)\right). \end{aligned}$$

The following estimate for $r(\Delta; \nu, b, a, x, y)$ holds: For any integer $K \geq 0$ there exist nonnegative integer $M(K)$ and constant $C_K > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta r(\Delta; \nu, b, a, x, y)| \leq C_K A_{M(K)} \tag{2.13}$$

if $|\alpha|, |\beta| \leq K$. Both $M(K)$ and C_K may depend on K but are independent of Δ and of J.

Theorem 2.12. *In the case $F(\gamma) \equiv 1$,*

$$\begin{aligned} &I[1](\Delta; \nu, b, a, x, y) \\ &\quad = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} D(\Delta; b, a, x, y)^{-1/2} \left(1 + \nu^{-1}(b-a)^2 r(\Delta; \nu, b, a, x, y)\right). \end{aligned}$$

Here $r(\Delta; \nu, b, a, x, y)$ satisfies the same estimate as (2.13).

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[2] A sharper result is given in [11].

Corollary 2.13. *Under the same assumption as in Theorem* 2.11,

$$I[F_\Delta](\Delta;\nu,b,a,x,y)=\Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2}e^{i\nu S(\gamma^*)}D(\Delta;b,a,x,y)^{-1/2}g(\Delta;\nu,b,a,x,y).$$

Here $g(\Delta;\nu,b,a,x,y)$ is a function with the the following property:
For any integer $m\geq 0$ there exist $M(m)$ and C_m independent of Δ, J such that if α, $\beta\leq m$ then

$$|\partial_x^\alpha\partial_y^\beta g(\Delta;\nu,b,a,x,y)|\leq C_m A_{M(m)}. \tag{2.14}$$

The right hand side of (2.14) remains bounded if $|\Delta|\to 0$.

In view of discussions above we introduce new norms. Let Δ be a division of interval $[a,b]$ and $\{x_j\}_{j=0}^{J+1}$ be as above. For nonnegative number m, constant $X>0$ and nonnegative integer K we define a norm of functional $F(\gamma)$ by the following equality

$$\|F\|_{\{m,K,X,\Delta\}}=\sup_{\substack{\alpha_0\leq K,\dots,\alpha_{J+1}\leq K\\(x_{J+1},\dots,x_0)\in\mathbf{R}^{J+2}}}(1+|x_{J+1}|+\dots+|x_0|)^{-m}\Big|\prod_{j=0}^{J+1}X^{-|\alpha_j|}\partial_{x_j}^{\alpha_j}F(\gamma_\Delta)\Big|, \tag{2.15}$$

Moreover, we define

$$\|F\|_{\{m,K,X\}}=\sup_{\Delta}\|F\|_{\{m,K,X,\Delta\}}, \tag{2.16}$$

where sup is taken over all divisions Δ of interval $[a,b]$.

Remark 2. Assumption 2.10 is equivalent to the assumption that for any nonnegative integer $K=0,1,2,3,\dots$ there exists constants $X_K>0$ such that

$$A_K=\|F\|_{\{0,K,X_K\}}<\infty. \tag{2.17}$$

Kumano-go [16] generalized Theorem 2.11 above in the following way.

Theorem 2.14. *Assume that $F(\gamma)$ satisfies the following condition: There exist a constant $m\geq 0$ and a positive sequence $\{X_K;\,K=0,1,2,3,\dots\}$ such that*

$$\|F\|_{\{m,K,X_K\}}<\infty. \tag{2.18}$$

Then Theorem 2.11 *is true except for the remainder estimate: For any $K=0,1,2,\dots$ there exists $M(K)$ such that*

$$(1+|x|+|y|)^{-m}|\partial_x^\alpha\partial_y^\beta r(\Delta;\nu,b,a,x,y)|\leq C_K\|F\|_{\{m,M(K),X_{M(K)}\}} \tag{2.19}$$

if $|\alpha|,|\beta|\leq K$. Both $M(K)$ and C_K may depend on K but are independent of Δ and of J.

Corollary 2.15. *Assume that $F(\gamma)$ satisfies the following condition: There exist a constant $m \geq 0$ and a positive sequence $\{X_K; K = 0,1,2,3,\dots\}$ such that*

$$\|F\|_{\{m,K,X_K\}} < \infty. \tag{2.20}$$

Then conclusion of Corollary 2.13 is true except for the estimate: For any $K = 0,1,2,\dots$ there exists $M(K)$ such that

$$(1+|x|+|y|)^{-m}|\partial_x^\alpha \partial_y^\beta g(\Delta; \nu, b, a, x, y)| \leq C_K \|F\|_{\{m,M(K),X_{M(K)}\}} \tag{2.21}$$

if $|\alpha|, |\beta| \leq K$. Both $M(K)$ and C_K may depend on K but are independent of Δ and of J.

§ 2.5. Proof of Main Theorem

We give an outline of the proof of Theorem 2.11. We begin with the simplest case. Let Δ be the simplest division of $[a,b]$ such that

$$\Delta : a < T < b. \tag{2.22}$$

We consider piecewise classical path γ_Δ. We write $\tau_1 = T - a$, $\tau_2 = b - T$ and $y = \gamma_\Delta(a)$, $z = \gamma_\Delta(T)$, $x = \gamma_\Delta(b)$. We can write

$$S(x,z,y) = S_\Delta(x,z,y) = S_1(z,y) + S_2(x,z),$$

where

$$S_1(z,y) = \int_a^T L(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t))dt = \frac{|z-y|^2}{2\tau_1} + \phi_1(T,a,z,y) \tag{2.23}$$

$$S_2(x,z) = \int_T^b L(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t))dt = \frac{|x-z|^2}{2\tau_2} + \phi_2(b,T,x,z). \tag{2.24}$$

Further we consider

$$I = \left(\frac{\nu}{2\pi i \tau_1}\right)^{1/2} \left(\frac{\nu}{2\pi i \tau_2}\right)^{1/2} \int_{\mathbf{R}} F(x,z,y) e^{i\nu S(x,z,y)} dz,$$

here we assume $F(x,z,y)$ and its all derivatives are uniformly bounded on $\mathbf{R}^3$.

Let z^* be the critical point of the phase and define $D_{z^*}(S : x,y)$ by

$$D_{z^*}(S : x,y) = \frac{\tau_1\tau_2}{\tau_1+\tau_2} \operatorname{Hess}_{z^*} S(x,z,y)\Big|_{z=z^*}.$$

We can write

$$D_{z^*}(S : x,y) = 1 + \tau_1\tau_2 d(S : x,y). \tag{2.25}$$

For any $K \geq 0$ we have the estimate

$$|\partial_x^\alpha \partial_y^\beta d(S : x,y)| \leq C_K, \tag{2.26}$$

as far as α, $\beta \leq K$. Regarding $\nu(\tau_1^{-1} + \tau_2^{-1})$ as a large parameter we apply stationary phase method and we have

Lemma 2.16 (cf. [7]).

$$I=\left(\frac{\nu}{2\pi i(\tau_1+\tau_2)}\right)^{1/2} e^{i\nu S(x,z^*,y)} D_{z^*}(S:x,y)^{-1/2}$$

$$\times\left[F(x,z^*,y)+\frac{i\tau_1\tau_2\partial_z^2 F(x,z^*,y)}{2\nu(\tau_1+\tau_2)D_{z^*}(S:x,y)}+\nu^{-1}\tau_1\tau_2 b(\nu,\tau_1,\tau_2,x,y)\right]. \tag{2.27}$$

Moreover, for any $m\geq 0$, there exist a constant $C_m>0$ and an integer $M(m)\geq 0$ such that as far as $|\alpha_2|\leq m$, $|\alpha_0|\leq m$

$$\left|\partial_x^{\alpha_2}\partial_y^{\alpha_0} b(\nu,\tau_1,\tau_2,x,y)\right| \tag{2.28}$$

$$\leq C_m \max \sup_{z\in\mathbf{R}} \left|\partial_x^{\beta_2}\partial_z^{\beta_1}\partial_y^{\beta_0} F(x,z,y)\right|. \tag{2.29}$$

Here max *is taken for all β_1 with $|\beta_1|\leq M(m)$ and $\beta_2\leq\alpha_2$, $\beta_0\leq\alpha_0$.*

$F(x,z^*,y)$ is the amplitude of the main term and other is the remainder.

