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To what extent can the Hamiltonian of vortices illustrate the mean field of equilibrium vortices ?[1]

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Abstract

The purpose of this note is to give a review on our recent study on the asymptotic non-degeneracy for the Gel'fand problem in two space dimensions, which suggests a deep link between the Hamiltonian of vortices and the mean field of equilibrium vortices. We also give a new simpler proof of the asymptotic formula for the solutions of the linealized Gel'fand problem, which is used to get the asymptotic non-degeneracy results.

1 Introduction

In this article, we are concerned with *the Gel'fand problem* in two space dimensions:

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. Let $\{\lambda_n\}_{n\in\mathbf{N}}$ be a sequence satisfying $\lambda_n \downarrow 0$ and $u_n = u_n(x)$ be a solution to (1.1) for $\lambda = \lambda_n$. The fundamental facts concerning the asymptotic behavior of u_n are established by the pioneering work of Nagasaki and Suzuki:

Fact 1.1 ([26]). Let $\Sigma_n = \lambda_n \int_\Omega e^{u_n}$. Then $\{\Sigma_n\}$ accumulate to Σ_∞ which is either

(i) 0, (ii) $8\pi m$ ($m \in \mathbf{N}$), or (iii) $+\infty$.

According to these cases, the (sub-)sequence of solutions $\{u_n\}$ behave as follows:

(i) uniform convergence to 0,

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(ii) m-point blow-up, that is, there is a blow-up set $\mathscr{S} = \{\kappa_1, \ldots, \kappa_m\} \subset \Omega$ of distinct m-points such that

$$u_n \longrightarrow u_\infty(x) = 8\pi \sum_{j=1}^{m} G(x, \kappa_j) \quad \text{locally uniformly,} \tag{1.2}$$

where $G(x, y)$ is the Green function of $-\Delta$ under the Dirichlet condition, that is,

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega, \quad G(\cdot, y) = 0 \quad \text{on } \partial\Omega.$$

(iii) entire blow-up, that is, $u_n(x) \longrightarrow +\infty$ for every $x \in \Omega$.

Moreover, the blow-up points κ_j $(j = 1, \cdots, m)$ in case (ii) satisfy the relations

$$\nabla \left(K(x, \kappa_j) + \sum_{1 \le k \le m,\, k \ne j} G(x, \kappa_k) \right) \Bigg|_{x = \kappa_j} = 0, \tag{1.3}$$

where $K(x, y) = G(x, y) - \frac{1}{2\pi} \log |x - y|^{-1}$.

$K(x, y)$ is called the regular part of the Green function $G(x, y)$. Here we set $R(x) = K(x, x)$ and introduce the function

$$H^m(x_1, \ldots, x_m) = \frac{1}{2} \sum_{j=1}^{m} R(x_j) + \frac{1}{2} \sum_{1 \le j,k \le m,\, j \ne k} G(x_j, x_k),$$

which we call *the Hamiltonian.* Since $G(x, y) = G(y, x)$ and $K(x, y) = K(y, x)$, the relation (1.3) means that $\mathscr{S} \in \Omega^m$ is *a critical point* of the function H^m of $2m$-variables. Therefore we are able to say that *the limit function of* $\{u_n\}$ *blows up at a critical point of the Hamiltonian* H^m. Concerning this link between H^m and $\{u_n\}$, recently we get the following result:

Theorem 1.2 ([15]). Suppose $\mathscr{S}$ in (ii) of Theorem 1.1 is a *non-degenerate* critical point of H^m. Then the associated u_n for $n \gg 1$ is a *non-degenerate* critical point of the functional

$$F_{\lambda_n}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda_n \int_\Omega e^u dx.$$

It is easy to see that the equation (1.1) is the Euler-Lagrange equation of the functional F_λ. Therefore we may say that Theorem 1.2 insists deeper links between the functional F_λ and the function H^m than Theorem 1.1.

This kind of result is sometimes called *the asymptotic non-degeneracy* and the above theorem has been already established by Gladiali and Grossi [11] for the case $m = 1$. Several studies also exist for other kind of equations (e.g., [14], [32]), but they also consider the 1-point blow-up cases.

See also [12] for further correspondence concerning the Morse indices between F_λ and H^m for the case $m = 1$, whose extension to general m cases seems next target for us.

2 A short note on H^m

The Hamiltonian function H^m is rather popular in fluid mechanics. This is the Kirchhoff-Routh path function of *vortices* in two-dimensional incompressible non-viscous fluid, see [19, 25, 23] or [10, Chapter 15] for example.

Formally speaking, N-vortices is a set $\{(x_j(t), \Gamma_j)\}_{j=1,\cdots,N} (\subset \Omega \times (\boldsymbol{R} \setminus \{0\}))$ that forms a vorticity field $\omega(x,t) = \sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ satisfying the Euler vorticity equation

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\omega = 0, \tag{2.1}$$

where $\boldsymbol{v} = \nabla^\perp \int_\Omega G(x,y)\omega(y,t)dy$ is the velocity field of the fluid. Here $\nabla^\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)$ and we assumed that Ω is simply connected for simplicity. δ_p is the Dirac measure supported at the point $p\,(\in \Omega)$ and Γ_j is the intensity (circulation) of the vortex at $x_j(t)$. From the Kelvin circulation law, the intensity Γ_j is considered to be conserved. From other several physical considerations, the form $\sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ is considered to be preserved during the time evolution.

