数理解析研究所講究録
第 1797 巻 2012 年 167-186

Phase Space Feynman Path Integrals – Calculation Examples via Piecewise Bicharacteristic Paths

By

Naoto KUMANO-GO*

Abstract

This survey of [19] is base on the introductory talk at RIMS. Because the RIMS Kôkyûroku gives us a chance for introducing the examples which are not suited for publication in ordinary journal, we introduce the calculation examples of the fundamental solutions for some equations, using the phase space path integral [19] via piecewise bicharacteristic paths.

§ 1. Introduction

Let $T>0$ and $x\in\mathbf{R}^{d}$. Let $U(T,0)$ be the fundamental solution for the Schrödinger equation

$$\left(i\hbar\partial_{T}-H(T,x,\frac{\hbar}{i}\partial_{x})\right)U(T,0)=0,\ \ U(0,0)=I, \tag{1.1}$$

with the Planck parameter $0<\hbar<1$. By the Fourier transform with respect to $x_{0}\in\mathbf{R}^{d}$ and the inverse Fourier transform with respect to $\xi_{0}\in\mathbf{R}^{d}$, the identity operator I is given by

$$Iv(x)=v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}v(x_{0})dx_{0}d\xi_{0},$$

and the Hamilton operator $H(T,x,\frac{\hbar}{i}\partial_{x})$ is given by

$$H(T,x,\frac{\hbar}{i}\partial_{x})v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}H(T,x,\xi_{0})v(x_{0})dx_{0}d\xi_{0}.$$

When T is small, we consider the function $U(T,0,x,\xi_{0})$ satisfying

$$U(T,0)v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}U(T,0,x,\xi_{0})v(x_{0})dx_{0}d\xi_{0}. \tag{1.2}$$

2010 Mathematics Subject Classification(s): 81S40 ; 35S30 ; 81Q20 ; 58D30
Key Words: Path integrals, Fourier integral operators, Semiclassical approximation
Supported by JSPS KAKENHI(C)21540196

*Division of Liberal Arts, Kogakuin University, 1-24-2 Nishi-shinjuku, Shinjuku-ku, Tokyo 163-8677, Japan.

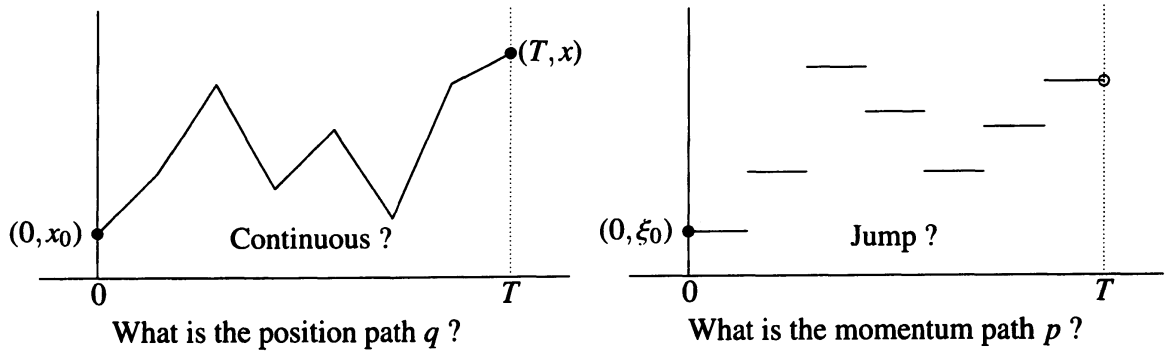


Figure 1.

Using the phase space path integral in R. P. Feynman [8, Appendix B], we formally write

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]. \tag{1.3}$$

Here $q:[0,T]\to\mathbf{R}^d$ is a position path with $q(0)=x_0$ and $q(T)=x$, $p:[0,T)\to\mathbf{R}^d$ is a momentum path with $p(0)=\xi_0$, $\phi[q,p]$ is the action of Hamiltonian type along the phase space path $(q.p)$ defined by

$$\phi[q,p]=\int_{[0,T)}p(t)\cdot dq(t)-\int_{[0,T)}H(t,q(t),p(t))dt,$$

and the phase space path integral $\int\sim\mathcal{D}[q,p]$ is a sum over all the paths (q,p) (Figure 1). However, in the sense of mathematics, the measure $\mathcal{D}[q,p]$ of the path integral (1.3) does not exist. Furthermore, in the sense of the uncertain principle, we can not have the position $q(t)$ and the momentum $p(t)$ at the same time t.

In [19], when the time interval $[0,T]$ is small, using piecewise bicharacteristic paths, we proved the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p] \tag{1.4}$$

with general functional $F[q,p]$ as integrand. More precisely, we gave a fairly general class $\mathcal{F}$ of functionals $F[q,p]$ such that for any $F[q,p]\in\mathcal{F}$, the time slicing approximation of (1.4) converges uniformly on compact subsets with respect to $(x,\xi_0,x_0)\in\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d$. Our approach via piecewise bicharacteristic paths is a little different from the known approaches. In this article, as a merit of piecewise bicharacteristic paths, we give the calculation examples of the fundamental solutions for some equations.

Remark. Using Fourier integral operators, H. Kitada–H. Kumano-go [17] proved the convergence of the time slicing approximation of (1.3). We regard (1.3) as a particular case of (1.4) with $F[q,p]=1$. In this sense, [19] has its origin in [17].

Remark. Using broken line paths of position and piecewise constant paths of momentum, W. Ichinose [14] gave some functionals $F[q,p] = \prod_{k=1}^{K} B_k(q(\tau_k), p(\tau_k))$, $0 < \tau_1 < \tau_2 < \cdots < \tau_K < T$ for which the time slicing approximations of (1.4) diverge as an operator. We exclude these functional from our class $\mathcal{F}$.

Remark. Inspired by the forward and backward approach of K. L. Chung–J.-C. Zambrini [4, §2.4], we use left-continuous paths and right-continuous paths. Furthermore, inspired by L. S. Shulman [24, §31], we pay attention to the operations which are valid in the phase space path integrals.

Since [8, Appendix B], the phase space path integral (1.3) has been rediscovered repeatedly (cf. W. Tobocman [25], H. Davies [6], C. Garrod [10]) and developed in various forms (cf. L. S. Schulman [24, §31], H. Kleinert [21], C. Grosche–F. Steiner [12], P. Cartier–C. DeWitt-Morette [3, §3.4], J. R. Klauder [20, §6.2]). For giving a well-defined mathematical meaning, various approaches have been proposed. C. DeWitt-Morette–A. Maheshwari–B. Nelson [7] and M. M. Mizrahi [23] introduced the formulation without limiting procedure. K. Gawedzki [11] used the technique analogous to that used by K. Itô [15]. I. Daubechies–J. R. Klauder [5] presented the phase space path integral via analytic continuation from Wiener measure. Furthermore, S. Albeverio–G. Guatteri–S. Mazzucchi [2] (cf. [1, §10.5.3], [22, §3.3]) realized the phase space path integral as an infinite dimensional oscillatory integral. O. G. Smolyanov–A. G. Tokarev–A. Truman [26] formulated the phase space path integral via Chernoff formula. For the main part of [8], G. W. Johnson–M. Lapidus [16] and T. L. Gill–W. W. Zachary [13] developed Feynman's operational calculus.

§ 2. Existence of Phase Space Path Integrals

We explain the existence of the phase space path integrals (1.4) step by step.

§ 2.1. Assumption for Hamiltonian

Our assumption for the Hamiltonian function $H(t,x,\xi)$ of (1.1) are the following.