Corollary 2.17. *If $F(\gamma)\equiv 1$,*

$$I=\left(\frac{\nu}{2\pi i(\tau_1+\tau_2)}\right)^{1/2} e^{i\nu S(x,z^*,y)} D_{z^*}(S:x,y)^{-1/2}[1+\nu^{-1}\tau_1\tau_2 b(\nu,\tau_1,\tau_2,x,y)], \tag{2.30}$$

and for any α, β there exists a constant $C_{\alpha\beta}>0$ such that

$$|\partial_x^{\alpha}\partial_y^{\beta} b(\nu,\tau_1,\tau_2,x,y)|\leq C_{\alpha\beta}.$$

Outline of Proof of Theorem 2.11

We first perform integration by x_1. Next we carefully treat integration by x_2 and so on. We successively treat integration by $x_1,x_2,x_3,\dots,x_J$. At each step we apply stationary phase method and use a small trick at each step.

The part of the right hand side of (2.5) which is related to x_1 is

$$I_1=\left(\frac{\nu}{2\pi i\tau_2}\right)^{1/2}\left(\frac{\nu}{2\pi i\tau_1}\right)^{1/2}\int_{\mathbf{R}} F_\Delta(x_{J+1},x_J,\dots,x_2,x_1,x_0)e^{i\nu(S_{2,1}(x_2,x_1)+S_{1,0}(x_1,x_0))}\,dx_1. \tag{2.31}$$

We apply Lemma2.16. Then

$$I_1=\left(\frac{\nu}{2\pi i(\tau_1+\tau_2)}\right)^{1/2}$$

$$\times e^{i\nu S_{2,0}(x_2,x_0)}\Big(P_1[F](x_{J+1},x_J,\dots,x_2,x_0)+R_1[F](x_{J+1},x_J,\dots,x_2,x_0)\Big). \tag{2.32}$$

$P_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is the main term. $R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is the remainder.

Let Δ_2 be a new division of $[a,b]$ such that

$$\Delta_2: a=T_0<T_2<T_3<\cdots<T_J<T_{J+1}=b. \tag{2.33}$$

Then the main term is expressed as

$$P_1[F](x_{J+1},x_J,\dots,x_2,x_0) = F_{\Delta_2}(x_{J+1},x_J,\dots,x_2,x_0)D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1/2}.$$

As a result of (2.27) and (2.28), $R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ can be written

$$\begin{aligned}(2.34)\qquad & R_1[F](x_{J+1},x_J,\dots,x_2,x_0)\\ &= D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1/2}\\ &\times\Big(\frac{\tau_1\tau_2}{2\nu(\tau_1+\tau_2)}D_{x_1^*}(S_{2,1}+S_{1,0};x_2,x_0)^{-1}\partial_{x_1}^2 F_\Delta(x_{J+1},x_J,\dots,x_2,x_1^*,x_0)\\ &\quad+\frac{(\tau_1\tau_2)}{\nu}b(\nu,x_{J+1},x_J,\dots,x_2,x_0)\Big).\end{aligned}$$

$R_1[F](x_{J+1},x_J,\dots,x_2,x_0)$ is a complicated function with respect to x_2 but is relatively simple with respect to variables $(x_{J+1},x_J,\dots,x_3,x_0)$. In fact, we have the following fact. For all $m\geq 0$ there exist constant $C_m>0$ and an integer $M(m)>0$ such that if $|\alpha_0|,|\alpha_2|\leq m$, then for any $\beta_{J+1},\beta_J,\dots,\beta_3$

$$\begin{aligned}(2.35)\qquad & |\partial_{x_{J+1}}^{\beta_{J+1}}\partial_{x_J}^{\beta_J}\dots\partial_{x_3}^{\beta_3}\partial_{x_2}^{\alpha_2}\partial_{x_0}^{\alpha_0}b(\nu,x_{J+1},x_J,\dots,x_2,x_0)|\\ &\leq C_m\max_{\substack{\gamma\leq M(m)\\ \alpha_0'\leq\alpha_0,\alpha_2'\leq\alpha_2}}\ \sup_{x_1\in\mathbf{R}}\big|\partial_{x_{J+1}}^{\beta_{J+1}}\partial_{x_J}^{\beta_J}\dots\partial_{x_3}^{\beta_3}\partial_{x_2}^{\alpha_2'}\partial_{x_0}^{\alpha_0'}\partial_{x_1}^{\gamma}F(\gamma_\Delta)\big|.\end{aligned}$$

Here we must note that the differential operator with respect to x_j for $j\geq 3$ is the same on both sides of the above inequality (2.35).

Remark 3. The remainder term (2.34) is small, $O(\nu^{-1}\min\{\tau_1,\tau_2\})$. In particular if $F(\gamma)\equiv 1$ the remainder term is $O(\nu^{-1}\tau_1\tau_2)$.

Next we treat integration with respect to variable x_2.

- Integrate $P_1[F]$ by x_2. We get as result $P_2P_1[F]+R_2P_1[F]$. $P_2P_1[F]$ is principal part and $R_2P_1[F]$ is the remainder.
- But do not integrate $R_1[F]$ by x_2 and leave it.

Next we treat integration by x_3, In the following expression the left of the symbol $\longrightarrow$ means operation and the right of the symbol $\longrightarrow$ is the result of operation:

- we integrate $P_2P_1[F]$ by $x_3 \longrightarrow P_3P_2P_1[F]+R_3P_2P_1[F]$.
- We integrate $R_1[F]$ by $x_3 \longrightarrow P_3R_1[F]+R_3R_1[F]$.
- We do not integrate $R_2P_1[F]$ by x_3.

When we treat integration by x_4,

- Integrate $P_3P_2P_1[F] \longrightarrow P_4P_3P_2P_1[F]+R_4P_3P_2P_1[F]$.
- Integrate $P_3R_1[F] \longrightarrow P_4P_3R_1[F]+R_4P_3R_1[F]$.
- Integrate $R_2P_1[F] \longrightarrow P_4R_2P_1[F]+R_4R_2P_1[F]$.

- Do not integrate $R_3P_2P_1[F]$ by x_4.
- Do not integrate $R_3R_1[F]$ by x_4.

etc.

Repeating this operation, $I[F_\Delta](\Delta;\nu,b,a,x,y)$ is expressed as a sum of many terms.

$$I[F](\Delta;\nu,b,a,x,y) = A_0(\Delta;\nu,b,a,x,y) + \sum{}' A_{js_\ell,js_{\ell-1},\dots,js_1}. \tag{2.36}$$

Here $A_0(\Delta;\nu,b,a,x,y)$ is the main term through all steps, i.e.

$$A_0(\Delta;\nu,b,a,x,y) = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} P_J P_{J-1}\dots P_1[F].$$

The sum $\sum'$ expresses taking sum over some sequences $\{js_\ell, js_{\ell-1},\cdots,js_1\}$ which is a subsequence of the sequence $\{J,J-1,J-2,\dots,1\}$ and $A_{js_\ell,js_{\ell-1},\dots,js_1}$ is the term which came from skipping integration with respect to variables $x_{js_\ell},x_{js_{\ell-1}},\cdots,x_{js_1}$.