It is true that the model "vortices" made many success to understand the motion of real fluid, but it should be noticed that the velocity field $\boldsymbol{v} = \sum_{j=1}^N \Gamma_j \nabla^\perp G(x, x_j(t))$ determined by the vorticity field $\sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ makes the kinetic energy $\frac{1}{2}\int_\Omega |\boldsymbol{v}|^2 dx$ infinite. Moreover it is difficult to understand, even in the sense of distributions, how it satisfies the vorticity equation (2.1). Nevertheless the motion of vortices have been "known" from 19th century. Indeed, they are considered to move according to the following equations:

$$\Gamma_i \frac{dx_i}{dt} = \nabla_i^\perp H^{N,\boldsymbol{\Gamma}}(x_1, \cdots, x_N) \left(= \left(\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,2}}, -\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,1}} \right) \right), \tag{2.2}$$

where

$$H^{N,\mathbf{\Gamma}}(x_1,\cdots,x_N)=\frac{1}{2}\sum_{j=1}^{N}\Gamma_j^2K(x_j,x_j)+\frac{1}{2}\sum_{1\le j,k\le N,j\ne k}\Gamma_j\Gamma_kG(x_j,x_k)$$

and $x_i=(x_{i,1},x_{i,2})$. It is easy to see that the value of $H^{N,\mathbf{\Gamma}}$ is preserved under the time evolution of vortices. Therefore $H^{N,\mathbf{\Gamma}}$ is called the *Hamiltonian.*

H^m referred in Fact 1.1 corresponds to the special case $N=m$ and $\mathbf{\Gamma}=(\Gamma_1,\cdots,\Gamma_m)=(1,\cdots,1)$, that is, m-vortices of *one kind.* Therefore $\mathscr{S}$ in Fact 1.1, that is, the possible blow-up set of the solution sequence of the Gel'fand problem is a critical point of H^m, that is, the Hamiltonian of m-vortices of *one kind.*

3 On the contrary...

It should be remarked that we are able to get the Gel'fand problem from this special Hamiltonian H^m. Indeed suppose all the intensities of vortices is equivalent to some constant Γ. Then the Hamiltonian of m-vortices $H^{m,\mathbf{\Gamma}}$ reduces to Γ^2H^m. In this situation, the Gibbs measure associated to this Hamiltonian is given as follows:

$$\mu^m=\frac{e^{-\tilde\beta\Gamma^2H^m(x_1,\cdots,x_m)}}{\int_{\Omega^m}e^{-\tilde\beta\Gamma^2H^m(x_1,\cdots,x_m)}dx_1\cdots dx_m}dx_1\cdots dx_m,$$

where $\tilde\beta$ is a parameter called the inverse temperature. The canonical Gibbs measure is considered in statistical mechanics to give the possibility of the state for given *energy* H^m under the fixed (inverse) temperature. If $\tilde\beta>0$ (as usual), the low-energy states are likely to occur. On the contrary, if $\tilde\beta<0$ (*negative* temperature cases), the high energy states have more possibility to occur, which is considered to give some reason why there are often observed large-scale long-lived structures in two-dimensional turbulence. One of the most famous example of such structures is the Jupiter's great red spot. The idea to relate such structures to negative temperature states of equilibrium vortices is first proposed by Onsager [29], see [9] for the development of his ideas.

Using the canonical Gibbs measure, we are able to get the probability (density) of the first vortex observed at $x_1\in\Omega$ from

$$\rho^m(x_1)=\int_{\Omega^{m-1}}\mu^m dx_2\cdots dx_m,$$

which is equivalent to every vortices from the symmetry of H^m. Now we assume that total vorticity is equivalent to 1, that is, $\Gamma = \frac{1}{m}$ and suppose $\tilde{\beta} = \tilde{\beta}_\infty \cdot m$ for some fixed $\tilde{\beta}_\infty \in (-8\pi, +\infty)$. Then we get ρ satisfying the following equation at the limit of ρ^m as $m \longrightarrow \infty$:

$$\rho(x) = \frac{e^{-\tilde{\beta}_\infty G\rho(x)}}{\int_\Omega e^{-\tilde{\beta}_\infty G\rho(x)} dx}, \tag{3.1}$$

where G is the Green operator given by $G\rho(x) = \int_\Omega G(x,y)\rho(y)dy$ ([2, Thorem 2.1]). This ρ is called the *mean field* of the equilibrium vortices of one kind. It should be remarked that when the solution of (3.1) is unique, ρ^m weakly converges to ρ, and not unique, to some *superposition* of ρ. The solution of (3.1) is know to unique if Ω is simply connected [31].

These argument was established mathmatically rigorously by Caglioti-Lions-Marchioro-Pulvirenti [2] and Kiessling [17] independently based on the argument developped by Messer-Sphon [24], see also [23, 21]. We note that the equations similar to (3.1) are derived by several authors under several physically reasonable assumptions and arguments in several situations, e.g., the system of vortices of neutral and two kinds, that means there exist same numbers of vortices with positive or negative intensities with the same absolute value, was considered in [16, 30].