Assumption 1 (Hamiltonian function). *$H(t,x,\xi)$ is a real-valued function of (t,x,ξ) in $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$, and for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ is continuous. For any non-negative integer k, there exists a positive constant κ_k such that*

$$|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)| \le \kappa_k(1+|x|+|\xi|)^{\max(2-|\alpha+\beta|,0)}, \tag{2.1}$$

for any multi-indices α, β with $|\alpha+\beta| = k$.

A typical example of the Hamiltonian operator $H(t,x,\frac{\hbar}{i}\partial_x)$ of (1.1) is the following.

Example 1 (Hamiltonian operator).

$$H(t,x,\frac{\hbar}{i}\partial_x) = \sum_{j,k=1}^{d}(a_{j,k}(t)\frac{\hbar}{i}\partial_{x_j}\frac{\hbar}{i}\partial_{x_k} + b_{j,k}(t)x_j\frac{\hbar}{i}\partial_{x_k} + c_{j,k}(t)x_jx_k)$$
$$+\sum_{j=1}^{d}(a_j(t)\frac{\hbar}{i}\partial_{x_j} + b_j(t)x_j) + c(t,x).$$

Here $a_{j,k}(t)$, $b_{j,k}(t)$, $c_{j,k}(t)$, $a_j(t)$, $b_j(t)$ and $\partial_x^\alpha c(t,x)$ are real-valued continuous bounded functions.

§ 2.2. We can produce many functionals $F[q,p] \in \mathcal{F}$

Typical examples of the functionals $F[q,p]$ in our class $\mathcal{F}$ are the following.

Example 2 ($F[q,p] \in \mathcal{F}$).

(1) *Let $m \geq 0$. Let $B(t,x)$ be a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ such that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous and satisfies $|\partial_x^\alpha B(t,x)| \leq C_\alpha(1+|x|)^m$ with a positive constant C_α. Then, the value at time t, $0 \leq t \leq T$,*

$$F[q,p] = B(t,q(t)) \in \mathcal{F}.$$

In particular, $F[q,p] = 1 \in \mathcal{F}$.

(2) *Let $m \geq 0$ and $0 \leq T' \leq T'' \leq T$. Let $B(t,x,\xi)$ be a function of $(t,x,\xi) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d$ such that for any multi-indices α, β, $\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)$ is continuous and satisfies $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \leq C_{\alpha,\beta}(1+|x|+|\xi|)^m$ with a positive constant $C_{\alpha,\beta}$. Then*

$$F[q,p] = \int_{[T',T'')} B(t,q(t),p(t))dt \in \mathcal{F}.$$

Furthermore, if $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \leq C_{\alpha,\beta}$, then

$$F[q,p] = e^{\int_{[T',T'')} B(t,q(t),p(t))dt} \in \mathcal{F}.$$

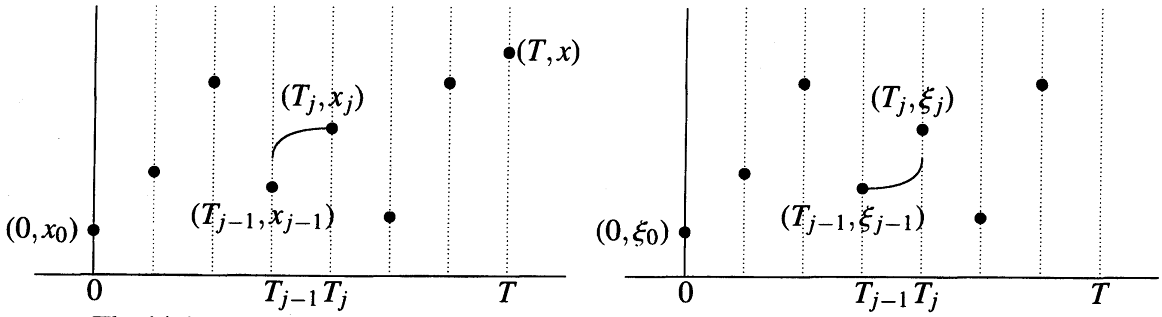
For simplicity, we will state the definition of the class $\mathcal{F}$ in §6. Even if we do not state the definition of $\mathcal{F}$ here, we can produce many functionals $F[q,p] \in \mathcal{F}$, applying Theorem 1 to Example 2.

Theorem 1 (Algebra). *If $F[q,p] \in \mathcal{F}$ and $G[q,p] \in \mathcal{F}$, then $F[q,p]+G[q,p] \in \mathcal{F}$ and $F[q,p]G[q,p] \in \mathcal{F}$.*

§ 2.3. Bicharacteristic paths

Let $\Delta_{T,0} = (T_{J+1}, T_J, \ldots, T_1, T_0)$ be any division of the interval $[0,T]$ given by

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0. \tag{2.2}$$



The bicharacteristic path $\bar{q}_{T_j,T_{j-1}}$ The bicharacteristic path $\bar{p}_{T_j,T_{j-1}}$

Figure 2.

Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Set $x_{J+1} = x$. Let $x_j \in \mathbf{R}^d$ and $\xi_j \in \mathbf{R}^d$ for $j = 1,2,\ldots,J$. Assume that $\kappa_2 d|\Delta_{T,0}| < 1/2$. Then we can define the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}} = \bar{q}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1})$ and $\bar{p}_{T_j,T_{j-1}} = \bar{p}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1})$, $T_{j-1}\le t\le T_j$ by the canonical equation

$$\begin{aligned}
&\partial_t\bar{q}_{T_j,T_{j-1}}(t) = (\partial_\xi H)(t,\bar{q}_{T_j,T_{j-1}},\bar{p}_{T_j,T_{j-1}}),\\
&\partial_t\bar{p}_{T_j,T_{j-1}}(t) = -(\partial_x H)(t,\bar{q}_{T_j,T_{j-1}},\bar{p}_{T_j,T_{j-1}}),\quad T_{j-1}\le t\le T_j,
\end{aligned}\tag{2.3}$$

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$. Note that $\bar{q}_{T_j,T_{j-1}}(T_{j-1})$ and $\bar{p}_{T_j,T_{j-1}}(T_j)$ are independent of x_{j-1} and ξ_j (Figure 2).

§ 2.4. Piecewise bicharacteristic paths

Using the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}}$ and $\bar{p}_{T_j,T_{j-1}}$ of (2.3), we define the piecewise bicharacteristic paths $q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t,x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)$ and $p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t,x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0)$ by

$$\begin{aligned}
&q_{\Delta_{T,0}}(t) = \bar{q}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1}),\quad T_{j-1}< t\le T_j,\quad q_{\Delta_{T,0}}(0)=x_0,\\
&p_{\Delta_{T,0}}(t) = \bar{p}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1}),\quad T_{j-1}\le t< T_j
\end{aligned}\tag{2.4}$$

for $j = 1,2,\ldots,J,J+1$ (Figure 3). Then the functionals $\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ become the functions $\phi_{\Delta_{T,0}}$, $F_{\Delta_{T,0}}$ given by

$$\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}] = \phi_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0),\tag{2.5}$$

$$F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0).\tag{2.6}$$

§ 2.5. Phase space path integrals exist

Our result about the existence of phase space path integrals is the following.