We can show the term $A_0(\Delta;\nu,b,a,x,y)$ coincides with the main term of stationary phase method of $I[F_\Delta](\Delta;\nu,b,a,x,y)$ with respect to whole variables $(x_J,x_{J-1},\dots,x_1)$. That is

$$A_0(\Delta;\nu,b,a,x,y) = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} D(\Delta;b,a,x,y)^{-1/2} F(\gamma^*).$$

The term $A_{js_\ell,js_{\ell-1},\dots,js_1}$ is of the following form:

$$\begin{aligned} A_{js_\ell,js_{\ell-1},\dots,js_1} &= \nu^{-\ell}\prod_{k=1}^{\ell}\left(\frac{\nu}{2\pi i(T_{js_k+1}-T_{js_k})}\right)^{1/2}\int_{\mathbf{R}^\ell} e^{\nu i S_{js_\ell,js_{\ell-1},\dots,js_1}(x_{js_\ell},x_{js_{\ell-1}},\dots x_{js_1})} \\ &\quad\times a_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)\prod_{k=1}^{\ell} dx_{js_k}. \end{aligned} \tag{2.37}$$

Here

$$S_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0) = \sum_{k=1}^{\ell}\left(S_{js_{k+1},js_k}(x_{js_{k+1}},x_{js_k}) + S_{js_k,js_{k-1}}(x_{js_k},x_{js_{k-1}})\right),$$

and $a_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)$ is a function satisfying the following estimate: For any $m\geq 0$, there exist $K(m)$ and $C(m)>0$ such that as long as $|\alpha_{js_k}|\leq m,(k=1,2,\dots,\ell)$ and $|\alpha_0|\leq m, |\alpha_{J+1}|\leq m$

$$\left|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}\prod_{k=1}^{\ell}\partial_{x_{js_k}}^{\alpha_{js_k}} a_{js_\ell,js_{\ell-1},\dots,js_1}(x_{J+1},x_{js_\ell},\dots,x_{js_1},x_0)\right| \leq C(m)\left(\prod_{k=1}^{\ell}\tau_{js_k}\right)A_{K(m)}X_{K(m)}^{\ell}. \tag{2.38}$$

Now we apply Kumano-go& Taniguchi theorem to the right hand side of (2.37). We can prove that

$$A_{js_\ell,js_{\ell-1},\dots,js_1} = \nu^{-\ell}\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(\gamma^*)} b_{js_\ell,js_{\ell-1},\dots,js_1}(\Delta;\nu,b,a,x,y),$$

and we have the estimate

$$\left|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0} b_{j_{s_\ell},j_{s_{\ell-1}},\dots,j_{s_1}}(\Delta;\nu,b,a,x,y)\right| \le C_1(m)^\ell C(m) A_{K(m)} X_{K(m)}^\ell \prod_{k=1}^{\ell} \tau_{j_{s_k}}.$$

From here we have

$$\sum{}' A_{j_{s_k},j_{s_{k-1}},\dots,j_{s_1}} = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x.y)} c(\Delta;\nu,b,a,x,y),$$

where

$$c(\Delta;\nu,b,a,x,y) = \sum{}' \nu^{-\ell} b_{j_{s_\ell},j_{s_{\ell-1}},\dots,j_{s_1}}(\Delta;\nu,b,a,x,y),$$

and we have that

$$\begin{aligned}
\left|\partial_x^{\alpha_{J+1}}\partial_y^{\alpha_0} c(\Delta;\nu,b,a,x,y)\right| &\le \sum{}' \nu^{-\ell} C_1(m)^\ell C(m) A_{K(m)} X_{K(m)}^\ell \prod_{k=1}^{\ell} \tau_{j_{s_k}} \\
&\le C(m) A_{K(m)} \left[\prod_{j=1}^{J} \left(1+\nu^{-1} C_1(m) X_{K(m)} \tau_j\right) - 1\right] \\
&\le \nu^{-1} C'(m) A_{K(m)} X_{K(m)} (b-a).
\end{aligned}$$

with some constant C'(m) independent of Δ and of J.

Theorem 2.11 is now proved. Similarly we can prove Theorem 2.12 ([5]).

§ 3. Application to Feynman Path Integral

§ 3.1. Existence of $\lim_{|\Delta|\to 0} D(\Delta,b,a,x,y)$

We shall prove that the limit

$$\lim_{|\Delta|\to 0} I[1](\Delta;\nu,b,a,x,y)$$

exists (cf. [9], [6] and also [13]). Existence of $\lim_{|\Delta|\to 0} I[F](\Delta;\nu,b,a,x,y)$ for more general $F(\gamma)$ is proved in [16]. See also [10].

We begin with

Theorem 3.1. *The limit*

$$D(b,a,x,y) = \lim_{|\Delta|\to 0} D(\Delta,b,a,x,y) \tag{3.1}$$

exists and

$$D(b,a,x,y) = 1 + (b-a)^2 d(b,a,x,y). \tag{3.2}$$

For any $K \ge 0$ there exists $C_K > 0$ such that for all α and β with $|\alpha|, |\beta| \le K$,

$$|\partial_x^\alpha \partial_y^\beta d(b,a,x,y)| \le C_K.$$

Remark 4. 1. As one can see from next Theorem, $D(\Delta, b, a, x, y)$ converges uniformly together with its all derivatives with respect to (x, y).

2. $D(b, a, x, y)$ is called Van Bleck-Morette determinant (cf. [19], [9]).

To prove Theorem 3.1, we have only to prove the following Theorem (cf. [8], [13] or [9]).

Theorem 3.2. *Assume $|b-a| \leq \mu_1$. Let Δ be an arbitrary division of $[a, b]$. Let Δ' be an arbitrary refinement of Δ. Define $d(\Delta, \Delta'; x, y)$ by the following equality.*

$$\frac{D(\Delta'; b, a, x, y)}{D(\Delta; b, a, x, y)} = 1 + |\Delta|(b-a)\, d(\Delta, \Delta'; x, y).$$

Then for any α and β, there exists $C_{\alpha,\beta}$ independent of Δ, Δ' and of (a, b, x, y) such that

$$|\partial_x^\alpha \partial_y^\beta d(\Delta, \Delta'; x, y) \leq C_{\alpha,\beta}. \tag{3.3}$$

Proof. We prove Theorem 3.2 through several steps. Let Δ be

$$\Delta : a = T_0 < T_1 < T_2 < \cdots < T_J < T_{J+1} = b$$

and Δ' its refinement

$$\begin{aligned}
\Delta' : a = T_0 = T_{1,0} &< T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} \\
&= T_1 = T_{2,0} < T_{2,1} < \cdots\cdots < T_{2,p_2} < T_{2,p_2+1} \\
&= T_2 = T_{3,0} < \cdots < T_J < T_{J+1,1} < T_{J+1,2} < \cdots \\
&\qquad \cdots < T_{J+1,p_{J+1}} < T_{J+1,p_{J+1}+1} = T_{J+1} = b.
\end{aligned}$$

Set $\tau_j = T_j - T_{j-1}$ $\tau_{j,k} = T_{j,k} - T_{j,k-1}$.

The piecewise classical path corresponding to division Δ' is denoted by

$$\gamma_{\Delta'}(x_{J+1}, x_{J+1,p_{J+1}}, \ldots, x_J, \ldots, x_1, x_{1,p_1}, \ldots, x_{1,1}, x_0)(t),$$

which will be abbreviated to $\gamma_{\Delta'}(t)$. Its action is

$$S_{\Delta'}(x_{J+1}, x_{J+1,p_{J+1}}, \ldots, x_J, \ldots, x_1, x_{1,p_1}, \ldots, x_{1,1}, x_0).$$

In the following, we use a special sequence of refinements $\Delta^{(0)}$, $\Delta^{(1)}$, $\Delta^{(2)}$, $\ldots$, $\Delta^{(J+1)}$ of Δ such that $\Delta^{(0)} = \Delta$, $\Delta^{(J+1)} = \Delta'$ and $\Delta^{(k)}$ is a refinement of $\Delta^{(k-1)}$.