We also note that (3.1) means $u := -\tilde{\beta}_\infty G\rho$ and $\beta := -\tilde{\beta}_\infty$ satisfy

$$-\Delta u = \beta \frac{e^u}{\int_\Omega e^u\, dx} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Therefore each solution of (3.2) is linked to that of the Gel'fand problem (1.1) under the relation $\beta / \int_\Omega e^u dx = \lambda$, that is, $\beta = \lambda \int_\Omega e^u dx (= \Sigma)$. The behaviors of the sequences of solutions of (3.2) with $\beta > 0$ (that is, $\tilde{\beta}_\infty = -\beta$ is negative !) are now well studied by several authors [1, 20, 18, 5, 28] etc, see also recent development around this problem for [6, 7, 8, 22] and references therein. Especially based on the argument in [1] (see also [28]), we are able to get a subsequence satisfying $\int_\Omega e^{u_n}\, dx \longrightarrow \infty$ if $\{(u_n, \beta_n)\}$ is a sequence of solutions of (3.2) satisfying that $\{u_n\}$ is unbounded in $L^\infty(\Omega)$ and $\{\beta_n\}$ is bounded. Therefore, the behaviors of unbounded sequence of solutions of (3.2) with bounded $\{\beta_n\}$ reduce to those of (1.1) satisfying $\lambda_n = \beta_n / \int_\Omega e^{u_n}\, dx \longrightarrow 0$. Consequently we return to the situation of Fact 1.1 and we are able to represent the conclusion of Fact 1.1 as follows:

> The mean fields generated by equilibrium vortices of *one kind* with negative temperature converge only to the stationary vortices of *one kind*.

I consider that Theorem 1.2 is a next answer to the question "To what extent can the Hamiltonian of vortices illustrate the mean field of equilibrium vortices ?"

4 Sketch of the proof of Theorem 1.2

Similarly to [11], we prove Theorem 1.2 arguing by contradiction. For this purpose we assume the existence of a sequence $\{v_n\}$ of non-degenerate critical point of F_{λ_n} as $n \longrightarrow \infty$. Using the standard arguments, v_n is a non-trivial solution of the linealized problem of (1.1):

$$-\Delta v = \lambda_n e^{u_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

Without loss of generality we may assume that $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

Taking sufficiently small $\bar{R} > 0$, we may assume that for each κ_j there exists a sequence $\{x_{j,n}\}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_{\bar{R}}(x_{j,n})} u_n(x) \longrightarrow \infty.$$

Then we re-scale u_n and v_n around $x_{j,n}$ as follows:

$$\begin{aligned} \widetilde{u}_{j,n}(\widetilde{x}) &= u_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0), \\ \widetilde{v}_{j,n}(\widetilde{x}) &= v_n(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0), \end{aligned} \tag{4.2}$$

where the scaling parameter $\delta_{j,n}$ is chosen to satisfy $\lambda_n e^{u_n(x_{j,n})}\delta_{j,n}^2 = 1$. From the standard argument based on the estimate concerning the blow-up behavior of u_n [18] (see Corollary 5.3 below) we know $\delta_{j,n} \longrightarrow 0$. Moreover the classification result of the solutions of (1.1) and (4.1) in the whole space [3, 4], there exist $\boldsymbol{a}_j \in \mathbf{R}^2$, $b_j \in \mathbf{R}$ for each j and subsequences of u_n and v_n satisfying

$$\widetilde{u}_{j,n} \longrightarrow \log \frac{1}{\left(1 + \frac{|\widetilde{x}|^2}{8}\right)^2}, \quad \widetilde{v}_{j,n} \longrightarrow \frac{\boldsymbol{a}_j \cdot \widetilde{x}}{8 + |x|^2} + b_j \frac{8 - |\widetilde{x}|^2}{8 + |\widetilde{x}|^2},$$

locally uniformly. We shall show $\boldsymbol{a}_j = \mathbf{0}$ and $b_j = 0$.

The proof is divided into 3 steps:

Step 1: We show the following asymptotic behavior for (a subsequence of) v_n:

$$\frac{v_n}{\lambda_n^{\frac{1}{2}}} \longrightarrow 2\pi \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \tag{4.3}$$

locally uniformly in $\overline{\Omega}\backslash \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)$, where $C_j > 0$ is some constant.
Step 2: Using the fact that $\mathscr{S}$ is a non-degenerate critical point of H^m, we show $\boldsymbol{a}_j = \boldsymbol{0}$ for every j.
Step 3: We show $b_j = 0$ for every j and consequently we show the uniform convergence $v_n \longrightarrow 0$ in Ω, which contradicts $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

