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TURBULENT CASCADES IN PHYSICAL SCALES OF 3D INCOMPRESSIBLE FLUID FLOWS

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ABSTRACT. The purpose of this survey article is to present a recent progress in the study of turbulent cascades in *physical scales* of 3D incompressible fluid flows, as well as to indicate some directions of the current research.

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1. INTRODUCTION

Despite many advances in analysis, modeling, and experimental study of 3D incompressible fluid flows, the problem of understanding turbulence is still largely open. Fundamentally, the phenomenological theory Kolmogorov introduced in 1940s ([Ko41-1, Ko41-2]) remains the most influential due to its simplicity and overall good fit with experimental data ([GSM62, ZTMW94]). The challenges however remain the same: to connect the phenomenology to the underlying laws of motion and to understand mechanisms leading to formation, persistence, and disappearance of turbulent regions within a fluid flow (see e.g. [Fr95, FMRT, DP01]). Mathematically, these problems are linked to open questions in the theory of the Navier-Stokes (NSE) and Euler (EE) equations; particularly relevant are the issues of uniqueness and regularity as well as qualitative behavior of the solutions.

Naturally, this is a very active area of research in applied mathematics. On one hand, improved hardware capabilities and advances in computational techniques made it possible to perform numerical simulations of turbulent flow at very high resolutions (see, e.g. [DYS08, WK09, CBBL10]), on the other, new analytical tools allowed proofs of stronger regularity criteria and better understanding of scaling properties of solutions to NSE and EE (e.g. [L-R02, ISS03, ChKaLe07, Ku08, GrGu10-2, DLS10, CS11]), and as a consequence, led to partial confirmations of important tenets of turbulence theories directly from the equations of motion (see, e.g. [FMRT01, CCFS08, EA09, RDD11]). Most of the analytical approaches use Harmonic Analysis techniques in Fourier scales. This framework proved very effective in establishing new global regularity properties of NSE and EE and led to better understanding of homogeneous isotropic turbulence phenomenology in the context of periodic and whole space domains.

Another approach is to study the role local geometry and coherent structures play in turbulent behavior as well as possible singularity formations ([Ch94, MKO94, AK98, LM00, CF01, Gr01, MB02, CFL04, Shvy09, GM11]), for example, it was established that Hölder coherence of the vorticity direction in regions of high fluid intensity prevents singularity formation ([CoFe93, daVeigaBe02, Gr09]). Up to now however, there was no effective way to consider turbulent phenomenology in this context. The main obstacle seemed to be the lack of effective (amenable to analysis) definition of a scale in a physical domain setting.

In this article, we survey a recent approach that allows understanding turbulent phenomenology in the context of physical scales – without assuming homogeneity and isotropy – by directly zooming in to the regions of intense fluid activity.

2. ENSEMBLE AVERAGES; PHYSICAL SCALES

The ensemble averaging process to be described in this section in some detail was introduced in a recent study of turbulent cascades in physical scales of 3D incompressible flows [DaGr11-1, DaGr11-2, DaGr11-3, DaGr11-5]. However, it is a mathematical entity of its own, and it is instructive to present it as such.

A natural way of actualizing a concept of scale in a PDE model is to measure distributional derivatives of a quantity with respect to the scale. Let x_0 be in $B(0, R_0)$ (R_0 being the *integral scale*, $B(0, 2R_0)$ contained in Ω where Ω is the global spatial domain) and $0 < R \leq R_0$. Considering a locally integrable physical density of interest f on a ball of radius $2R$, $B(x_0, 2R)$, a *local physical scale* R – associated to the point x_0 – is realized via bounds on distributional derivatives of f where a test function ψ is a refined – smooth, non-negative, equal to 1 on $B(x_0, R)$ and featuring optimal bounds on the derivatives over the outer R-layer – cut-off function on $B(x_0, 2R)$. More explicitly, $|(D^\alpha f, \psi)| \leq \int_{B(x_0,2R)} |f||D^\alpha \psi| \leq \left(c(\alpha)\frac{1}{R^{|\alpha|}}|f|, \psi^{\delta(\alpha)}\right)$ for some $c(\alpha) > 0$ and $\delta(\alpha)$ in $(0, 1)$. This – via duality – leads to various estimates on distributional derivatives of f in L^p, reminiscent of Bernstein inequalities in the Littlewood-Paley decomposition of a tempered distribution.