We define $\Delta^{(1)}$ by

$$\begin{aligned}
\Delta^{(1)} : a = T_0 = T_{1,0} &< T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} \\
&= T_1 < T_2 < \cdots < T_J < T_{J+1} = b.
\end{aligned}$$

$\Delta^{(1)}$ is different from Δ only in $[T_0,T_1]$ where $\Delta^{(1)}$ and Δ' define the same division.
We denote by $\gamma_{\Delta^{(1)}}(x_{J+1},x_J,\dots,x_1,x_{1,p_1},\dots,x_{1,1},x_0)$ the piecewise classical path corresponding to division $\Delta^{(1)}$.

We define $\Delta^{(2)}$ so that $\Delta^{(2)}$ is different from $\Delta^{(1)}$ only in $[T_1,T_2]$ and it defines the same division as Δ' in $[T_1,T_2]$. $\Delta^{(2)}$ is

$$\begin{aligned}\Delta^{(2)}: a = T_0 = T_{1,0} &< T_{1,1} < \cdots < T_{1,p_1} < T_{1,p_1+1}\\ &= T_1 = T_{2,0} < T_{2,1} < \cdots < T_{2,p_2} < T_{2,p_2+1}\\ &= T_2 < T_3 < \cdots < T_J < T_{J+1} = b.\end{aligned}$$

Similarly , $\Delta^{(j)}$ is defined for $j = 3,4,\dots,J$.

We compare $D(\Delta^{(j)};b,a,x,y)$ with $D(\Delta^{(j-1)};b,a,x,y)$.

We claim that for $j = 1,2,\dots,J+1$

$$\begin{aligned}D(\Delta^{(j)};b,a,x,y) &= D(\Delta^{(j-1)};b,a,x,y)D(\delta_j;T_j,T_{j-1},x_j^*,x_{j-1}^*)\\ &= D(\Delta^{(j-1)};b,a,x,y)\left(1+\tau_j^2 d(\delta_j;T_j,T_{j-1},x,y)\right).\end{aligned} \tag{3.4}$$

Here δ_j denotes the division of $[T_{j-1},T_j]$

$$\delta_j : T_{j-1} = T_{j,0} < T_{j,1} < \cdots < T_{j,p_j} < T_{j,p_j+1} = T_j. \tag{3.5}$$

For any any α,β there exists $C_{\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_y^\beta d(\delta_j;T_j,T_{j-1},x,y)\right| \le C_{\alpha\beta}. \tag{3.6}$$

Let us admit the claim to be true for the moment. Then it follows from (3.4) that

$$D(\Delta';b,a,x,y) = D(\Delta;b,a,x,y)\prod_{j=1}^{J+1}\left(1+\tau_j^2 d(\delta_j;T_j,T_{j-1},x,y)\right).$$

We define $d(\Delta,\Delta';b,a,x,y)$ by

$$\prod_{j=1}^{J+1}\left(1+\tau_j^2 d(\delta_j;T_j,T_{j-1},x,y)\right) = 1+|\Delta|(b-a)d(\Delta,\Delta';b,a,x,y).$$

Then estimate (3.3) holds. Therefore, Theorem 3.2 is proved once the claim is proved.

We prove the claim for $j = 1$. As we defined by (3.5)

$$\delta_1 : a = T_0 = T_{1,0} < T_{1,1} < T_{1,2} < \cdots < T_{1,p_1} < T_{1,p_1+1} = T_1.$$

Let $\gamma_{\delta_1}(x_{1,p_1+1},x_{1,p_1},\dots,x_{1,1},x_{1,0})$ be the piecewise classical path such that $\gamma_{\delta_1}(T_{1,j}) = x_{1,j}$, $j = 0,1,\dots,p_1+1$. We write its action by

$$S_{\delta_1}(x_{1,p_1+1},x_{1,p_s},\dots,x_{1,1},x_{1,0}) = \sum_{k=1}^{p_1+1} S(T_{1,k},T_{1,k-1},x_{1,k},x_{1,k-1}).$$

The action of $S(\gamma_{\Delta^{(1)}})$ is written as

$$S(\gamma_{\Delta^{(1)}}) = S_{\Delta^{(1)}}(x_{J+1},x_J,\dots,x_1,x_{1,p_1},\dots,x_{1,1},x_0)$$
$$= \sum_{j=2}^{J+1} S(T_j,T_{j-1},x_j,x_{j-1}) + \sum_{k=1}^{p_1+1} S(T_{1,k},T_{1,k-1},x_{1,k},x_{1,k-1})$$
$$= \sum_{j=2}^{J+1} S(T_j,T_{j-1},x_j,x_{j-1}) + S_{\delta_1}(x_{1,p_1+1},x_{1,p_1},\dots,x_{1,1},x_{1,0}). \tag{3.7}$$

In calculating $\det(\mathrm{Hess}\, S_{\Delta^{(1)}})$, we first fix $(x_{J+1},x_J,\dots,x_1,x_0)$ and consider the critical point $(x^*_{1,p_1},\dots,x^*_{1,1})$ with respect to $(x_{1,p_1},\dots,x_{1,1})$.

$$\det(\mathrm{Hess}_{(x^*_{1,p_1},\dots,x^*_{1,1})} S_{\Delta^{(1)}}(x_{J+1},x_J,\dots,x_1,x^*_{1,p_1},\dots,x^*_{1,1},x_0))$$
$$= \det(\mathrm{Hess}_{(x^*_{1,p_1},\dots,x^*_{1,1})} S_{\delta_1}(x_{1,p_1+1},x^*_{1,p_1},\dots,x^*_{1,1},x_{1,0}))$$
$$= \frac{\tau_1}{\prod_{k=1}^{p_1+1}\tau_{1,j}} D(\delta_1;T_1,T_0,x_1,x_0). \tag{3.8}$$

Since

$$S_{\delta_1}(x_{1,p_1+1},x^*_{1,p_1},\dots,x^*_{1,1},x_{1,0}) = S(T_1,T_0,x_1,x_0),$$

we know that for fixed $(x_{J+1},\dots,x_1,x_0)$

$$S_{\Delta^{(1)}}(x_{J+1},x_J,\dots,x_1,x^*_{1,p_1},\dots,x^*_{1,1},x_0) = \sum_{j=2}^{J+1} S(T_j,T_{j-1},x_j,x_{j-1}) + S(T_1,T_0,x_1,x_0)$$
$$= S_\Delta(x_{J+1},x_J,\dots,x_1,x_0). \tag{3.9}$$

Calculation shows

$$\det(\mathrm{Hess}_{(x^*_J,\dots,x^*_1,x^*_{1,p_1},\dots,x^*_{1,1})} S_{\Delta^{(1)}}) = \det(\mathrm{Hess}_{x^*_J,\dots,x^*_1} S_\Delta) \times \det(\mathrm{Hess}_{(x^*_{1,p_1},\dots,x^*_{1,1})} S_{\delta_1}\big|_{x_1=x^*_1}).$$

It follows from this and Theorem 2.9 applied to δ_1 that

$$D(\Delta^{(1)};b,a,x,y) = D(\Delta;b,a,x,y)D(\delta_1;T_1,T_0,x^*_1,y)$$
$$= D(\Delta;b,a,x,y)\left(1+\tau_1^2 d(\delta_1;T_1,T_0,x,y)\right). \tag{3.10}$$

For any α,β there exists $C_{\alpha\beta}>0$ such that

$$\left|\partial_x^\alpha\partial_y^\beta d(\delta_1;T_1,T_0,x,y)\right| \le C_{\alpha\beta}. \tag{3.11}$$

Similarly, the claim can be proved for $j>1$. Hence Theorem 3.2 is proved. □

§ 3.2. Convergence of Feynman Path Integral

Next we prove $\lim_{|\Delta|\to 0} I[1](\Delta;\nu,b,a,x,y)$ exists. Existence of $\lim_{|\Delta|\to 0} I[F](\Delta;\nu,b,a,x,y)$ for more general $F(\gamma)$ is proved in [16]. See also [10].