We explain Step 1 in detail in the next section. Here we remark further on Step 2, which is based on the simple observation that

$$-\Delta u_{x_i} = \lambda e^u u_{x_i}, \tag{4.4}$$

holds for every solution u of (1.1), that is, $u_{x_i} = \frac{\partial u}{\partial x_i}$ is always a solution of (4.1) except for the boundary condition. Then using the Green identity, we get

$$\frac{1}{\lambda_n^{\frac{1}{2}}} \int_{\partial B_R(\kappa_j)} \left(\frac{\partial}{\partial \nu} (u_n)_{x_i} v_n - (u_n)_{x_i} \frac{\partial}{\partial \nu} v_n \right) d\sigma = 0. \tag{4.5}$$

for every κ_j and sufficiently small $R (> 2\bar{R}) > 0$. From the know asymptotic behaviors (1.2) (4.3) of u_n and v_n, we are able to see that the limit of the above identity (4.5) is a linear combination of the integration

$$I_{ij} := \int_{\partial B_R(z_1)} \left\{ \frac{\partial}{\partial \nu_x} G_{x_i}(x, z_2) G_{y_j}(x, z_3) - G_{x_i}(x, z_2) \frac{\partial}{\partial \nu_x} G_{y_j}(x, z_3) \right\} d\sigma_x.$$

We are able to calculate this as

$$I_{ij} = I_{ij}(z_1, z_2, z_3) = \begin{cases} 0 & (z_1 \neq z_2, \ z_1 \neq z_3), \\ \frac{1}{2} R_{x_i x_j}(z_1) & (z_1 = z_2 = z_3), \\ G_{x_i y_j}(z_1, z_3) & (z_1 = z_2, \ z_1 \neq z_3), \\ G_{x_i x_j}(z_1, z_2) & (z_1 \neq z_2, \ z_1 = z_3), \end{cases} \tag{4.6}$$

which is a *localized* versions of the known integral identity for the Green function:

$$-\int_{\partial\Omega} G_{x_i}(x, y) \frac{\partial}{\partial \nu_x} G_{y_j}(x, y) d\sigma_x = \frac{1}{2} R_{x_i x_j}(y),$$

see [11] for example.

Collecting the limit of (4.5) for all $j = 1, \cdots, m$, we get

$$0 = 16\pi^2 \, \mathrm{Hess} H^m|_{(x_1, \cdots, x_m) = (\kappa_1, \cdots, \kappa_m)} \ {}^t(C_1 \boldsymbol{a}_1, \cdots, C_m \boldsymbol{a}_m).$$

This gives $\boldsymbol{a}_j = \boldsymbol{0}$ from the assumption that $\mathrm{Hess} H^m$ is invertible at $\mathscr{S}$.

5 Another proof of the asymptotic formula

In this section, we give another simpler proof for the asymptotic formula (4.3), which is different from the original one in [15]. First we recall several facts necessary for the proof of (4.3) from Section 4 of [15].

Fact 5.1 ([18], see [15, Theorem 4.1]). For every fixed $0 < R \ll 1$, there exists a constant C independent of j and $n \gg 1$ such that

$$\left| u_n(x) - \log \frac{e^{u_n(x_{j,n})}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_{j,n})} |x - x_{j,n}|^2\right)^2} \right| \leq C \quad \forall x \in B_R(x_{j,n}).$$

Corollary 5.2 ([15, Corollary 4.2]). *For fixed R, there exists a constant C satisfying*

$$\left| \widetilde{u}_{j,n}(\widetilde{x}) - \log \frac{1}{\left(1 + \frac{1}{8}|\widetilde{x}|^2\right)^2} \right| \leq C \quad \forall \widetilde{x} \in B_{\frac{R}{\delta_{j,n}}}(0)$$

for every j.

Corollary 5.3 ([15, Corollary 4.3]). *For each j there exists a constant $C_j > 0$ and a subsequence of $\delta_{j,n}$ satisfying*

$$\delta_{j,n} = C_j \lambda_n^{\frac{1}{2}} + o\left(\lambda_n^{\frac{1}{2}}\right) \quad \text{as } n \longrightarrow \infty.$$

From the Green's representation formula, we divide v_n into several parts as follows[3]:

$$\begin{aligned} v_n(x) &= \int_\Omega G(x,y)\lambda_n e^{u_n(y)} v_n(y)dy \\ &= \sum_{j=1}^m \int_{B_{\bar{R}}(x_{j,n})} G(x,y)\lambda_n e^{u_n(y)} v_n(y)dy \\ &\quad + \int_{\Omega \setminus \cup_{j=1}^m B_{\bar{R}}(x_{j,n})} G(x,y)\lambda_n e^{u_n(y)} v_n(y)dy =: \sum_{j=1}^m \psi_{j,n} + \psi_{0,n}. \end{aligned}$$

Recall that u_n is bounded outside from $\kappa_1, \ldots, \kappa_m$ and we derive

$$\|\psi_{0,n}\|_{L^\infty\left(\Omega \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)\right)} = O(\lambda_n) = o\left(\lambda_n^{\frac{1}{2}}\right).$$

First we show the following pre-formula to (4.3):

[3]We note that in [15] we use a cut-off function to localize the integration around $x_{j,n}$, which seems rather complicated from now. Therefore here we simplify the presentation and, just to be sure, prove several lemmas equivalent to those exist in [15].

Proposition 5.4 (cf. [15, Proposition 4.4][4]). *For each* j,

$$\psi_{j,n}(x) = G(x, x_{j,n})\gamma_{j,n} + 2\pi \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n})\delta_{j,n} + o(\delta_{j,n})$$

uniformly for all $x \in \overline{\Omega} \backslash B_{2\bar{R}}(\kappa_j)$, *where*

$$\gamma_{j,n} = \int_{B_{3\bar{R}}(x_{j,n})} \lambda_n e^{u_n(y)} v_n(y) dy.$$

Proof. For simplicity, we shall omit j in several characters, e.g., ψ_n as $\psi_{j,n}$, $\widetilde{u}_n$ as $\widetilde{u}_{j,n}, \cdots$. Without loss of generality, furthermore, we may assume $\kappa_j = 0$.