The precise requirements on the cut-offs are as follows. The spatiotemporal cut-off ϕ is of a product form, $\phi = \phi_{x_0,R,T} = \psi\eta$, where $\eta = \eta_T(t) \in \mathcal{D}(0,T)$ and $\psi = \psi_{x_0,R}(x) \in \mathcal{D}(B(x_0,2R))$ satisfying $0 \le \eta \le 1, \eta = 1$ on $(T/3, 2T/3)$, $\frac{|\eta'|}{\eta^{\rho_1}} \le \frac{C_0}{T}$ and $0 \le \psi \le 1$, $\psi = 1$ on $B(x_0,R)$, $\frac{|\nabla\psi|}{\psi^{\rho_2}} \le \frac{C_0}{R}$, $\frac{|\Delta\psi|}{\psi^{2\rho_2-1}} \le \frac{C_0}{R^2}$, for some $\frac{1}{2} < \rho_1, \rho_2 < 1$. Sometimes, the above conditions on η are replaced by $\eta = \eta_T(t) \in C^\infty(0,T)$ supplemented with $0 \le \eta \le 1$, $\eta = 0$ on $(0, T/3)$, $\eta = 1$ on $(2T/3, T)$, $\frac{|\eta'|}{\eta^{\rho_1}} \le \frac{C_0}{T}$ (e.g., in the study of the 3D enstrophy cascade). The case $x_0 = 0$ and $R = R_0$ corresponds to the integral domain cut-off ϕ_0, $\phi_0 = \eta\psi_0$. (There are additional technical conditions for x_0 near the boundary of $B(0,R_0)$, c.f. [DaGr11-1].)

A *physical scale* R – associated to the integral domain $B(0,R_0)$ – is realized via suitable ensemble-averaging of the localized quantities with respect to '(K_1,K_2)-covers at scale R'.

Let K_1 and K_2 be two positive integers, and $0 < R \le R_0$. A cover $\{B(x_i,R)\}_{i=1}^n$ of the integral domain $B(0,R_0)$ is a (K_1,K_2)-*cover at scale* R if

$$\left(\frac{R_0}{R}\right)^3 \le n \le K_1\left(\frac{R_0}{R}\right)^3,$$

and any point x in $B(0,R_0)$ is covered by at most K_2 balls $B(x_i,2R)$. The parameters K_1 and K_2 represent the maximal *global* and *local multiplicities*, respectively.

For a physical density of interest f, consider time-averaged, per unit mass – spatially localized to the cover elements $B(x_i,R)$ – local quantities $\hat{f}_{x_i,R,T}$,

$$\hat{f}_{x_i,R,T} = \frac{1}{T}\int_0^T \frac{1}{R^3}\int_{B(x_i,2R)} f(x,t)\phi^\delta_{x_i,R,T}(x,t)\,dx\,dt$$

for some $0 < \delta \le 1$, and denote by $\langle F\rangle_R$ the *ensemble average* given by

$$\langle F\rangle_R = \frac{1}{n}\sum_{i=1}^n \hat{f}_{x_i,R,T}\,.$$

The key feature of the ensemble averages $\{\langle F\rangle_R\}_{0<R\le R_0}$ is that $\langle F\rangle_R$ being *stable*, i.e., nearly independent on a particular choice of the cover – with the fixed parameters K_1 and K_2 – indicates there are no significant sign-fluctuations of the density f at scales comparable or greater than R. On the other hand, if f does exhibit significant sign-fluctuations at scales comparable or greater than R, suitable rearrangements of the cover elements up to the maximal multiplicities will result in the range of $\langle F\rangle_R$ containing an interval of the form $(-M, M)$ for a large M.

Consequently, for an *a priori* sign-varying density, the ensemble averaging process acts as a *coarse detector of the sign-fluctuations at scale R*. (The larger the maximal multiplicities K_1 and K_2, the finer detection.)