Theorem 3.3. [3] *The limit*

$$K(\nu,b,a,x,y) = \lim_{|\Delta|\to 0} I[1](\Delta,\nu,b,a,x,y) \tag{3.12}$$

exists. Moreover $K(\nu,b,a,x,y)$ is of the form:

$$K(\nu,b,a,x,y) = \Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2} e^{i\nu S(b,a,x,y)} D(b,a,x,y)^{-1/2}(1+\nu^{-1}r(\nu,b,a,x,y)). \tag{3.13}$$

For any α, β there exist a positive constant $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_y^\beta r(\nu,b,a,x,y)| \le C_{\alpha\beta}. \tag{3.14}$$

Remark 5. 1. Moreover $I[1](\Delta,\nu,b,a,x,y)$ converges uniformly together with its all derivatives with respect to (x,y).

2. The function $K(\nu,b,a,x,y)$ is the fundamental solution (Feynman propagator) of Schrödinger equation (cf. [5], [6], [9] and [13]).

3. (3.13) and (3.14) prove semi-classical asymptotic formula for Fundamental solution (Feynman propagator) of Schrödinger equation. This is another proof of famous formula of Birkhoff [2] (cf. [9]. See also [19]).

We have only to prove that $I[1](\Delta;\nu,b,a,x,y)$ is a Cauchy net with respect to $|\Delta|$.

Theorem 3.4. [4] *Assume that $|b-a| \le \mu_1$. Let Δ be an arbitrary division of the interval $[a,b]$ and Δ' be its arbitrary refinement. Then*

$$\begin{aligned} &I[1](\Delta';\nu,b,a,x,y) - I[1](\Delta;\nu,b,a,x,y) \\ &\qquad = \Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2} e^{i\nu S(b,a,x,y)} D(\Delta;b,a,x,y)^{-1/2} q(\Delta,\Delta',\nu,b,a,x,y). \end{aligned} \tag{3.15}$$

Moreover, for all α, β, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_y^\beta q(\Delta,\Delta';\nu,b,a,x,y)| \le C_{\alpha,\beta}|\Delta|(b-a). \tag{3.16}$$

Proof. We prove the theorem along the same line as the proof of Theorem 3.2. We again use the sequence of refinements $\Delta^{(0)}$, $\Delta^{(1)}$,$\Delta^{(2)}$, ..., $\Delta^{(J+1)}$ of Δ which appeared in the proof of Theorem 3.2. We claim that for $k = 1,2,\dots,J+1$

$$\begin{aligned} &I[1](\Delta^{(k)};\nu,b,a,x,y) - I[1](\Delta^{(k-1)};\nu,b,a,x,y) \\ &\qquad = \Big(\frac{\nu}{2\pi i(b-a)}\Big)^{1/2} e^{i\nu S(b,a,x,y)} D(\Delta;b,a,x,y)^{-1/2} q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y). \end{aligned} \tag{3.17}$$

[3] For more information see [9] and [12].
[4] cf.[8] and [13]

For any α,β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y)\right| \le C_{\alpha,\beta}\tau_k^2.$$

Let us admit the claim to be true for a moment. Then we have

$$\begin{aligned}
(3.18)\qquad & I[1](\Delta';\nu,b,a,x,y) - I[1](\Delta;\nu,b,a,x,y) \\
&= \sum_{k=1}^{J+1}\left(I[1](\Delta^{(k)};\nu,b,a,x,y) - I[1](\Delta^{(k-1)};\nu,b,a,x,y)\right) \\
&= \sum_{k=1}^{J+1}\left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y) \\
&= \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)}D(\Delta;b,a,x,y)^{-1/2}q(\Delta,\Delta';\nu,b,a,x,y).
\end{aligned}$$

Here

$$q(\Delta,\Delta';\nu,b,a,x,y) = \sum_{k=1}^{J+1} q(\Delta^{(k)},\Delta^{(k-1)};\nu,b,a,x,y).$$

For any α,β there exists a positive constant $C_{\alpha,\beta}$ such that

$$(3.19)\qquad \left|\partial_{x_1}^{\alpha}\partial_{x_0}^{\beta}q(\Delta,\Delta';\nu,b,a,x,y)\right| \le C_{\alpha,\beta}\sum_{k=1}^{J+1}\tau_k^2.$$

Therefore Theorem 3.4 is proved once we admit the claim.

Proof of the Claim for $k=1$. First we compare $I[1](\Delta^{(1)};\nu,b,a,x,y)$ with $I[1](\Delta;\nu,b,a,x,y)$. Using (3.7), we have

$$\begin{aligned}
(3.20)\qquad & I[1](\Delta^{(1)};\nu,b,a,x,y) \\
&= \prod_{j=2}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\mathbf{R}^J}\exp(i\nu\sum_{j=2}^{J+1}S(T_j,T_{j-1},x_j,x_{j-1}))\prod_{j=1}^{J}dx_j \\
&\times \prod_{k=1}^{p_1+1}\left(\frac{\nu}{2\pi i\tau_{1,k}}\right)^{1/2}\left[\int_{\mathbf{R}^{p_1}}\exp(i\nu S_{\delta_1}(x_{1,p_1+1},x_{1,p_1},\dots,x_{1,1},x_{1,0}))\prod_{k=1}^{p_1}dx_{1,k}\right]
\end{aligned}$$

Let $x_{1,k}^* = \gamma_\Delta(T_{1,k})$ for $1\le k\le p_1$. Then it is the critical point with respect to $(x_{1,p_1},\dots,x_{1,1})$ of $S_{\delta_1}(x_{1,p_1+1},x_{1,p_1},\dots,x_{1,1},x_{1,0})$, and just as in (3.9) the critical value is

$$S_{\delta_1}(x_{1,p_1+1},x_{1,p_1}^*,\dots,x_{1,1}^*,x_{1,0}) = S(T_1,T_0,x_1,x_0).$$

We fix $(x_J,\dots,x_1)$ and integrate with respect to $(x_{1,p_1},\dots,x_{1,1})$ in (3.20). Then it follows from Theorem 2.11 and the fact above that

$$\begin{aligned}
& I[1](\Delta^{(1)};\nu,b,a,x,y) \\
&= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\mathbf{R}^J}F_{\Delta^{(1)}/\Delta}(x_{J+1},\dots,x_0)\exp(i\nu S_\Delta(x_{J+1},x_J,\dots,x_1,x_0))\prod_{j=1}^{J}dx_j \\
&= I[F_{\Delta^{(1)}/\Delta}](\Delta;\nu,b,a,x,y),
\end{aligned}$$

with

$$F_{\Delta^{(1)}/\Delta}(\nu, x_{J+1}, \ldots, x_0) = D(\delta_1; T_1, T_0, x_1, y)^{-1/2}\big(1 + \frac{\tau_1^2}{\nu} r_{\Delta^{(1)}/\Delta}(\nu, T_1, T_0, x_1, y))\big).$$

Here $D(\delta_1; T_1, T_0, x_1, y)$ is given by (2.9) and used in (3.10). So we know that it is of the following form:

$$D(\delta_1; T_1, T_0, x_1, y) = 1 + \tau_1^2 d(\delta_1; T_1, T_0, x_1, x_0). \tag{3.21}$$

This means that we have

$$F_{\Delta^{(1)}/\Delta}(\nu, x_{J+1}, x_J, \ldots, x_1, x_0) = 1 + \tau_1^2 f_{\Delta^{(1)}/\Delta}(\nu, T_1, T_0, x_1, x_0),$$

and we have the estimate for $f_{\Delta^{(1)}/\Delta}(\nu, T_1, T_0, x_1, x_0)$: For any α, β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|\partial_{x_1}^{\alpha} \partial_{x_0}^{\beta} f_{\Delta^{(1)}/\Delta}(\nu, T_1, T_0, x_1, x_0)\right| \le C_{\alpha,\beta}.$$