For every $x \in \overline{\Omega} \backslash B_{2\bar{R}}(0)$ and $y \in B_{\bar{R}}(x_n)$, Taylor's theorem guarantees

$$G(x, y) = G(x, x_n) + \nabla_y G(x, x_n)(y - x_n) + s(x, \eta, y - x_n)$$

with

$$s(x, \eta, y - x_n) = \frac{1}{2} \sum_{1 \le k, l \le 2} G_{y_k y_l}(x, \eta)(y_k - x_{n,k})(y_l - x_{n,l})$$

and $\eta = \eta(n, y) \in B_{\bar{R}}(x_n)$. Therefore it hold that

$$\begin{aligned}
\psi_n(x) = & \int_{B_{\bar{R}}(x_n)} G(x, y) \lambda_n e^{u_n(y)} v_n(y) dy \\
= & G(x, x_n) \int_{B_{\bar{R}}(x_n)} \lambda_n e^{u_n(y)} v_n(y) dy \\
& + \nabla_y G(x, x_n) \cdot \int_{B_{\bar{R}}(x_n)} (y - x_n) \lambda_n e^{u_n(y)} v_n(y) dy \\
& + \int_{B_{\bar{R}}(x_n)} s(x, \eta, y - x_n) \lambda_n e^{u_n(y)} v_n(y) dy \\
=: & I_1 + I_2 + I_3.
\end{aligned}$$

Obviously it holds that

$$\begin{aligned}
\int_{B_{\bar{R}}(x_n)} \lambda_n e^{u_n(y)} v_n(y) dy &= \int_{B_{3\bar{R}}(x_n)} - \int_{B_{3\bar{R}}(x_n) \backslash B_{\bar{R}}(0)} \\
&= \gamma_n + O(\lambda_n) = \gamma_n + o(\delta_n)
\end{aligned}$$

and consequently

$$I_1 = G(x, x_n)\gamma_n + o(\delta_n).$$

[4] This is essentialy the same one to [15, Proposition 4.4] and we prove here similarly (and slightly simply) based on the argument in the proof of [12, Proposition 6.4]. We note that we have completely different proof for this lemma [27] based on the idea in the proof of [11, Lemma 6 (p.1345)].

uniformly for $x \in \overline{\Omega} \backslash B_{2\bar{R}}(0)$.

On the other hand, we get

$$I_2 = \nabla_y G(x, x_n) \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \delta_n \widetilde{y} e^{\widetilde{u}_n(\widetilde{y})} \widetilde{v}_n d\widetilde{y},$$

by (4.2). From Corollary 5.2, we can apply the dominated convergence theorem:

$$\begin{aligned} &\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \widetilde{y}_j e^{\widetilde{u}_n(\widetilde{y})} \widetilde{v}_n(\widetilde{y}) d\widetilde{y} \\ &\longrightarrow \int_{\mathbf{R}^2} \widetilde{y}_j \left\{ \boldsymbol{a} \cdot \nabla \left(-\frac{1}{4} e^U \right) + b \operatorname{div} \left(\frac{1}{2} \widetilde{y} e^U \right) \right\} d\widetilde{y} = 2\pi a_j, \end{aligned}$$

which implies

$$I_2 = 2\pi \boldsymbol{a} \cdot \nabla_y G(x, x_n) \delta_n + o(\delta_n).$$

Finally we use

$$\sup_{\substack{x \notin B_{2\bar{R}}(0),\, \eta \in B_{\bar{R}}(x_n) \\ 1 \le k,l \le 2}} |G_{y_k y_l}(x, \eta)| \le C < \infty$$

for some constant C independent of $n \gg 1$ to estimate

$$\begin{aligned} |I_3| &\le C \lambda_n \int_{B_{\bar{R}}(x_n)} |y - x_n|^2 e^{u_n(y)} dy \\ &\le C \bar{R}^\rho \lambda_n \int_{B_{\bar{R}}(x_n)} |y - x_n|^{2-\rho} e^{u_n(y)} dy \\ &\le C \bar{R}^\rho \delta_n^{2-\rho} \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} |\widetilde{y}|^{2-\rho} e^{\widetilde{u}_n(\widetilde{y})} dy \end{aligned}$$

for some $\rho \in (0, 1)$. Using Corollary 5.2 again, we get the following limit since $\rho \in (0, 1)$:

$$\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} |\widetilde{y}|^{2-\rho} e^{\widetilde{u}_n(\widetilde{y})} d\widetilde{y} \longrightarrow \int_{\mathbf{R}^2} |\widetilde{y}|^{2-\rho} e^{U(\widetilde{y})} d\widetilde{y} < \infty.$$

Consequently we get $I_3 = O(\delta_n^{2-\rho}) = o(\delta_n)$ and the conclusion. □

Proposition 5.4 and Corollary 5.3 imply the following *pre*-asymptotic formula:

$$v_n(x) = \sum_{j=1}^{m} \gamma_{j,n} G(x, x_{j,n}) + 2\pi \lambda_n^{\frac{1}{2}} \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n}) + o\left(\lambda_n^{\frac{1}{2}}\right) \quad (5.1)$$

uniformly in $x \in \overline{\Omega} \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)$. We also get the similar formula for ∇v_n from the Green representation formula for ∇v_n and consequently the above formula holds in $C^1\left(\overline{\Omega} \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)\right)$.