As expected, for a non-negative density f, all the averages are comparable to each other throughout the full range of scales R, $0 < R \le R_0$. In particular, they are all comparable to the simple average over the integral domain; $\frac{1}{K_*}F_0 \le \langle F\rangle_R \le K_* F_0$ for all $0 < R \le R_0$, where $F_0 = \frac{1}{T}\int \frac{1}{R_0^3}\int f(x,t)\phi_0^\delta(x,t)\,dx\,dt$, and $K_* = K_*(K_1,K_2) > 1$. More generally, the above bound holds for any non-negative distribution (integration being replaced by duality); in fact, this was a key technical ingredient in the proof of existence of 3D inviscid cascade presented in [DaGr11-2].

3. TURBULENT CASCADES IN PHYSICAL SCALES

One of the hallmarks of turbulence phenomenology pioneered by Kolmogorov and Onsager in the 1940's [Ko41-1, Ko41-2, Ko41-3, On45, On49] is existence of the inertial range, the range of scales in which the dissipation effects are strongly dominated by the nonlinear (inertial) transfer of averaged energy from larger to smaller scales. This process is also referred to as 'energy cascade'. In addition, the cascade is thought to be highly local-in-scale, i.e., the averaged energy flux at the given scale is supposed to be significantly correlated only with the fluxes at the nearby scales. A great mathematical challenge has been to establish existence and locality of turbulent cascades directly from the mathematical model, namely the 3D NSE.

Prior to [DaGr11-1], the only rigorous mathematical work on existence of the energy cascade was the work by Foias, Manley, Rosa and Temam [FMRT01] in which the authors obtained existence of the energy cascade in the Fourier space – i.e., in the wavenumbers – in the setting of infinite-time averages of Leray solutions to the 3D NSE satisfying the global energy inequality. The averaging process utilized in [FMRT01] was global-in-scale and did not lead to a proof of locality. On the other hand, mathematical evidence of locality was first presented by L'vov and Falkovich [LF92], and later by Eyink [E05, EA09] and Cheskidov, Constantin, Friedlander and Shvydkoy [CCFS08].

In the case of 'decaying turbulence' (zero driving force, non-increasing global energy), [DaGr11-1] presented a proof of both existence and locality of the energy cascade in the physical scales of the flow, under an assumption plausible in the regions of intense fluid activity. In the language of turbulence, the condition translates to the requirement that Taylor micro-scale be dominated by the integral scale of the flow. The result is obtained in the realm of distributional solutions to the 3D NSE satisfying the local energy inequality. The main idea of the proof was to first 'discretize' the flow by localizing it to spatial balls at the given scale, and then 'reconstruct' it via a suitably defined ensemble-averaging process – described in the previous section – capable of detecting significant sign-fluctuations of the physical density of interest (the case in point being the energy flux); the key device in the reconstruction process being the local energy inequality. This furnished the first proof of existence of the energy cascade in physical space, as well as the only mathematical setting in which both existence and locality of the cascade were derived simultaneously from the 3D NSE.

Let x_0 be in $B(0, R_0)$ (R_0 being the integral scale, $B(0, 2R_0) \subset \Omega$ where Ω is the global spatial domain) and $0 < R \le R_0$, and let u be a weak solution to the 3D NSE on $\Omega \times (0, 2T)$ satisfying the *local energy inequality*,

$$\nu \iint |\nabla u|^2 \phi \le \iint \frac{1}{2}|u|^2(\partial_t \phi + \nu \triangle \phi) + \iint \Big(\frac{1}{2}|u|^2 + p\Big) u \cdot \nabla \phi, \tag{3.1}$$

for any nonnegative test function ϕ (e.g., a suitable weak solution). Physically, the term $\nu \iint |\nabla u|^2 \phi$ represents the local spatiotemporal energy dissipation rate due to viscosity, while a (non-negative) defect in the local energy inequality due to possible singularities/lack of smoothness can be interpreted as the local spatiotemporal anomalous dissipation. For a refined cut-off function $\phi = \phi_{x_0,R,T}$ on $B(x_0, 2R) \times (0, 2T)$ denote by $\Phi_{x_0,R}$ a local *inward* flux around x_0 at scale R,