Now we can write

$$\begin{aligned} I[1](\Delta^{(1)}; \nu, b, a, x, y) - I[1](\Delta; \nu, b, a, x, y) &= I[F_{\Delta^{(1)}/\Delta} - 1](\Delta; \nu, b, a, x, y) \\ &= \tau_1^2 I[f_{\Delta^{(1)}/\Delta}](\Delta; \nu, b, a, x, y). \end{aligned} \tag{3.22}$$

We can apply corollary 2.13 to the right hand side of above equation and obtain

$$\tau_1^2 I[f_{\Delta^{(1)}/\Delta}](\Delta, \nu, b, a, x, y) = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} D(\Delta; b, a, x, y)^{-1/2} q(\Delta^{(1)}, \Delta; \nu, b, a, x, y).$$

Here $q(\Delta^{(1)}, \Delta; \nu, b, a, x, y)$ has the following property: For any α, β there exists $C_{\alpha,\beta}$ such that

$$\left|\partial_{x_1}^{\alpha} \partial_{x_0}^{\beta} q(\Delta^{(1)}, \Delta; \nu, b, a, x, y)\right| \le C_{\alpha,\beta} \tau_1^2. \tag{3.23}$$

This means that

$$\begin{aligned} &I[1](\Delta^{(1)}; \nu, b, a, x, y) - I[1](\Delta; \nu, b, a, x, y) \\ &\quad = \left(\frac{\nu}{2\pi i(b-a)}\right)^{1/2} e^{i\nu S(b,a,x,y)} D(\Delta; b, a, x, y)^{-1/2} q(\Delta^{(1)}, \Delta; \nu, b, a, x, y). \end{aligned} \tag{3.24}$$

The claim for $k = 1$ is proved. Similarly, we can prove the claim for $k > 1$. Theorem 3.4 is proved. □

§ 4. Integration by Parts Formula for Feynman Path Integrals

§ 4.1. Some Operators of Trace Class

Let $\mathcal{H}_0 = H_0^1(a, b)$ be the Sobolev space of order 1 with vanishing boundary condition. Let $\rho\colon \mathcal{H}_0 \to L^2(a, b)$ be the canonical embedding and $\rho^*\colon L^2(a, b) \to \mathcal{H}_0$ its adjoint.

Remark 6. It is well known that $\rho\rho^* = G_0$, where G_0 is the Green operator of Dirichlet boundary value problem of ordinary differential equation:

$$-\frac{d^2}{dt^2}u(t) = f(t), \qquad \text{and} \qquad u(a) = 0 = u(b). \tag{4.1}$$

Proposition 4.1. *Let $B\colon L^2(a,b) \to L^2(a,b)$ be a bounded linear operator. Then both of linear operators $\rho^*B\rho\colon \mathcal{H}_0 \to \mathcal{H}_0$ and $\rho\rho^*B\colon L^2 \to L^2$ are of trace class. Their traces are equal, i.e.*

$$\mathrm{tr}\rho^*B\rho = \mathrm{tr}\rho\rho^*B. \tag{4.2}$$

Let B be a bounded linear operator in $L^2(a,b)$. It is clear from the previous Proposition that the linear mapping $\rho\rho^*B\colon L^2(a,b) \to L^2(a,b)$ has an integral kernel, i.e. there exists $k(s,t) \in L^2([a,b]\times[a,b])$ such that for all $f \in L^2(a,b)$,

$$\rho\rho^*Bf(s) = \int_a^b k(s,t)f(t)dt. \tag{4.3}$$

Proposition 4.2. *$k(s,t)$ has the following properties:*

1. *Restriction $k(s,s)$ of $k(s,t)$ to the diagonal subset of $[a,b]\times[a,b]$ is well-defined for almost all s, and $\int_a^b |k(s,s)|^2 ds < \infty$.*
2. *Moreover,*

$$\mathrm{tr}\rho\rho^*B = \int_a^b k(s,s)ds. \tag{4.4}$$

§ 4.2. Divergence Operator

We fix $(x,y) \in \mathbf{R}^2$, and we set $\mathcal{H}_{xy} = \{\gamma \in H^1(a,b);\ \gamma(a) = y, \gamma(b) = x\}$. $\mathcal{H}_{xy}$ is an infinite dimensional C^∞ manifold. The tangent space of $\mathcal{H}_{xy}$ at point $\gamma \in \mathcal{H}_{xy}$ is identified with $\mathcal{H}_0$.

Let p be a continuous mapping $p\colon \mathcal{H}_{xy} \ni \gamma \mapsto p(\gamma) \in \mathcal{H}_0$. Since $p(\gamma) \in \mathcal{H}_0$, $p(\gamma)$ is expressed as a function $p(\gamma,s)$ of variable s, i.e. $p(\gamma,s) = \rho p(\gamma)(s)$. It satisfies $p(\gamma,a) = 0 = p(\gamma,b)$, it is an absolute continuous function of s, its derivative $\partial_s p(\gamma,s)$ exists almost everywhere and $\partial_s p(\gamma,s)$ satisfies

$$\int_a^b |\partial_s p(\gamma,s)|^2 dt < \infty.$$

We regard $p(\gamma)$ as a vector field on $\mathcal{H}_{xy}$, because $\mathcal{H}_0$ is the tangent space to $\mathcal{H}_{xy}$ at γ.

Definition 4.3 (Admissible Vector Field). We say that $p(\gamma)$ is an admissible vector field if $p(\gamma)$ has the following properties:

1. There exits a C^1 mapping $q\colon \mathcal{H}_{x,y} \to L^2(0,T)$ such that

$$p(\gamma) = \rho^*q(\gamma), \quad (\gamma \in \mathcal{H}_{x,y}). \tag{4.5}$$

2. The Fréchet differential $Dq(\gamma)\colon \mathcal{H}_0 \ni h \mapsto Dq(\gamma)[h] \in L^2(0,T)$ can be boudedly extended to a bounded linear operator $B(\gamma)$ in $L^2(0,T)$; that is, for any $h \in \mathcal{H}_0$,

$$Dq(\gamma)[h] = B(\gamma)\rho h. \tag{4.6}$$

It is clear from Definition 4.3 that for any $h \in \mathcal{H}_0$,

$$Dp(\gamma)[h] = \rho^* Dq(\gamma)[h] = \rho^* B(\gamma)\rho h. \tag{4.7}$$

Definition 4.4 (Divergence of an Admissible Vector Field). Suppose that $p(\gamma)$ is an admissible vector field. Then we define its divergence $\operatorname{Div} p(\gamma)$ at γ in the following way:

$$\operatorname{Div} p(\gamma) = \operatorname{tr} Dp(\gamma) = \operatorname{tr} \rho^* B(\gamma)\rho. \tag{4.8}$$

Proposition 4.2 enables us to use another expression of $\operatorname{Div} p(\gamma)$.

Proposition 4.5. *Suppose that $p(\gamma)$ is an admissible vector field. Then there exists the kernel function $k_\gamma(s,t)$ of the map $\rho\rho^* B(\gamma)$, and*

$$\operatorname{Div} p(\gamma) = \operatorname{tr} \rho\rho^* B(\gamma) = \int_a^b k_\gamma(s,s)\,ds. \tag{4.9}$$

Remark 7. One can see our definition (4.8) of Div is akin to but slightly different from that of "divergence" in [18], [17].