Similarly, we are able to get the following asymptotic formula for u_n, which is a finer version of (1.2):

$$u_n(x) = \sum_{j=1}^{m} \sigma_{j,n} G(x, x_{j,n}) + o\left(\lambda_n^{\frac{1}{2}}\right) \tag{5.2}$$

in $C^1\left(\overline{\Omega} \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)\right)$, where

$$\sigma_{j,n} = \int_{B_{3\bar{R}}(x_{j,n})} \lambda_n e^{u_n(y)} dy. \tag{5.3}$$

We know $\sigma_{j,n} \longrightarrow 8\pi$ from Fact 1.1, see also Remark 5.6 below.

To get the finer asymptotic formula (4.3) we need to get

$$\gamma_{j,n} = o\left(\lambda_n^{\frac{1}{2}}\right) \tag{5.4}$$

for some subsequence. In the previous paper [15], we prove (5.4) by contradiction argument supposing $\limsup_{n \longrightarrow \infty} \lambda_n^{\frac{1}{2}} / |\gamma_{j,n}| < \infty$. In this note, we prove (5.4) by a completely different direct argument using bi-linearly generalized version of the Pohozaev identity:

Proposition 5.5. *For every $p \in \mathbf{R}^2$, $R > 0$, and f, $g \in C^2\left(\overline{B_R(p)}\right)$, the following identity holds:*

$$\begin{aligned} &\int_{B_R(p)} \{[(x-p) \cdot \nabla f] \Delta g + \Delta f [(x-p) \cdot \nabla g]\} \\ &\quad = R \int_{\partial B_R(p)} \left(2 \frac{\partial f}{\partial \nu} \frac{\partial g}{\partial \nu} - \nabla f \cdot \nabla g\right). \end{aligned} \tag{5.5}$$

Proof.

$$\begin{aligned} &\int_{B_R(p)} \{[(x-p) \cdot \nabla f] \Delta g + \Delta f [(x-p) \cdot \nabla g]\} \\ &= \int_{\partial B_R(p)} \left\{[(x-p) \cdot \nabla f] \frac{\partial g}{\partial \nu} + \frac{\partial f}{\partial \nu}[(x-p) \cdot \nabla g]\right\} \\ &\quad - \int_{B_R(p)} \{\nabla [(x-p) \cdot \nabla f] \cdot \nabla g + \nabla f \cdot \nabla [(x-p) \cdot \nabla g]\} \\ &= R \int_{\partial B_R(p)} 2 \frac{\partial f}{\partial \nu} \frac{\partial g}{\partial \nu} - \int_{B_R(p)} [2 \nabla f \cdot \nabla g + (x-p) \cdot \nabla (\nabla f \cdot \nabla g)] \end{aligned}$$

$$= R\int_{\partial B_R(p)} 2\frac{\partial f}{\partial \nu}\frac{\partial g}{\partial \nu} - \int_{\partial B_R(p)} [(x-p)\cdot\nu]\nabla f\cdot\nabla g$$
$$= R\int_{\partial B_R(p)} \left(2\frac{\partial f}{\partial \nu}\frac{\partial g}{\partial \nu} - \nabla f\cdot\nabla g\right).$$

□

Applying $p = x_{j,n}$, $R = 3\bar{R}$, $f = u_n$, and $g = v_n$ to the lefthand side of (5.5), we get

$$\int_{B_{3\bar{R}}(x_{j,n})} \{[(x-x_{j,n})\cdot\nabla u_n]\Delta v_n + \Delta u_n[(x-x_{j,n})\cdot\nabla v_n]\}$$
$$= \int_{B_{3\bar{R}}(x_{j,n})} \{[(x-x_{j,n})\cdot\nabla u_n](-\lambda_n e^{u_n} v_n) - \lambda_n e^{u_n}[(x-x_{j,n})\cdot\nabla v_n]\}$$
$$= -\int_{B_{3\bar{R}}(x_{j,n})} (x-x_{j,n})\cdot\nabla\left(\lambda_n e^{u_n} v_n\right)$$
$$= -\int_{\partial B_{3\bar{R}}(x_{j,n})} [(x-x_{j,n})\cdot\nu]\lambda_n e^{u_n} v_n + 2\int_{B_{3\bar{R}}(x_{j,n})} \lambda_n e^{u_n} v_n$$
$$= 2\gamma_{j,n} + O(\lambda_n) = 2\gamma_{j,n} + o\left(\lambda_n^{\frac{1}{2}}\right).$$

Therefore we get the following representation formula of $\gamma_{j,n}$ from (5.5):

$$\gamma_{j,n} = \frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} \left(2\frac{\partial u_n}{\partial \nu}\frac{\partial v_n}{\partial \nu} - \nabla u_n\cdot\nabla v_n\right) + o\left(\lambda_n^{\frac{1}{2}}\right). \tag{5.6}$$