$$\Phi_{x_0,R}(t) = -\int \Big((u \cdot \nabla)u + \nabla p\Big) \cdot u\,\phi\,dx = \int \Big(\frac{1}{2}|u|^2 + p\Big) u \cdot \nabla \phi\,dx, \tag{3.2}$$

by $\hat{\Phi}_{x_0,R,T}$ a local time-averaged flux, by $\hat{e}_{x_0,R,T}$ a local time-averaged energy and by $\hat{\varepsilon}_{x_0,R,T}$ a local time-averaged total (viscosity plus anomalous) energy dissipation rate, all per unit mass,

$$\hat{\Phi}_{x_0,R,T} = \frac{1}{TR^3}\int \Phi_{x_0,R}(t), \quad \hat{e}_{x_0,R,T} = \frac{1}{TR^3}\iint \frac{1}{2}|u|^2\phi^{2\delta-1}, \tag{3.3}$$
$$\hat{\varepsilon}_{x_0,R,T} = \frac{1}{TR^3}\iint \frac{1}{2}|u|^2(\partial_t\phi + \nu\triangle\phi) + \hat{\Phi}_{x_0,R,T}\,.$$

Also denote by e and ε the time-averaged energy and total energy dissipation rate per unit mass associated to the integral domain on $(0, 2T)$, $e_0 = \hat{e}_{0,R_0,T}$ and $\varepsilon_0 = \hat{\varepsilon}_{0,R_0,T}$. Then, the main result describing the universality of the averaged flux *per unit mass* $\langle\Phi\rangle_R$ throughout the inertial range, and implying existence of the energy cascade in decaying turbulence (in the viscous case) can be stated as follows [DaGr11-1, DaGr11-3].

Theorem 3.1. *Let $\gamma \in (0,1)$ and c, K suitable constants depending only on the cover parameters K_1 and K_2. Define a modified Taylor length scale τ_0 by $\tau_0 = \left[\left(\frac{R_0^2}{T} + \nu\right)\frac{e_0}{\varepsilon_0}\right]^{1/2}$, and assume that $\tau_0^2 < \frac{\gamma}{c}R_0^2$. Then,*

$$\frac{1}{K}(1-\gamma)\,\varepsilon \le \langle\Phi\rangle_R \le K(1+\gamma)\,\varepsilon \tag{3.4}$$

for all R inside the inertial range determined by $\left[\left(\frac{\tau_0^2}{\frac{\gamma}{c}(R_0^2+\nu T) - \tau_0^2}\,\nu T\right)^{1/2}, R_0\right]$.

Denoting the averaged flux at scale R by $\langle\tilde{\Phi}\rangle_R$, $\langle\tilde{\Phi}\rangle_R = R^3\langle\Phi\rangle_R$, the following manifestation of locality of the averaged flux throughout the inertial range is immediate, $\frac{1}{K^2}\frac{1-\gamma}{1+\gamma}\left(\frac{R}{r}\right)^3 \le \frac{\langle\tilde{\Phi}\rangle_R}{\langle\tilde{\Phi}\rangle_r} \le K^2\frac{1+\gamma}{1-\gamma}\left(\frac{R}{r}\right)^3$. In particular, both the infrared and the ultraviolet locality propagates *exponentially* along the *dyadic scale*, as predicted by the phenomenology, $\frac{1}{K^2}\frac{1-\gamma}{1+\gamma}2^{-3k} \le \frac{\langle\tilde{\Phi}\rangle_R}{\langle\tilde{\Phi}\rangle_{2^kR}} \le K^2\frac{1+\gamma}{1-\gamma}2^{-3k}$.

Onsager in 1949 wrote that "...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without aid of the viscosity, is available" [On49]. More precisely, Onsager conjectured that the minimal spatial regularity of (weak) solutions to the 3D Euler equations (the inviscid model) needed to conserve the energy is $\left(\frac{1}{3}\right)^+$, and that in the case the energy is not conserved, the 'anomalous dissipation' – the dissipation due to the lack of regularity – triggers the energy cascade which then continues indefinitely all the way to the zero scale. The main mathematical works confirming Onsager's $\left(\frac{1}{3}\right)^+$-criticality conjecture are the works by Eyink [E94], Constantin, E. and Titi [CET94], Duchon and Robert [DR00], and Cheskidov, Constantin, Friedlander and Shvydkoy [CCFS08]. On the other hand, there had been no rigorous mathematical works showing that the anomalous dissipation is indeed capable of triggering the cascade.