§ 4.3. Integration by Parts Formula

Suppose a functional $F\colon \mathcal{H}_{xy} \to \mathbf{C}$ has Fréchet differential $DF(\gamma)$ at $\gamma \in \mathcal{H}_{xy}$ and that there is a density function $f_\gamma \in L^2(a,b)$ such that

$$DF(\gamma)[h] = \int_a^b f_\gamma(s)\rho h(s)\,ds, \qquad (\forall h \in \mathcal{H}_0). \tag{4.10}$$

Then we denote $f_\gamma(s)$ by $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$ or by $\dfrac{\delta}{\delta\gamma(s)}F(\gamma)$, i.e.

$$DF(\gamma)[h] = \int_a^b \frac{\delta F(\gamma)}{\delta\gamma(s)}\rho h(s)\,ds, \qquad (\forall h \in \mathcal{H}_0). \tag{4.11}$$

Definition 4.6. Let m be a non-negative constant. We say a functional $F(\gamma)$ a m-smooth functional if $F(\gamma)$ satisfies all of the following conditions:

F1 Functional $F(\gamma)$ is a infinitely differentiable map from $H^1(a,b)$ to $\mathbf{C}$.

F2 At every $\gamma \in \mathcal{H}_{xy}$ functional $F(\gamma)$ has Fréchet derivative $DF(\gamma)$ with density functional $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$, i.e.

$$DF(\gamma)[h] = \int_a^b \frac{\delta F(\gamma)}{\delta\gamma(s)}\rho h(s)\,ds, \qquad (\forall \gamma \in \mathcal{H}_{xy}, \forall h \in \mathcal{H}_0). \tag{4.12}$$

F3 $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$ is a continuos function of s if $\gamma \in \mathcal{H}_{xy}$ is fixed. It is infinitely differentiable with respect to $\gamma \in \mathcal{H}_{xy}$ if s is fixed.

F4 For any nonnegative integer K there exists a positive constants X_K such that

$$\sup_{s\in[a,b]} \left\| \frac{\delta F(\gamma)}{\delta\gamma(s)} \right\|_{\{m,K,X_K\}} < \infty. \tag{4.13}$$

Remark 8. An m-smooth functional is Feynman path integrable (cf. [16] and [10]).

In accordance with this notation we use also the following notation.
Let $p(\gamma)$ be a vector field as in Definition 4.3 above. Then we use the symbol $\dfrac{\delta q(\gamma)}{\delta\gamma}$ for $B(\gamma)$; that is

$$Dq(\gamma)[h] = \frac{\delta q(\gamma)}{\delta\gamma}\rho h, \qquad (\forall h \in \mathcal{H}_0).$$

Thus

$$Dp(\gamma)[h] = \rho^* \frac{\delta q(\gamma)}{\delta\gamma}\rho h, \qquad (\forall h \in \mathcal{H}_0).$$

And we denote the kernel function $k_\gamma(s,t)$ of the map $\rho\rho^* B(\gamma) = \rho\rho^* \dfrac{\delta q(\gamma)}{\delta\gamma}$ by $\dfrac{\delta p(\gamma,s)}{\delta\gamma(t)}$ or $\dfrac{\delta}{\delta\gamma(t)} p(\gamma,s)$, i.e. for any $h \in \mathcal{H}_0$

$$\rho Dp(\gamma)[h](s) = \int_a^b \frac{\delta p(\gamma,s)}{\delta\gamma(t)}\rho h(t)\,dt. \tag{4.14}$$

Remark 9. Using this notation, we can write

$$\operatorname{Div} p(\gamma) = \int_a^b \frac{\delta p(\gamma,s)}{\delta\gamma(s)}\,ds, \tag{4.15}$$

for a vector field $p(\gamma)$ as above.

Definition 4.7. Let m' be a nonnegative number. We say that the vector field $p(\gamma)$ is an m'-admissible vector field if it has all the following properties:

P1 $p(\gamma)$ is admissible; that is, there is a C^1 mapping $q\colon \mathcal{H}_{xy} \to L^2(0,T)$ such that $p(\gamma) = \rho^* q(\gamma)$ for $\gamma \in \mathcal{H}_{x,y}$, and for all $h \in \mathcal{H}_0$, $Dq(\gamma)[h] = \dfrac{\delta q(\gamma)}{\delta\gamma}\rho h$, where $\dfrac{\delta q(\gamma)}{\delta\gamma}$ is a bounded linear operator in $L^2(a,b)$.

P2 The kernel function $\dfrac{\delta p(\gamma,s)}{\delta\gamma(t)}$ of $\rho\rho^* \dfrac{\delta q(\gamma)}{\delta\gamma}$ is continuous in $[0,T] \times [0,T]$. For each $K = 0,1,2,\ldots$, there exist positive numbers Y_K and B_K such that

$$\begin{aligned} B_K \geq{} & \sup_{s\in[0,T]} \|p(\gamma,s)\|_{\{m',K,Y_K\}} + \sup_{s\in[0,T]} \|\partial_s p(\gamma,s)\|_{\{m',K,Y_K\}} \\ & + \sup_{t\in[0,T]} \left\| \partial_s \frac{\delta p(\gamma,s)}{\delta\gamma(t)} \right\|_{\{m',K,Y_K\}}. \end{aligned} \tag{4.16}$$

Theorem 4.8 (Integration by Parts Formula). *Let $b-a\le\mu_0$. Assume $F(\gamma)$ is m-smooth functional and $p(\gamma)$ is an m'-admissible vector field on $\mathcal{H}_{xy}$ with some m and m'. Further assume that* $\operatorname{Div} p(\gamma)$ *is m'-smooth and $DF(\gamma)[p(\gamma)]$, $DS(\gamma)[p(\gamma)]$ are F-integrable. Then the following integration by parts formula holds*:

$$
\begin{aligned}
(4.17)\qquad &\int_\Omega DF(\gamma)[p(\gamma)]e^{ivS(\gamma)}\mathcal{D}(\gamma)\\
&\qquad= -\int_\Omega F(\gamma)\operatorname{Div} p(\gamma)e^{ivS(\gamma)}\mathcal{D}(\gamma) - iv\int_\Omega F(\gamma)DS(\gamma)[p(\gamma)]e^{ivS(\gamma)}\mathcal{D}(\gamma).
\end{aligned}
$$

Remark 10 (cf. [16]). If $p(\gamma,s)$ is independent of γ, i.e. $p(\gamma)=h$ then $\operatorname{Div} p(\gamma)=0$ and the formula above reduces to

$$
\int_{\Omega_{xy}} DF(\gamma)[h]e^{ivS(\gamma)}\mathcal{D}(\gamma) = -iv\int_{\Omega_{xy}} F(\gamma)DS(\gamma)[h]e^{ivS(\gamma)}\mathcal{D}(\gamma).
$$

§ 4.4. Sketch of Proof

We write

$$
(4.18)\qquad N(\Delta)=\prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}.
$$

We also write $y_{\Delta,j}=p(\gamma_\Delta,T_j)$ for $j=0,1,\dots,J+1$. Clearly $y_{\Delta,0}=0=y_{\Delta,J+1}$. Since definition of oscillatory integral on finite dimensional space $\mathbf{R}^J$ implies that

$$
(4.19)\qquad \int_{\mathbf{R}^J}\sum_{j=1}^J\frac{\partial}{\partial x_j}\left(F(\gamma_\Delta)y_{\Delta,j}e^{ivS(\gamma_\Delta)}\right)\prod_{j=1}^J dx_j=0,
$$

we have

$$
\begin{aligned}
(4.20)\qquad &N(\Delta)\int_{\mathbf{R}^J}\sum_{j=1}^J\partial_{x_j}(F(\gamma_\Delta))y_{\Delta,j}e^{ivS(\gamma_\Delta)}\prod_{j=1}^J dx_j\\
&= -N(\Delta)\int_{\mathbf{R}^J}F(\gamma_\Delta)\sum_{j=1}^J\partial_{x_j}(y_{\Delta,j})e^{ivS(\gamma_\Delta)}\prod_{j=1}^J dx_j\\
&\quad - ivN(\Delta)\int_{\mathbf{R}^J}F(\gamma_\Delta)\sum_{j=1}^J y_{\Delta,j}\partial_{x_j}S(\gamma_\Delta)e^{ivS(\gamma_\Delta)}\prod_{j=1}^J dx_j.
\end{aligned}
$$