We use this formula (5.6) to get (5.4). We note that the asymptotic formulas (5.1)(5.2) hold on $\partial B_{3\bar{R}}(x_{j,n})$ for $n \gg 1$ and therefore we are able to insert them into (5.6). To this purpose, we extract singular parts around $x_{j,n}$ from the asymptotic formulas (5.1) and (5.2).

$$\nabla u_n(x) = -\frac{\sigma_{j,n}}{2\pi}\cdot\frac{\nu}{3\bar{R}} + \nabla h^1_{j,n}(x) + o\left(\lambda_n^{\frac{1}{2}}\right) \tag{5.7}$$

$$\begin{aligned}\nabla v_n(x) = &-\frac{\gamma_{j,n}}{2\pi}\cdot\frac{\nu}{3\bar{R}} + C_j\frac{\boldsymbol{a}_j - 2(\boldsymbol{a}_j\cdot\nu)\nu}{(3\bar{R})^2}\lambda_n^{\frac{1}{2}} \\ &+ \nabla h^2_{j,n}(x) + \lambda_n^{\frac{1}{2}}\nabla h^3_{j,n}(x) + o\left(\lambda_n^{\frac{1}{2}}\right)\end{aligned} \tag{5.8}$$

on $\partial B_{3\bar{R}}(x_{j,n})$, where

$$h^1_{j,n}(x) = \sigma_{j,n}K(x, x_{j,n}) + \sum_{1\le i\le m, i\ne j} \sigma_{i,n}G(x, x_{j,n})$$
$$h^2_{j,n}(x) = \gamma_{j,n}K(x, x_{j,n}) + \sum_{1\le i\le m, i\ne j} \gamma_{i,n}G(x, x_{j,n}),$$
$$h^3_{j,n}(x) = 2\pi C_j \boldsymbol{a}_j \cdot \nabla_y K(x, x_{j,n}) + 2\pi \sum_{1\le i\le m, i\ne j} C_i \boldsymbol{a}_i \nabla_y G(x, x_{j,n}).$$

Inserting these into the representation formula (5.6) of $\gamma_{j,n}$, we get the conclusion. Indeed, we have

$$\begin{aligned}
&\frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} 2\frac{\partial u_n}{\partial \nu}\frac{\partial v_n}{\partial \nu} \\
&= \frac{\sigma_{j,n}\gamma_{j,n}}{2\pi} + \frac{\sigma_{j,n}C_j\lambda_n^{\frac{1}{2}}}{2\pi(3\bar{R})^2}\int_{\partial B_{3\bar{R}}(x_{j,n})} \boldsymbol{a}_j\cdot\nu - \frac{\sigma_{j,n}}{2\pi}\int_{\partial B_{3\bar{R}}(x_{j,n})}\frac{\partial\left(h^2_{j,n} + \lambda_n^{\frac{1}{2}}h^3_{j,n}\right)}{\partial\nu} \\
&\quad - \frac{\gamma_{j,n}}{2\pi}\int_{\partial B_{3\bar{R}}(x_{j,n})}\frac{\partial h^1_{j,n}(x)}{\partial\nu} - \frac{C_j\lambda_n^{\frac{1}{2}}}{3\bar{R}}\int_{\partial B_{3\bar{R}}(x_{j,n})}(\boldsymbol{a}_j\cdot\nu)\frac{\partial h^1_{j,n}(x)}{\partial\nu} \\
&\quad + \frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} 2\frac{\partial h^1_{j,n}(x)}{\partial\nu}\frac{\partial\left(h^2_{j,n} + \lambda_n^{\frac{1}{2}}h^3_{j,n}\right)}{\partial\nu} + o\left(\lambda_n^{\frac{1}{2}}\right) \\
&= \frac{\sigma_{j,n}\gamma_{j,n}}{2\pi} - \frac{C_j\lambda_n^{\frac{1}{2}}}{3\bar{R}}\int_{\partial B_{3\bar{R}}(x_{j,n})}(\boldsymbol{a}_j\cdot\nu)\frac{\partial h^1_{j,n}(x)}{\partial\nu} \\
&\quad + \frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} 2\frac{\partial h^1_{j,n}(x)}{\partial\nu}\frac{\partial\left(h^2_{j,n} + \lambda_n^{\frac{1}{2}}h^3_{j,n}\right)}{\partial\nu} + o\left(\lambda_n^{\frac{1}{2}}\right)
\end{aligned}$$

from the divergence formula and the fact that $h^i_{j,n}$ $(i = 1, 2, 3)$ is harmonic in $B_{3\bar{R}}(x_{j,n})$.