Adopting the method introduced in [DaGr11-1] to 3D inviscid flows, [DaGr11-2] presented – under an assumption that the anomalous dissipation in the region of interest is strong enough with respect to the energy – a proof of both existence and locality of the energy cascade in the 3D Euler flows extending *ad infinitum* confirming the dynamical part of the Onsager's conjecture. As a matter of fact – in the case of a spatially isolated singularity – provided the strict energy inequality holds on some neighborhood of the singular curve, the cascade condition will be satisfied on any small enough (in the spatial coordinates) neighborhood of the curve.

Let u be a (hypothetical) weak L^3 in the space-time solution to the 3D Euler satisfying the local energy inequality,

$$\iint \frac{1}{2}|u|^2\partial_t\phi + \iint\left(\frac{1}{2}|u|^2 + p\right)u\cdot\nabla\phi \ge 0. \tag{3.5}$$

Keeping the notation in line with the viscous case – and focusing on $B(x_0, R)$ – denote by $\hat{\epsilon}_{x_0,R,T}$ the local time-averaged anomalous dissipation per unit mass associated to $B(x_0, R)$, and define $\langle\Phi\rangle_R$, e, and ε as in the viscous case, setting $\nu = 0$. In the same spirit, define the anomalous inviscid Taylor scale τ_0 by $\tau_0 = \left(\frac{R_0^2}{T}\frac{e_0}{\varepsilon_0}\right)^{1/2}$. Then, the main result of [DaGr11-2] implying existence of the inviscid energy cascade in decaying turbulence states the following.

Theorem 3.2. *Let $\gamma \in (0,1)$ and c, K as in Theorem 3.1. Assume that $\tau_0^2 < \frac{\gamma}{c} R_0^2$ Then,*

$$\frac{1}{K}(1-\gamma)\,\varepsilon_0 \le \langle\Phi\rangle_R \le K(1+\gamma)\,\varepsilon_0 \tag{3.6}$$

for all R inside the inertial range determined by $(0,\ R_0]$.

Note that in the inviscid case, once the energy cascade commences, it continues indefinitely towards the zero-scale as predicted by Onsager. Consequently, the ultraviolet locality in the inviscid case is more pronounced.

A short follow-up note [DaGr11-3] presented the first scale-to-scale evidence of 'dissipation anomaly' in physical scales of 3D turbulent incompressible flows. Dissipation anomaly – non-vanishing of the averaged energy dissipation rate in the infinite Reynolds number limit – plays a key role in both empirical and phenomenological turbulence; in fact, it is often referred to as 'zeroth law of turbulence'.

Let $\{u^\nu\}$ be a family of weak solutions to the 3D NSE satisfying the local energy inequality converging strongly in $L^3(\mathbb{R}^3 \times (0, 2T))^3$ to a (hypothetical) weak L^3 solution to the 3D Euler u. Recall that Duchon and Robert [DR00] noticed that in this case the inviscid limit u also satisfies the local energy inequality (3.5). Henceforth, a superscript ν will be used to denote the energy, the flux and the (total) energy dissipation rate corresponding to a solution u^ν, while the quantities related to u will be denoted as in the inviscid section. The main result of [DaGr11-3] reads as follows.

Theorem 3.3. *Suppose that the inviscid cascade condition, $\tau_0 < \sqrt{\gamma/c}R_0$, holds for the inviscid limit u. Then, for every R^*, $0 < R^* < R_0$, there exists ν^*, $\nu^* > 0$, such that for any ν, $0 < \nu \le \nu^*$,*

$$\frac{1}{K_\gamma}\varepsilon_0 \le \langle\Phi^\nu\rangle_R \le K_\gamma\varepsilon_0 \tag{3.7}$$

throughout the inertial range determined by $[R^*, R_0]$, where $K_\gamma = \dfrac{K}{1-\gamma}$.