Theorem 4.8 follows from the formula above, because we can prove the following three propositions:

Proposition 4.9.

$$
(4.21)\qquad \lim_{|\Delta|\to0}N(\Delta)\int_{\mathbf{R}^J}F(\gamma_\Delta)\sum_{j=1}^J y_{\Delta,j}\partial_{x_j}S(\gamma_\Delta)e^{ivS(\gamma_\Delta)}\prod_{j=1}^J dx_j=\int_\Omega F(\gamma)DS(\gamma)[p(\gamma)]e^{ivS(\gamma)}\mathcal{D}(\gamma).
$$

Proposition 4.10.

$$\lim_{|\Delta|\to 0} N(\Delta)\int_{\mathbf{R}^J}\sum_{j=1}^{J}\partial x_j(F(\gamma_\Delta))y_{\Delta,j}e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j=\int_\Omega DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{4.22}$$

Proposition 4.11.

$$\lim_{|\Delta|\to 0} N(\Delta)\int_{\mathbf{R}^J}F(\gamma_\Delta)\sum_{j=1}^{J}\partial_{x_j}(y_{\Delta,j})e^{(i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j=\int_\Omega F(\gamma)\operatorname{Div}p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{4.23}$$

Proof of these propositions is a long story. We omit it here. It will be published elsewhere.

§ 4.5. An Application to Semiclassical Asymptotic Formula

We always assume $b-a<\mu_0$. Let $F(\gamma)$ be an m-smooth functional. Then semiclassical asymptotic formula was proved by Kumano-go [16]:

$$\begin{aligned}\int F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma] \\ &= \left(\frac{-i\nu}{2\pi(b-a)}\right)^{1/2}D(b,a,x,y)^{-1/2}e^{i\nu S(\gamma^*)}\big(F(\gamma^*)+\nu^{-1}r(\nu.b,a,x,y)\big).\end{aligned} \tag{4.24}$$

where γ^* is the classical path connecting (b,x) and (a,y) in time-space.

If $F(\gamma^*)=0$, then the main term of the asymptotic formula vanishes. What happens in that case? Integration by parts formula enables us to get informations even in this case.

Suppose $F(\gamma)$ is an m-smooth functional and $F(\gamma^*)=0$. Then

$$F(\gamma)=\int_0^1 DF(\gamma_\theta)[\gamma-\gamma^*]\,d\theta,$$

where $\gamma_\theta=\theta\gamma+(1-\theta)\gamma^*$ and $DF(\gamma_\theta)$ is the Fréchet differential of $F(\gamma)$ at γ_θ. In other words,

$$F(\gamma)=\int_0^1\int_a^b\frac{\delta F(\gamma_\theta)}{\delta\gamma(s)}(\rho\gamma(s)-\rho\gamma^*(s))\,ds\,d\theta. \tag{4.25}$$

We set

$$\zeta(\gamma,t)=\int_0^1\frac{\delta F(\gamma_\theta)}{\delta\gamma(t)}\,d\theta. \tag{4.26}$$

Since $DS(\gamma^*)=0$, we have for for all $h\in\mathcal{H}_0$

$$DS(\gamma)[h]=DS(\gamma)[h]-DS(\gamma^*)[h]=(\gamma-\gamma^*,h)_{\mathcal{H}_0}-(\widetilde{W}(\gamma)(\rho\gamma-\rho\gamma^*),\rho h)_{L^2(a,b)}. \tag{4.27}$$

Here $(,)_{L^2(a,b)}$ and $(,)_{\mathcal{H}_0}$ are inner products in Hilbert spaces $L^2(a,b)$ and $\mathcal{H}_0$, respectively. $\widetilde{W}(\gamma)$ is multiplication operator $L^2(a,b)\ni h(s)\mapsto\widetilde{W}(\gamma,s)h(s)\in L^2(a,b)$, where

$$\widetilde{W}(\gamma,s)=\int_0^1\partial_x^2V(s,\gamma_\theta(s))\,d\theta.$$

We can show $I - \tilde{W}(\gamma)\rho\rho^*$ is an invertible operator in $L^2(a,b)$. We set

$$p(\gamma) = \rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma). \tag{4.28}$$

By definition of $p(\gamma)$ we have

Proposition 4.12. *The following equality holds:*

$$DS(\gamma)[p(\gamma)] = F(\gamma). \tag{4.29}$$

We can prove

Proposition 4.13. *If $F(\gamma)$ is an m-smooth functional, then $p(\gamma)$ is an m-admissible vector field.*

Proposition 4.14. *The following equality holds:*

$$\int_\Omega F(\gamma)e^{ivS(\gamma)}\mathcal{D}[\gamma] = \int_\Omega DS(\gamma)[p(\gamma)]e^{ivS(\gamma)}\mathcal{D}[\gamma]. \tag{4.30}$$

By applying integration by parts formula, we have

Theorem 4.15. *Suppose $F(\gamma)$ is an m-smooth functional with some $m \geq 0$ and $F(\gamma^*) = 0$. Set $\zeta(\gamma,t)$ be as above. Set*

$$p(\gamma) = \rho^*(I - W(\gamma)\rho\rho^*)^{-1}\zeta(\gamma). \tag{4.31}$$

Then $p(\gamma)$ is an m-admissible vector field on $\mathcal{H}_{xy}$. Moreover,

$$\int_{\Omega_{xy}} F(\gamma)e^{ivS(\gamma)}\mathcal{D}[\gamma] = -(iv)^{-1}\int_{\Omega_{xy}} \operatorname{Div} p(\gamma)e^{ivS(\gamma)}\mathcal{D}[\gamma]. \tag{4.32}$$

Theorem 4.16. *Under the same assumption the following asymptotic formula holds:*

$$\begin{aligned} &\int_{\Omega_{xy}} F(\gamma)e^{ivS(\gamma)}\mathcal{D}[\gamma] \\ &= \left(\frac{-iv}{2\pi(b-a)}\right)^{1/2} D(b,a,x,y)^{-1/2}e^{ivS(\gamma^*)}\left(-(iv)^{-1}\operatorname{Div} p(\gamma^*) + v^{-2}r(v,b,a,x,y)\right). \end{aligned} \tag{4.33}$$

where the remainder term $r(v,b,a,x,y)$ has the property such that for α,β there exists a positive constant $C_{\alpha\beta}$

$$\left|\partial_x^\alpha \partial_y^\beta r(v,b,a,x,y)\right| \leq C_{\alpha\beta}(1+|x|+|y|)^m. \tag{4.34}$$

Let $G_{\gamma^*}(t,s)$ be the Green function of differential equation of Jacobi field:

$$-\Big(\frac{d^2}{dt^2}+\partial_x^2 V(t,\gamma^*(t))\Big)u(t)=f(t),$$
$$u(a)=0=u(b).$$

Calculation shows:

Theorem 4.17.

$$\text{(4.35)}\quad \operatorname{Div} p(\gamma^*)=\frac{1}{2}\int_a^b\int_a^b \frac{\delta}{\delta\gamma(t)}(G_{\gamma^*}(t,s)\frac{\delta F(\gamma^*)}{\delta\gamma(s)})\,ds\,dt$$
$$=\frac{1}{2}\int_a^b\int_a^b \frac{\delta G_{\gamma^*}(t,s)}{\delta\gamma(t)}\frac{\delta F(\gamma^*)}{\delta\gamma(s)}\,ds\,dt+\frac{1}{2}\int_a^b\int_a^b G_{\gamma^*}(t,s)\frac{\delta^2 F(\gamma^*)}{\delta\gamma(s)\delta\gamma(t)}\,ds\,dt.$$

We omit details of the proof. It will be published elsewhere.

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