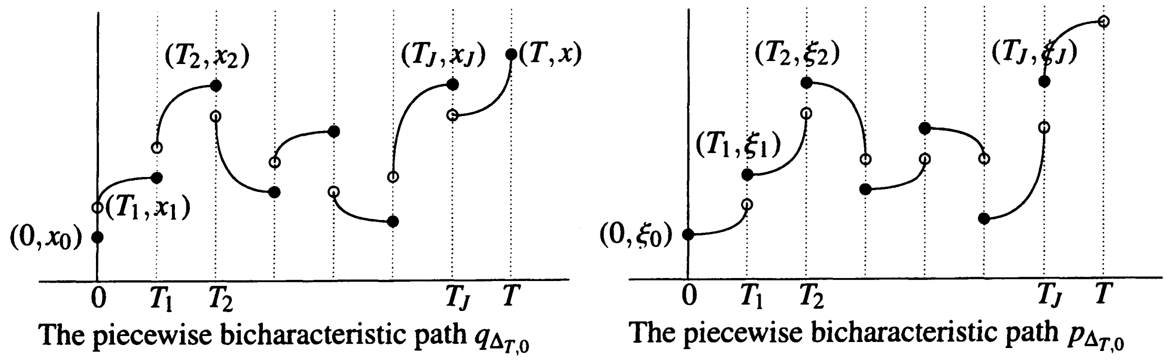


Figure 3.

Theorem 2 (Existence of phase space path integrals). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}$,*

$$\begin{aligned}(2.7)\qquad &\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]\\ &\equiv \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j d\xi_j\end{aligned}$$

converges uniformly on compact sets of $\mathbf{R}^{3d}$ with respect to (x,ξ_0,x_0), i.e., the phase space path integral (2.7) *is well-defined.*

Remark. Even when $F[q,p]=1$, each integral of the right hand side of (2.7) does not converge absolutely, i.e.,

$$\int_{\mathbf{R}^{2d}} d\xi_j dx_j = \infty .$$

Furthermore, the number J of integrals (division points) tends to ∞, i.e.,

$$\infty\times\infty\times\infty\times\infty\times\cdots\cdots\cdots, \quad J\to\infty.$$

We treat the multiple integral of (2.7) directly as an oscillatory integral (cf. H. Kumano-go [18, §1.6]) to keep the paths $q_{\Delta_{T,0}}$ and $p_{\Delta_{T,0}}$.

§ 3. Calculation Examples

As examples of (2.7), we calculate the fundamental solutions for some equations.

§ 3.1. Example: $d=1, H(t,x,\xi)=\xi^2/2, F[q,p]=1$

Note $(\partial_\xi H)=\xi$ and $(\partial_x H)=0$. By the canonical equation

$$\partial_t \bar{q}_{T_j,T_{j-1}}(t)=\bar{p}_{T_j,T_{j-1}}(t), \;\; \partial_t \bar{p}_{T_j,T_{j-1}}(t)=0, \;\; T_{j-1}\le t\le T_j$$

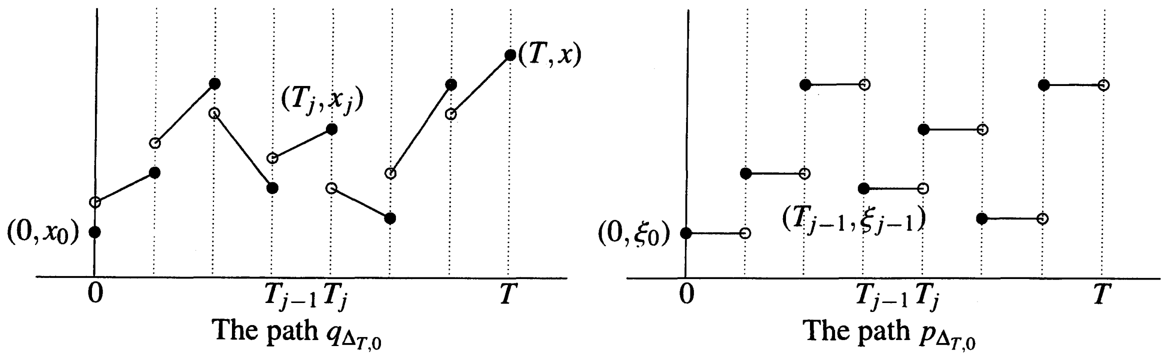


Figure 4.

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$, we have the bicharacteristic paths

$$\begin{aligned}\bar{q}_{T_j,T_{j-1}}(t) &= x_j - (T_j - t)\xi_{j-1}\,,\\ \bar{p}_{T_j,T_{j-1}}(t) &= \xi_{j-1}\,.\end{aligned}$$

Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (2.4) (Figure 4).

Then the functional $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ becomes the function

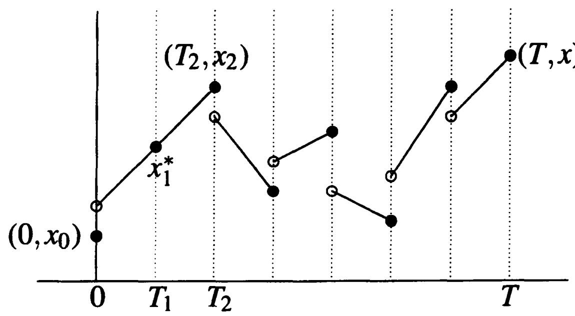
$$\begin{aligned}\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= \int_{[0,T)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}} - \int_{[0,T)} H(t, q_{\Delta_{T,0}}, p_{\Delta_{T,0}})\\ &= \sum_{j=1}^{J+1} \phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}),\end{aligned}$$

where

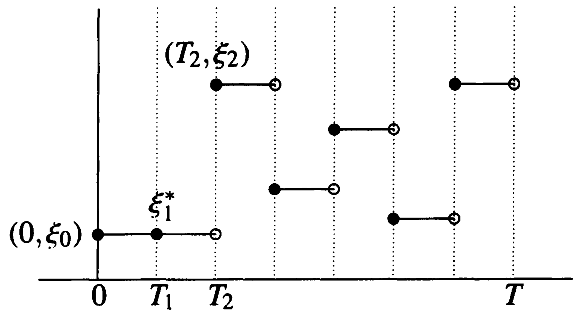
$$\begin{aligned}&\phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1})\\ &= \int_{[T_{j-1},T_j)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}} - \int_{[T_{j-1},T_j)} H(q_{\Delta_{T,0}}, p_{\Delta_{T,0}})dt\\ &= (\bar{q}_{T_j,T_{j-1}}(T_{j-1}) - x_{j-1}) \cdot \xi_{j-1}\\ &\quad + \int_{T_{j-1}}^{T_j} \bar{p}_{T_j,T_{j-1}} \cdot \partial_t \bar{q}_{T_j,T_{j-1}} dt - \int_{T_{j-1}}^{T_j} H(t, \bar{q}_{T_j,T_{j-1}}, \bar{p}_{T_j,T_{j-1}})dt\\ &= (x_j - (T_j - T_{j-1})\xi_{j-1} - x_{j-1}) \cdot \xi_{j-1} + \int_{T_{j-1}}^{T_j} \xi_{j-1}^2 dt - \int_{T_{j-1}}^{T_j} \xi_{j-1}^2/2dt\\ &= (x_j - x_{j-1}) \cdot \xi_{j-1} - \frac{t_j}{2}\xi_{j-1}^2\,.\end{aligned}$$

First we consider the oscillatory integral

$$\left(\frac{1}{2\pi\hbar}\right) \int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1) + \frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)} dx_1 d\xi_1\,.$$



The path $q_{(\Delta_{T,T_2},0)}$ and x_1^*



The path $p_{(\Delta_{T,T_2},0)}$ and ξ_1^*

Figure 5.