It is plain to verify that for all ν, $0 < \nu < \nu^*$, the energy cascade $\frac{1}{K}(1-\gamma)\,\varepsilon_0^\nu \le \langle\Phi^\nu\rangle_R \le K(1+\gamma)\,\varepsilon_0^\nu$ holds inside the ever-expanding inertial ranges $\left[\left(\frac{(\tau_0^\nu)^2}{\frac{\gamma}{c}(R_0^2+\nu T)-(\tau_0^\nu)^2}\,\nu T\right)^{1/2},\ R_0\right]$. This provides an *intrinsic version of Theorem* 2.3 – without an explicit reference to the vanishing viscosity limit (of course, $\varepsilon_0^\nu \to \varepsilon_0$).

The latest work in the series [DaGr11-5] merged the ideas and techniques from the previous work on localization of the geometric depletion of the nonlinearity in the 3D NSE [GrZh06, Gr09, GrGu10-1] with the ideas and techniques exposed in [DaGr11-1, DaGr11-2, DaGr11-3].

A role of the coherent vortex structures in turbulent flows was recognized as early as the 1500's in Leonardo da Vinci's "deluge" drawings. On the other hand, Kolmogorov's K41 phenomenology does not discern geometric structures; the K41 eddies are essentially amorphous. As stated by Frisch in his book *Turbulence*, The Legacy of A.N. Kolmogorov [Fr95], "Half a century after Kolmogorov's work on the statistical theory of fully developed turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno." A discrete statistical theory of 3D vortex filaments was presented by Chorin in [Ch94], while the first continuous statistical theory of 3D vortex filaments was given by P.-L. Lions and Majda in [LM00].

The enstrophy of the flow is defined as the L^2-norm of the vorticity; it is a natural candidate for encoding the information on the geometry of the flow in the theory of turbulent cascades. The main result in [DaGr11-5] states that $\frac{1}{2}$-Hölder coherence of the vorticity direction – a purely geometric condition – paired with a condition on a modified Kraichnan scale, and under a certain modulation assumption on evolution of the vorticity over the region of interest, leads to existence and scale-locality of 3D enstrophy cascade in physical scales of the flow. This provided a mathematical evidence that in contrast to 3D energy cascade, 3D enstrophy cascade is locally anisotropic, yielding a form of a reconciliation between Leonardo's and Kolmogorov's views on turbulence on the enstrophy level.

The vorticity-velocity formulation of the 3D Navier-Stokes equations (NSE) reads

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u - \triangle\omega \tag{3.8}$$

where u is the velocity and $\omega = \operatorname{curl} u$ is the vorticity of the fluid (the viscosity is set to 1).

Localization of evolution of the enstrophy to cylinder $B(x_0, 2R) \times (0, T)$ leads to the following version of local enstrophy flux,

$$\int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi)\, dx = -\int (u \cdot \nabla)\omega \cdot \phi\omega\, dx. \tag{3.9}$$

Since $\nabla\phi = (\nabla\psi)\eta$, and ψ can be constructed such that $\nabla\psi$ points inward, (3.9) represents *local inward enstrophy flux, at scale* R – more precisely, through the layer $S(x_0, R, 2R)$ – *around the point* x_0.

Consider a (K_1, K_2)-cover $\{B(x_i, R)\}_{i=1}^n$ at scale R, for some $0 < R \le R_0$. Local inward enstrophy fluxes, at scale R, associated to the cover elements $B(x_i, R)$, are then given by $\int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi_i)\, dx$, for $1 \le i \le n$. Assuming smoothness on $(0, T)$,

$$\begin{aligned} \int_0^t \int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi_i)\, dx\, ds &= \int \frac{1}{2}|\omega(x,t)|^2 \psi_i(x)\, dx + \int_0^t \int |\nabla\omega|^2 \phi_i\, dx\, ds \\ &- \int_0^t \int \frac{1}{2}|\omega|^2 ((\phi_i)_s + \triangle\phi_i)\, dx\, ds - \int_0^t \int (\omega \cdot \nabla)u \cdot \phi_i\, \omega\, dx\, ds, \end{aligned} \tag{3.10}$$