Similarly, we have

$$\begin{aligned}
&\frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} \nabla u_n\cdot\nabla v_n \\
&= \frac{\sigma_{j,n}\gamma_{j,n}}{4\pi} + \frac{C_j\lambda_n^{\frac{1}{2}}}{6\bar{R}}\int_{\partial B_{3\bar{R}}(x_{j,n})} \boldsymbol{a}_j\cdot\nabla h^1_{j,n} - \frac{C_j\lambda_n^{\frac{1}{2}}}{3\bar{R}}\int_{\partial B_{3\bar{R}}(x_{j,n})}(\boldsymbol{a}_j\cdot\nu)\frac{\partial h^1_{j,n}(x)}{\partial\nu} \\
&\quad + \frac{3\bar{R}}{2}\int_{\partial B_{3\bar{R}}(x_{j,n})} \nabla h^1_{j,n}\cdot\nabla\left(h^2_{j,n} + \lambda_n^{\frac{1}{2}}h^3_{j,n}\right) + o\left(\lambda_n^{\frac{1}{2}}\right).
\end{aligned}$$

We note that

$$\int_{\partial B_{3\bar{R}}(x_{j,n})} 2\frac{\partial h^1_{j,n}(x)}{\partial \nu}\frac{\partial\left(h^2_{j,n}+\lambda_n^{\frac{1}{2}}h^3_{j,n}\right)}{\partial \nu}-\int_{\partial B_{3\bar{R}}(x_{j,n})} \nabla h^1_{j,n}\cdot\nabla\left(h^2_{j,n}+\lambda_n^{\frac{1}{2}}h^3_{j,n}\right)=0$$

from the bi-linear Pohozaev identity (5.5) because $h^i_{j,n}$ $(i=1,\,2,\,3)$ is harmonic. We also note that

$$\begin{aligned}\frac{C_j\lambda_n^{\frac{1}{2}}}{6\bar{R}}\int_{\partial B_{3\bar{R}}(x_{j,n})}\boldsymbol{a}_j\cdot\nabla h^1_{j,n} &= \frac{C_j\lambda_n^{\frac{1}{2}}}{6\bar{R}}\left|\partial B_{3\bar{R}}\left(x_{j,n}\right)\right|\boldsymbol{a}_j\cdot\nabla h^1_{j,n}(x_{j,n})\\ &= \pi\lambda_n^{\frac{1}{2}}\boldsymbol{a}_j\cdot\nabla h^1_{j,n}(x_{j,n})=o\left(\lambda_n^{\frac{1}{2}}\right)\end{aligned}$$

from the mean value theorem for harmonic functions and the fact that

$$\nabla h^1_{j,n}(x_{j,n})\longrightarrow 8\pi\nabla\left(K(x,\kappa_j)+\sum_{1\le i\le m,\,i\neq j}G(x,\kappa_i)\right)\Bigg|_{x=\kappa_j}=0$$

from (1.3).

Consequently we get

$$\gamma_{j,n}=\frac{\sigma_{j,n}\gamma_{j,n}}{4\pi}+o\left(\lambda_n^{\frac{1}{2}}\right),$$

which means

$$\gamma_{j,n}=\frac{1}{1-\frac{\sigma_{j,n}}{4\pi}}o\left(\lambda^{\frac{1}{2}}\right)=o\left(\lambda_n^{\frac{1}{2}}\right) \tag{5.9}$$

since $\sigma_{j,n}\longrightarrow 8\pi$.

Using (5.9), we get the conclusion from the pre-formula (5.1) holds on $C^1\left(\overline{\Omega}\backslash\cup_{j=1}^m B_{2\bar{R}}\left(\kappa_j\right)\right)$.

Remark 5.6. Similar argument is also applicable to get the behavior of $\sigma_{j,n}$. Indeed, applying $p=x_{j,n}$, $R=3\bar{R}$, and $f=g=u_n$ to the lefthand side of (5.5), we get

$$\begin{aligned}&\int_{B_{3\bar{R}}(x_{j,n})} 2[(x-x_{j,n})\cdot\nabla u_n]\Delta u_n=-2\int_{B_{3\bar{R}}(x_{j,n})}(x-x_{j,n})\cdot\nabla\left(\lambda_n e^{u_n}\right)\\ &=-2\int_{\partial B_{3\bar{R}}(x_{j,n})}[(x-x_{j,n})\cdot\nu]\lambda_n e^{u_n}+4\int_{B_{3\bar{R}}(x_{j,n})}\lambda_n e^{u_n}\\ &=4\sigma_{j,n}+O(\lambda_n)=4\sigma_{j,n}+o\left(\lambda_n^{\frac{1}{2}}\right).\end{aligned}$$

Therefore we get the following representation formula of $\sigma_{j,n}$ from (5.5):

$$\sigma_{j,n} = \frac{3\bar{R}}{4} \int_{\partial B_{3\bar{R}}(x_{j,n})} \left(2 \left| \frac{\partial u_n}{\partial \nu} \right|^2 - |\nabla u_n|^2 \right) + o\left(\lambda_n^{\frac{1}{2}}\right). \tag{5.10}$$

Inserting (5.7) into this representation formula of $\sigma_{j,n}$, we get

$$\sigma_{j,n} = \frac{\sigma_{j,n}^2}{8\pi} + o\left(\lambda_n^{\frac{1}{2}}\right),$$

which means

$$\sigma_{j,n} = 8\pi + o\left(\lambda_n^{\frac{1}{2}}\right) \tag{5.11}$$

if only we show $\liminf_{n \longrightarrow \infty} \sigma_{j,n} > 0$ previously. We note that from the sharp estimate established in [5] it seems to hold that $\sigma_{j,n} = 8\pi + O(\lambda_n)$. Therefore the above estimate (5.11) is weaker than that in [5], but it is sufficient for the analysis in other relating problems [13].

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