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})(x_2,\xi_1^*,x_1^*,\xi_0)=0$ (Figure 5).

Then we have

$$\begin{aligned}
&\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\phi_{T_1,0}(x_1,\xi_0,x_0)\\
&=\phi_{T_2,0}(x_2,\xi_0,x_0)+\frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix}\cdot\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix},\\
&(-1)\det\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})=(-1)\det\begin{bmatrix}-t_2 & -1\\ -1 & 0\end{bmatrix}=1.
\end{aligned}$$

Lemma 3.1. *For any* 2×2 *real symmetric matrix* A, *we have*

$$\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\frac{1}{2}Ax\cdot x}dx=\sqrt{\frac{(2\pi\hbar i)^2}{\det A}}=\frac{2\pi\hbar}{\sqrt{(-1)\det A}}.$$

By lemma 3.1, we have

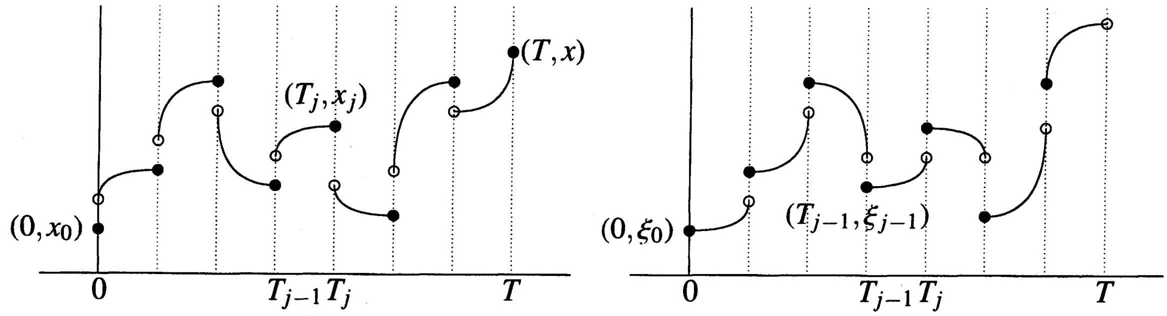
$$\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1d\xi_1=e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}.$$

Using this relation inductively, we have

$$\begin{aligned}
e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)&=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\\
&=\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^J dx_jd\xi_j\\
&=\lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\\
&=\exp\frac{i}{\hbar}\left((x-x_0)\cdot\xi_0-\frac{T}{2}\xi_0^2\right).
\end{aligned}$$

The operator $U(T,0)$ of (1.2) satisfies the Schrödinger equation

$$\left(i\hbar\partial_T+\hbar^2\triangle/2\right)U(T,0)=0,\ \ U(0,0)=I.$$



The path $q_{\Delta_{T,0}}$ The path $p_{\Delta_{T,0}}$

Figure 6.

§ 3.2. Example: $d=1, H(t,x,\xi)=x^2/2+\xi^2/2, F[q,p]=1$

Note $(\partial_\xi H)=\xi$ and $(\partial_x H)=x$. By the canonical equation

$$\partial_t \bar{q}_{T_j,T_{j-1}}(t)=\bar{p}_{T_j,T_{j-1}}(t)\,,\quad \partial_t \bar{p}_{T_j,T_{j-1}}(t)=-\bar{q}_{T_j,T_{j-1}}(t)\,,\quad T_{j-1}\le t\le T_j$$

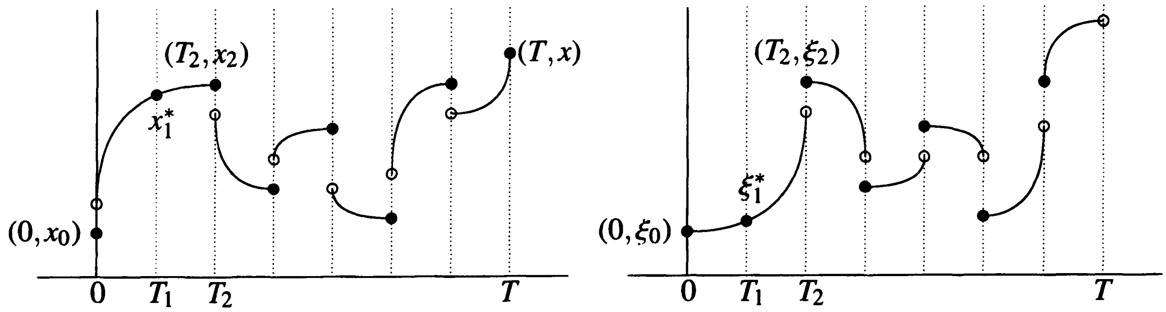
with $\bar{q}_{T_j,T_{j-1}}(T_j)=x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1})=\xi_{j-1}$, we have the bicharacteristic paths

$$\bar{q}_{T_j,T_{j-1}}(t)=\frac{x_j\cos(t-T_{j-1})-\xi_{j-1}\sin(T_j-t)}{\cos(T_j-T_{j-1})}\,,$$
$$\bar{p}_{T_j,T_{j-1}}(t)=\frac{-x_j\sin(t-T_{j-1})+\xi_{j-1}\cos(T_j-t)}{\cos(T_j-T_{j-1})}\,.$$

Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (2.4) (Figure 6).

Then the functional $\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ becomes the function

$$\begin{aligned}\phi_{\Delta_{T,0}}=\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]&=\int_{[0,T)}p_{\Delta_{T,0}}\cdot dq_{\Delta_{T,0}}-\int_{[0,T)}H(t,q_{\Delta_{T,0}},p_{\Delta_{T,0}})\\&=\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})\,,\end{aligned}$$



The path $q_{(\Delta_{T,T_2},0)}$ and x_1^* The path $p_{(\Delta_{T,T_2},0)}$ and ξ_1^*

Figure 7.

where

$$
\begin{aligned}
&\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1}) \\
&= \int_{[T_{j-1},T_j)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}} - \int_{[T_{j-1},T_j)} H(q_{\Delta_{T,0}}, p_{\Delta_{T,0}})dt \\
&= (\bar{q}_{T_j,T_{j-1}}(T_{j-1}) - x_{j-1}) \cdot \xi_{j-1} \\
&\quad + \frac{1}{2}\int_{T_{j-1}}^{T_j} \bar{p}_{T_j,T_{j-1}} \cdot \partial_t \bar{q}_{T_j,T_{j-1}} dt + \frac{1}{2}\Big[\bar{p}_{T_j,T_{j-1}} \cdot \bar{q}_{T_j,T_{j-1}}\Big]_{T_{j-1}}^{T_j} \\
&\quad - \frac{1}{2}\int_{T_{j-1}}^{T_j} \bar{q}_{T_j,T_{j-1}} \cdot \partial_t \bar{p}_{T_j,T_{j-1}} dt - \frac{1}{2}\int_{T_{j-1}}^{T_j} (\bar{q}^2_{T_j,T_{j-1}} + \bar{p}^2_{T_j,T_{j-1}})dt \\
&= -x_{j-1} \cdot \xi_{j-1} + \frac{\bar{p}_{T_j,T_{j-1}}(T_j)x_j + \bar{q}_{T_j,T_{j-1}}(T_{j-1})\xi_{j-1}}{2} \\
&= -x_{j-1} \cdot \xi_{j-1} + \frac{2x_j \cdot \xi_{j-1} - (x_j^2 + \xi_{j-1}^2)\sin(T_j - T_{j-1})}{2\cos(T_j - T_{j-1})}.
\end{aligned}
$$

First we consider the oscillatory integral

$$\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)} dx_1 d\xi_1 .$$

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0})(x_2,\xi_1^*,x_1^*,\xi_0) = 0$ (Figure 7).