for any t in $(2T/3, T)$ and $1 \le i \le n$. An explicit representation formula for the localized vortex-stretching term $(\omega \cdot \nabla)u \cdot \phi_i\, \omega$ obtained in [Gr09] reads

$$\begin{aligned} (\omega \cdot \nabla)u \cdot \phi_i\, \omega\, (x) &= -c\, P.V. \int_{B(x_0, 2r)} (\omega(x) \times \omega(y)) \cdot G_\omega(x,y)\, \phi_i^{\frac{1}{2}}(y)\, \phi_i^{\frac{1}{2}}(x)\, dy + \text{ LOT} \\ &= \text{VST} + \text{LOT} \end{aligned} \tag{3.11}$$

where ϵ_{jkl} is the Levi-Civita symbol, $\big(G_\omega(x,y)\big)_k = \dfrac{\partial^2}{\partial x_i \partial y_k} \dfrac{1}{|x-y|}\, \omega_i(x)$, and LOT denotes the lower order terms.

The key feature of the highest order term VST is that it faithfully reproduces the geometric structure of the 'free' vortex-stretching term $(\omega \cdot \nabla)u \cdot \omega$.

Denoting the time-averaged local fluxes per unit mass associated to the cover element $B(x_i, R)$ by $\hat{\Phi}_{x_i,R}$,

$$\hat{\Phi}_{x_i,R} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi_i)\, dx, \tag{3.12}$$

the main quantity of interest is the ensemble average of $\{\hat{\Phi}_{x_i,R}\}_{i=1}^n$,

$$\langle \Phi \rangle_R = \frac{1}{n} \sum_{i=1}^{n} \hat{\Phi}_{x_i,R}. \tag{3.13}$$

The assumptions guaranteeing the universality of the averaged enstrophy flux per unit mass $\langle \Phi \rangle_R$ across the range of scales – existence of the enstrophy cascade – are as follows [DaGr11-5].

(A1) Coherence Assumption Denote the vorticity direction field by ξ, and let $M > 0$ (large). Assume that there exists a positive constant C_1 such that

$$|\sin \varphi(\xi(x,t), \xi(y,t))| \le C_1 |x-y|^{\frac{1}{2}}$$

for any (x,y,t) in $\left(B(0,2R_0) \times B(0,2R_0 + R_0^{\frac{2}{3}}) \times (0,T)\right) \cap \{|\nabla u| > M\}$; $\frac{1}{2}$-Hölder coherence in the regions of intense fluid activity (large gradients). The previous local regularity results [GrZh06, Gr09] imply that – under (A1) – the *a priori* weak solution u is in fact smooth inside $B(0,2R_0) \times (0,T)$ and can – moreover – be smoothly continued (locally-in-space) past $t = T$; in particular, we can write (3.10) with $t = T$.

(A2) Modified Kraichnan Scale Denote by E_0 time-averaged enstrophy per unit mass associated with the integral domain $B(0,2R_0) \times (0,T)$,

$$E_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{2\rho-1} \, dx \, dt,$$

by P_0 a modified time-averaged palinstrophy per unit mass,

$$P_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, dt + \frac{1}{T} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x,T)|^2 \psi_0(x) \, dx$$

(the modification is due to the shape of the temporal cut-off η), and by σ_0 a corresponding modified Kraichnan scale, $\sigma_0 = \left(\frac{E_0}{P_0}\right)^{\frac{1}{2}}$. Then, the assumption (A2) is simply a requirement that the modified Kraichnan scale associated with the integral domain $B(0,2R_0) \times (0,T)$ be dominated by the integral scale, $\sigma_0 < \beta R_0$, for a suitable constant $\beta < 1$.

(A3) Localization of the Integral Domain and Modulation The general set up considered is one of the Leray solutions satisfying (A1). As already mentioned, (A1) implies smoothness; however, the control on regularity-type norms is only local. On the other hand, the energy inequality on the global spatiotemporal domain $\mathbb{R}^3 \times (0,T)$ implies $\int_0^T \int_{\mathbb{R}^3} |\omega|^2 \, dx \, dt < \infty$; localization of the integral domain will be determined by the condition

$$\int_0^T \int_{B(0,2R_0 + R_0^{\frac{2}{3}})} |\omega|^2 \, dx \, dt \le \frac{1}{C},$$

for a suitable constant $C > 1$.