Then we have

$$
\begin{aligned}
&\phi_{T_2,T_1}(x_2,\xi_1,x_1) + \phi_{T_1,0}(x_1,\xi_0,x_0) \\
&= \phi_{T_2,0}(x_2,\xi_0,x_0) + \frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0}) \begin{bmatrix} \xi_1 - \xi_1^* \\ x_1 - x_1^* \end{bmatrix} \cdot \begin{bmatrix} \xi_1 - \xi_1^* \\ x_1 - x_1^* \end{bmatrix}, \\
&(-1)\det \partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0}) \\
&= (-1)\det \begin{bmatrix} -\frac{\sin(T_2-T_1)}{\cos(T_2-T_1)} & -1 \\ -1 & -\frac{\sin(T_1-0)}{\cos(T_1-0)} \end{bmatrix} = \frac{\cos T_2}{\cos t_2 \cos t_1}.
\end{aligned}
$$

By Lemma 3.1, we have

$$\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1 d\xi_1$$
$$= e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}\left(\frac{\cos t_2\cos t_1}{\cos T_2}\right)^{1/2}.$$

Using this relation inductively and taking $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j \to 0$, we have

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]$$
$$= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J}dx_j d\xi_j$$
$$= \lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}\cos t_j}{\cos T}\right)^{1/2}$$
$$= \frac{1}{(\cos T)^{1/2}}\exp\frac{i}{\hbar}\Big(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0-(x^2+\xi_0^2)\sin T}{2\cos T}\Big).$$

The operator $U(T,0)$ of (1.2) satisfies the Schrödinger equation

$$\Big(i\hbar\partial_T+\hbar^2\triangle/2-x^2/2\Big)U(T,0)=0,\quad U(0,0)=I.$$

§ 3.3. Example: $d=1$, $H(t,x,\xi)=x^2/2+x\cdot\xi+\xi^2/2$, $F[q,p]=1$

Note $(\partial_\xi H)=x+\xi=(\partial_x H)$. By the canonical equation

$$\partial_t\bar q_{T_j,T_{j-1}}(t)=\bar q_{T_j,T_{j-1}}(t)+\bar p_{T_j,T_{j-1}}(t)=-\partial_t\bar p_{T_j,T_{j-1}}(t),\quad T_{j-1}\le t\le T_j$$

with $\bar q_{T_j,T_{j-1}}(T_j)=x_j$ and $\bar p_{T_j,T_{j-1}}(T_{j-1})=\xi_{j-1}$, we have the bicharacteristic paths

$$\bar q_{T_j,T_{j-1}}(t)=\frac{x_j(1+t-T_{j-1})-\xi_{j-1}(T_j-t)}{1+T_j-T_{j-1}},$$
$$\bar p_{T_j,T_{j-1}}(t)=\frac{-x_j(t-T_{j-1})+\xi_{j-1}(1+T_j-t)}{1+T_j-T_{j-1}}.$$

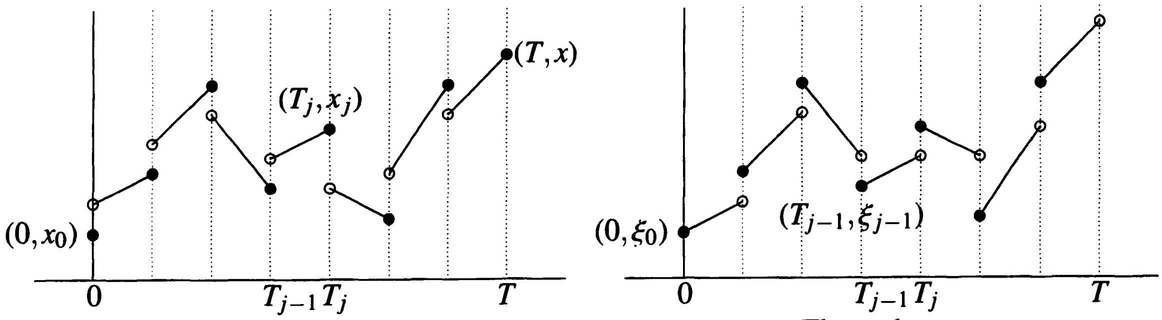
Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (2.4) (Figure 8).

Then the functional $\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ becomes the function

$$\phi_{\Delta_{T,0}}=\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]=\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1}),$$

where

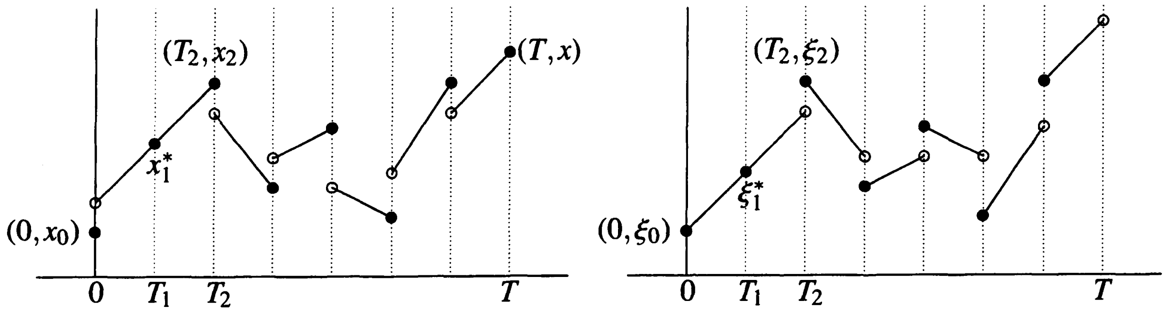
$$\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})$$
$$=-x_{j-1}\cdot\xi_{j-1}+\frac{2x_j\cdot\xi_{j-1}-(x_j^2+\xi_{j-1}^2)(T_j-T_{j-1})}{2(1+T_j-T_{j-1})}.$$



The path $q_{\Delta_{T,0}}$ The path $p_{\Delta_{T,0}}$

Figure 8.

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})(x_2,\xi_1^*,x_1^*,\xi_0)=0$ (Figure 9).



The path $q_{(\Delta_{T,T_2},0)}$ and x_1^* The path $p_{(\Delta_{T,T_2},0)}$ and ξ_1^*

Figure 9.

Then we have

$$\begin{aligned}
&\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\phi_{T_1,0}(x_1,\xi_0,x_0)\\
&=\phi_{T_2,0}(x_2,\xi_0,x_0)+\frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix}\cdot\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix},\\
&(-1)\det\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\\
&=(-1)\det\begin{bmatrix}-\frac{t_2}{1+t_2} & -1\\ -1 & -\frac{t_1}{1+t_1}\end{bmatrix}=\frac{1+T_2}{(1+t_2)(1+t_1)}.
\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}
&\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2}e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1d\xi_1\\
&=e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}\left(\frac{(1+t_2)(1+t_1)}{1+T_2}\right)^{1/2}.
\end{aligned}$$

Using this relation inductively and taking $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j \to 0$, we have

$$
\begin{aligned}
&e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p] \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J \int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J} dx_j d\xi_j \\
&= \lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}(1+t_j)}{1+T}\right)^{1/2} \\
&= \left(\frac{e^T}{\cos T}\right)^{1/2}\exp\frac{i}{\hbar}\Big(-x_0\cdot\xi_0 + \frac{2x\cdot\xi_0-(x^2+\xi_0^2)T}{2(1+T)}\Big).
\end{aligned}
$$

The operator $U(T,0)$ of (1.2) satisfies the equation

$$
\Big(i\hbar\partial_T + \hbar^2\triangle/2 - x\frac{\hbar}{i}\partial_x - x^2/2\Big)U(T,0) = 0,\ \ U(0,0) = I.
$$

§ 3.4. Example: $d=1$, $H(t,x,\xi) = x^2/2 - \xi^2/2$, $F[q,p]=1$

Note $(\partial_\xi H) = -\xi$ and $(\partial_x H) = x$. By the canonical equation

$$
\partial_t\bar{q}_{T_j,T_{j-1}}(t) = -\bar{p}_{T_j,T_{j-1}}(t),\ \ \partial_t\bar{p}_{T_j,T_{j-1}}(t) = -\bar{q}_{T_j,T_{j-1}}(t),\ \ T_{j-1}\le t\le T_j
$$

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$, we have the bicharacteristic paths

$$
\begin{aligned}
\bar{q}_{T_j,T_{j-1}}(t) &= \frac{x_j\cosh(t-T_{j-1}) + \xi_{j-1}\sinh(T_j-t)}{\cosh(T_j-T_{j-1})}, \\
\bar{p}_{T_j,T_{j-1}}(t) &= \frac{-x_j\sinh(t-T_{j-1}) + \xi_{j-1}\cosh(T_j-t)}{\cosh(T_j-T_{j-1})}.
\end{aligned}
$$

Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (2.4) (Figure 6).

Then the functional $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ becomes the function

$$
\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = \sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1}),
$$

where

$$
\begin{aligned}
&\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1}) \\
&= -x_{j-1}\cdot\xi_{j-1} + \frac{2x_j\cdot\xi_{j-1} - (x_j^2-\xi_{j-1}^2)\sinh(T_j-T_{j-1})}{2\cosh(T_j-T_{j-1})}.
\end{aligned}
$$

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0})(x_2,\xi_1^*,x_1^*,\xi_0) = 0$ (Figure 7).

Then we have

$$
\begin{aligned}
&\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\phi_{T_1,0}(x_1,\xi_0,x_0)\\
&=\phi_{T_2,0}(x_2,\xi_0,x_0)+\frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix}\cdot\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix},\\
&(-1)\det\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\\
&=(-1)\det\begin{bmatrix}\frac{\sinh(T_2-T_1)}{\cosh(T_2-T_1)} & -1\\ -1 & -\frac{\sinh(T_1-0)}{\cosh(T_1-0)}\end{bmatrix}=\frac{\cosh T_2}{\cosh t_2\cosh t_1}.
\end{aligned}
$$

By Lemma 3.1, we have

$$
\begin{aligned}
&\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2}e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1d\xi_1\\
&=e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}\left(\frac{\cosh t_2\cosh t_1}{\cosh T_2}\right)^{1/2}.
\end{aligned}
$$

Using this relation inductively and taking $|\Delta_{T,0}|=\max_{1\le j\le J+1}t_j\to 0$, we have

$$
\begin{aligned}
&e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\\
&=\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}}e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J}dx_jd\xi_j\\
&=\lim_{|\Delta_{T,0}|\to 0}e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}\cosh t_j}{\cosh T}\right)^{1/2}\\
&=\frac{1}{(\cosh T)^{1/2}}\exp\frac{i}{\hbar}\left(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0-(x^2-\xi_0^2)\sinh T}{2\cosh T}\right).
\end{aligned}
$$

The operator $U(T,0)$ of (1.2) satisfies the equation

$$
\left(i\hbar\partial_T-\hbar^2\Delta/2-x^2/2\right)U(T,0)=0,\quad U(0,0)=I.
$$

§ 3.5. Exception: $d=1$, $H(t,x,\xi)=-ix^2/2-i\xi^2/2$, $F[q,p]=1$

This complex-valued function $H(t,x,\xi)$ does not satisfy Assumption 1. However we can get the function $U(T,0,x,\xi_0)$ of the fundamental solution in a similar way:

Note $(\partial_\xi H)=-i\xi$ and $(\partial_x H)=-ix$. By the canonical equation

$$
\partial_t\bar{q}_{T_j,T_{j-1}}(t)=-i\bar{p}_{T_j,T_{j-1}}(t),\quad \partial_t\bar{p}_{T_j,T_{j-1}}(t)=i\bar{q}_{T_j,T_{j-1}}(t),\quad T_{j-1}\le t\le T_j
$$

with $\bar{q}_{T_j,T_{j-1}}(T_j)=x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1})=\xi_{j-1}$, we have the bicharacteristic paths

$$
\begin{aligned}
\bar{q}_{T_j,T_{j-1}}(t)&=\frac{x_j\cosh(t-T_{j-1})+i\xi_{j-1}\sinh(T_j-t)}{\cosh(T_j-T_{j-1})},\\
\bar{p}_{T_j,T_{j-1}}(t)&=\frac{ix_j\sinh(t-T_{j-1})+\xi_{j-1}\cosh(T_j-t)}{\cosh(T_j-T_{j-1})}.
\end{aligned}
$$

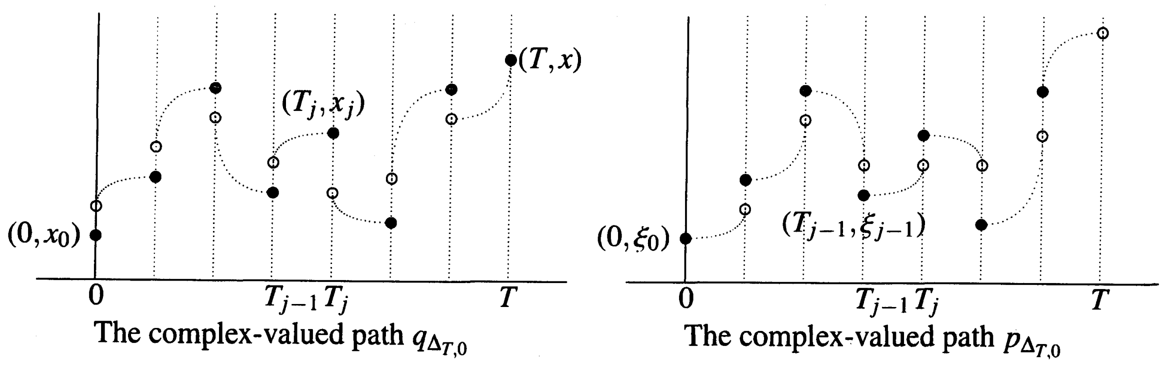


Figure 10.

Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (2.4) (Figure 10).

Then the functional $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ becomes the function

$$\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = \sum_{j=1}^{J+1} \phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}),$$

where

$$\begin{aligned}&\phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1})\\ &= -x_{j-1}\cdot\xi_{j-1} + \frac{2x_j\cdot\xi_{j-1} + i(x_j^2 + \xi_{j-1}^2)\sinh(T_j - T_{j-1})}{2\cosh(T_j - T_{j-1})}.\end{aligned}$$

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0})(x_2, \xi_1^*, x_1^*, \xi_0) = 0$ (Figure 11).

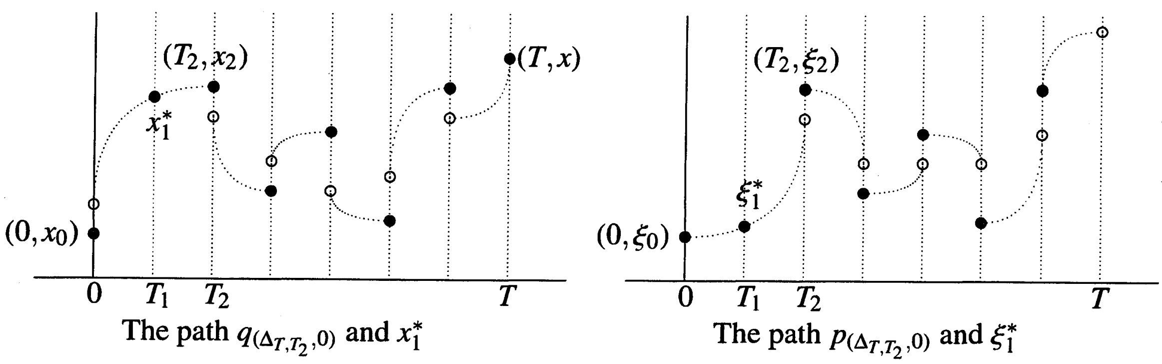


Figure 11.

Then we have

$$\begin{aligned}
&\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\phi_{T_1,0}(x_1,\xi_0,x_0)\\
&=\phi_{T_2,0}(x_2,\xi_0,x_0)+\frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix}\cdot\begin{bmatrix}\xi_1-\xi_1^*\\ x_1-x_1^*\end{bmatrix},\\
&(-1)\det\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1}+\phi_{T_1,0})\\
&=(-1)\det\begin{bmatrix} i\frac{\sinh(T_2-T_1)}{\cosh(T_2-T_1)} & -1\\ -1 & i\frac{\sinh(T_1-0)}{\cosh(T_1-0)}\end{bmatrix}=\frac{\cosh T_2}{\cosh t_2\cosh t_1}.
\end{aligned}$$

Performing the integration with respect to (ξ_1,x_1), we have

$$\begin{aligned}
&\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1 d\xi_1\\
&=e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}\left(\frac{\cosh t_2\cosh t_1}{\cosh T_2}\right)^{1/2}.
\end{aligned}$$

Using this relation inductively and taking $|\Delta_{T,0}|=\max_{1\le j\le J+1}t_j\to 0$, we have

$$\begin{aligned}
&e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\\
&=\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J}dx_j d\xi_j\\
&=\lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}\cosh t_j}{\cosh T}\right)^{1/2}\\
&=\frac{1}{(\cosh T)^{1/2}}\exp\frac{i}{\hbar}\Big(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0+i(x^2+\xi_0^2)\sinh T}{2\cosh T}\Big).
\end{aligned}$$

The operator $U(T,0)$ of (1.2) satisfies the heat equation

$$\Big(\hbar\partial_T-\hbar^2\triangle/2+x^2/2\Big)U(T,0)=0,\quad U(0,0)=I.$$

§ 4. Properties of Phase Space Path Integrals

We sketch some properties of (1.4) with general functional $F[q,p]$.

§ 4.1. Fubini-type theorem

Theorem 3 (Fubini-type). *Let T be sufficiently small. Let $m\ge 0$ and $0\le T'\le T''\le T$. Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous and satisfies $|\partial_x^\alpha B(t,x)|\le C_\alpha(1+$*

$|x|)^m$ *with a positive constant* C_α. *Then, for any* $F[q,p] \in \mathcal{F}$ *including* $F[q,p] = 1$, *we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]} \left(\int_{[T',T'')} B(t,q(t))dt \right) F[q,p]\mathcal{D}[q,p]$$
$$= \int_{[T',T'')} \left(\int e^{\frac{i}{\hbar}\phi[q,p]} B(t,q(t))F[q,p]\mathcal{D}[q,p] \right) dt\,.$$

Remark. We do not treat $B(t,q(t),p(t))$ at the time t.

Remark (Perturbation expansion formula). If $|\partial_x^\alpha B(t,x)| \le C_\alpha$, we have

$$\int e^{\frac{i}{\hbar}\phi[q,p]+\frac{i}{\hbar}\int_{[0,T)} B(\tau,q(\tau))d\tau} \mathcal{D}[q,p]$$
$$= \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \int_{[0,T)} d\tau_n \int_{[0,\tau_n)} d\tau_{n-1} \cdots \int_{[0,\tau_2)} d\tau_1$$
$$\times \int e^{\frac{i}{\hbar}\phi[q,p]} B(\tau_n,q(\tau_n))B(\tau_{n-1},q(\tau_{n-1}))\cdots B(\tau_1,q(\tau_1))\mathcal{D}[q,p]\,.$$

§ 4.2. Semiclassical approximation of Hamiltonian type

Let $4\kappa_2 dT < 1/2$. Let $q_{T,0} = q_{T,0}(t,x,\xi_0,x_0)$ and $p_{T,0} = p_{T,0}(t,x,\xi_0)$ be the piecewise bicharacteristic paths for the simplest division $0 < T$ (Figure 12).

For any $(x_{J+1},\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d$, there exists the stationary point $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ of the phase function $\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ given by

$$(\partial_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},\xi_J^*,x_J^*,\ldots,\xi_1^*,x_1^*,\xi_0) = 0\,.$$

Pushing the stationary point $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ into the Hessian matrix of $\phi_{\Delta_{T,0}}$, we define $D(T,x_{J+1},\xi_0)$ by

$$D(T,x_{J+1},\xi_0)$$
$$= \lim_{|\Delta_{T,0}|\to 0} (-1)^{dJ} \det(\partial^2_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*,\xi_0)\,.$$

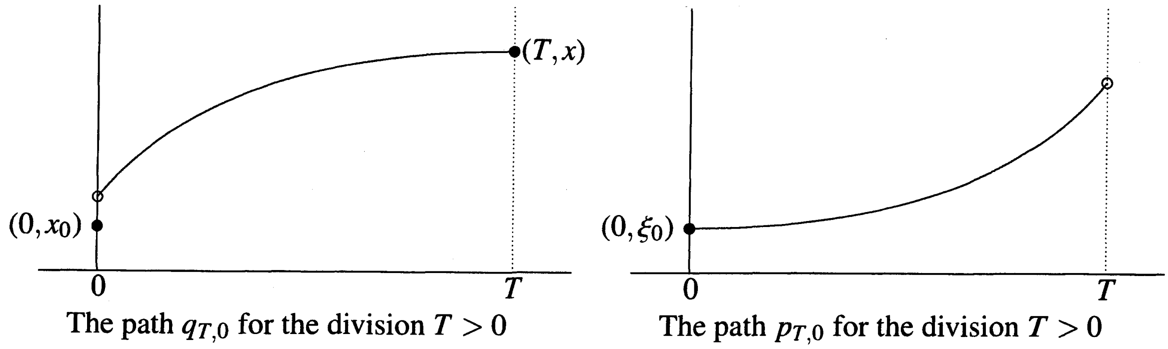


Figure 12.

Then the remainder estimate for the semiclassical approximation of Hamiltonian type as $\hbar \to 0$ are the following.

Theorem 4 (Semiclassical approximation of Hamiltonian type as $\hbar \to 0$). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}$, we have*

$$\begin{aligned}&\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]\\&= e^{\frac{i}{\hbar}\phi[q_{T,0},p_{T,0}]}\left(D(T,x,\xi_0)^{-1/2}F[q_{T,0},p_{T,0}]+\hbar\Upsilon(T,\hbar,x,\xi_0,x_0)\right).\end{aligned}$$

Here for any multi-indices α, β, the remainder term $\Upsilon(T,\hbar,x,\xi_0,x_0)$ satisfies

$$|\partial_x^\alpha\partial_{\xi_0}^\beta\Upsilon(T,\hbar,x,\xi_0,x_0)| \le C_{\alpha,\beta}(1+|x|+|\xi_0|+|x_0|)^m,$$

with a positive constant $C_{\alpha,\beta}$ independent of $0<\hbar<1$.

§ 5. Proof for Theorems 1, 2 and 4

We sketch the process of the proof for Theorems 1, 2 and 4. In order to prove the convergence of the multiple integral

$$\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j d\xi_j \tag{5.1}$$

as $|\Delta_{T,0}| \to 0$, we have only to add many assumptions for

$$F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\dots,x_1,\xi_0,x_0) = F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}].$$

The assumptions should be closed under addition and multiplication. Then $\mathcal{F}$ will be an algebra. Do not consider other things. Then $\mathcal{F}$ will become larger as a set. If lucky, $\mathcal{F}$ will contain at least one example $F[q,p]=1$ as the fundamental solution for the Schrödinger equation. Our proof consists of 3 steps. As the first step, we use an estimate of H. Kumano-go-Taniguchi's type [18, (6.94), p.360] to control (5.1) by C^J as $J \to \infty$ with a positive constant C. As the second step, we use a stationary phase method of Fujiwara's type [9] to control (5.1) by C independent of $J \to \infty$ with a positive constant C. At the last step, we add assumptions so that (5.1) converges as $|\Delta_{T,0}| \to 0$.

§ 6. Definition of Class $\mathcal{F}$

The definition of the class $\mathcal{F}$ of functionals $F[q,p]$ is the following.

Definition 1 (Class $\mathcal{F}$ of functionals $F[q,p]$). *Let $F[q,p]$ be a functional whose domain contains all the piecewise bicharacteristic paths $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ of* (2.4). *We say that $F[q,p]\in\mathcal{F}$ if $F_{\Delta_{T,0}}=F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ satisfies Assumption 2.*

Assumption 2. *Let $m\geq 0$. Let $u_j\geq 0$, $j=1,2,\dots,J,J+1$ are non-negative parameters depending on the division $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1}u_j=U<\infty$. For any integer $M\geq 0$, there exist positive constants A_M, X_M such that for any $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}|\leq M$, $j=1,2,\dots,J,J+1$ and any $1\leq k\leq J$,*

$$\tag{6.1} \begin{aligned} &|(\prod_{j=1}^{J+1}\partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}})F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0,x_0)|\\ &\leq A_M(X_M)^{J+1}(\prod_{j=1}^{J+1}(t_j)^{\min(|\beta_{j-1}|,1)})(1+\sum_{j=1}^{J+1}(|x_j|+|\xi_{j-1}|)+|x_0|)^m, \end{aligned}$$

$$\tag{6.2} \begin{aligned} &|(\prod_{j=1}^{J+1}\partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}})\partial_{x_k}F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0,x_0)|\\ &\leq A_M(X_M)^{J+1}u_k(\prod_{j\neq k}(t_j)^{\min(|\beta_{j-1}|,1)})(1+\sum_{j=1}^{J+1}(|x_j|+|\xi_{j-1}|)+|x_0|)^m. \end{aligned}$$

References

[1] Albeverio, S., Høegh-Krohn, and Mazzucchi, S., *Mathematical theory of Feynman path integrals*, Lecture notes of Math. **523**, Springer, Berlin, 1976 (The 2nd edition, 2008).

[2] Albeverio, S., Guatteri, G. and Mazzucchi, S., Phase space Feynman path integrals, *J. Math. Phys.* **43** (2002), 2847–2857.

[3] Cartier, P. and DeWitt-Morette, C., *Functional integration*, Cambridge university press (2006)

[4] Chung, K. L. and Zambrini, J.-C., *Introduction to Random Time and Quantum Randomness*, World Scientific Pub Co Inc, 2003.

[5] Daubechies, I. and Klauder, J. R., Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians. *J. Math. Phys.* **26** (1985), 2239-2256.

[6] Davies, H., Hamiltonian approach to the method of summation over Feynman histories, Proc. Camb. Phil. Soc. **59** (1963), 147–155.

[7] DeWitt-Morette, C., Maheshwari, A., and Nelson, B., Path integration in phase space, *Gen. Rel. and Grav.* 8 (1977), 581–593.

[8] Feynman, R. P., An operator calculus having applications in quantum electrodynamics, *Appendix B, Phys. Rev.* **84**, (1951), 108–236.

[9] Fujiwara, D., The stationary phase method with an estimate of the remainder term on a space of large dimension, *Nagoya Math. J.* **124** (1991), 61–97.

[10] Garrod, C., Hamiltonian path-integral methods, *Rev. Mod. Phys.* **38** (1966), 483–494.

[11] Gawedzki, K., Construction of quantum-mechanical dynamics by means of path integrals in phase space, *Rep. Math. Phys.* **6** (1974), 327–342.

[12] Grosche, C. and Steiner, F., Handbook of Feynman Path Integrals. Berlin-Heidelberg: Springer-Verlag, 1998.

[13] Gill, T. L. and Zachary, W. W., Foundations for relatives quantum theory. I. Feynman's operational calculus and the Dyson conjecture, *J. Math. Phys.* **43** (2002), 69–93.

[14] Ichinose, W., A mathematical theory of the phase space Feynman path integral of the functional, *Comm. Math. Phys.* **265** (2006), 739–779.

[15] Itô, K., Generalized uniform complex measure in Hilbert metric spaces and their applications to the Feynman path integrals, *Proc. 5th Berkeley Sympos. on Math. Statist. and Prob.* **2**, part 1, 1967, pp.145–161.

[16] Johnson, G. W. and Lapidus, M., *The Feynman Integral and Feynman's Operational Calculus*, Oxford University Press, New York, 2000.

[17] Kitada, H. and Kumano-go, H., A family of Fourier integral operators and the fundamental solution for a Schrödinger equation, *Osaka J. Math.* **18** (1981), 291-360.

[18] Kumano-go, H., *Pseudo-Differential Operators*, The MIT press, Cambridge, MA, 1981.

[19] Kumano-go, N. and Fujiwara, D., Phase space Feynman path integrals via piecewise bicharacteristic paths and their semiclassical approximations, Bull. Sci. math. **132** (2008), 313–357.

[20] Klauder, J. R., A Modern Approach to Functional Integration, Birkhauser, 2010.

[21] Kleinert, H., Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, World Scientific Pub Co Inc, 1990 (5th edition 2009).

[22] Mazzucchi, S., Mathematical Feynman Path Integrals and Their Applications, World Scientific Pub Co Inc, 2009.

[23] Mizrahi, M. M., Phase space path integrals, without limiting procedure, *J. Math. Phys.* **19** (1978), 298-307; Erratum, ibid. **21** (1980), 1965.

[24] Schulman, L. S., *Techniques and Applications of Path Integration*, Monographs and Texts in Physics and Astronomy, Wiley-Interscience, New York, 1981 (with new supplementary section, Dover Publications, Inc, Mineola, New York 2005).

[25] Tobocman, W., Transition amplitudes as sums over histories, *Nuovo Cim.* **3** (1956) 1213–1229.

[26] Smolyanov, O. G., Tokarev, A. G. and Truman, A., Hamiltonian Feynman path integrals via Chernoff formula, *J. Math. Phys.* **43** (2002), 5161-5171.