The modulation assumption on the evolution of local enstrophy on $(0,T)$ – consistent with the choice of the temporal cut-off η – reads

$$\int |\omega(x,T)|^2 \psi_0(x) \, dx \ge \frac{1}{2} \sup_{t \in (0,T)} \int |\omega(x,t)|^2 \psi_0(x) \, dx.$$

The purpose of (A3) is to have a suitable control on evolution of the enstrophy over the integral domain; uncontrolled fluctuations of the enstrophy would prevent the cascade.

Theorem 3.4. *Let u be a Leray solution on $\mathbb{R}^3 \times (0,T)$ with the initial vorticity ω_0 in the space of finite Radon measures. Suppose that u satisfies (A1)-(A3) on the spatiotemporal integral domain $B(0, 2R_0 + R_0^{\frac{2}{3}}) \times (0,T)$. Then,*

$$\frac{1}{4K_*} P_0 \leq \langle \Phi \rangle_R \leq 4K_* \, P_0$$

for all R, $\frac{1}{\beta}\sigma_0 \leq R \leq R_0$ ($K_* = K_*(K_1, K_2) > 1$).

The locality of the averaged enstrophy flux is analogous to the locality of the averaged energy flux (cf. [DaGr11-5]).

4. CURRENT RESEARCH

4.1. **Propagation of universality; applications.** There are several properties of the ensemble averaging process described in section 2 that appear plausible. Perhaps the most elemental one is that if the averages at a certain scale are 'stable', i.e., nearly independent of a particular choice of a (K_1, K_2)-cover (K_1 and K_2 fixed), then essentially the same universality property should propagate to larger scales. The preliminary calculations indicate that in order to obtain a precise quantitative propagation result, the universality assumption should hold on an interval, rather than at a single scale (this is to be expected due to the presence of smooth cut-offs with prescribed rates of change). Two types of results seem within reach. Let f be a locally integrable function (a density), and K_1 and K_2 two positive integers.

TYPE I. Assume that there exists $R_* > 0$ such that for any R in $[R^*, 2R_*)$ and any $(27K_1, 16K_2)$-cover at scale R, the averages $\langle F \rangle_R$ are all comparable to some value F_*; more precisely, $\frac{1}{C_1} F_* \leq \langle F \rangle_R \leq C_1 F_*$. Then, for all $R \geq 2R_*$ and all (K_1, K_2)-covers at scale R, the averages $\langle F \rangle_R$ satisfy $\frac{1}{C_2} F_* \leq \langle F \rangle_R \leq C_2 F_*$.

TYPE II. Assume that there exists $R_* > 0$ such that for any R in $(\frac{1}{2}R^*, \frac{5}{2}R_*)$ and any (K_1, K_2)-cover at scale R, $\frac{1}{C_1} F_* \leq \langle F \rangle_R \leq C_1 F_*$. Then, for all $R \geq \frac{5}{2}R_*$ and all (K_1, K_2)-covers at scale R, the averages $\langle F \rangle_R$ satisfy $\frac{1}{C_3}\left(\frac{R_*}{R}\right)^{C_4} F_* \leq \langle F \rangle_R \leq C_3 \left(\frac{R}{R^*}\right)^{C_4} F_*$.

Shortly, in a Type I result, the universality is assumed with respect to more refined covers, while in a Type II result, the non-exactness of the propagation caused by the smooth cut-offs is reflected in a correction to the universal value F_* by the ratio of the scales R and R^*.

An (immediate) application in view is to obtain an improved version of Theorem 3.4; more precisely, to essentially remove the assumption on the localization of the integral domain (the first part of (A3)). This was the only truly technical assumption – (A1) was simply a mathematical quantification of coherence, (A2) a condition on Kraichnan scale of the type appearing in 2D theory (cf. [FJMR02, DaGr11-4]), and the other part of (A3) a modulation condition preventing wild fluctuations of the vorticity magnitude that would be incompatible with universality of the cascade.

4.2. **Scaling laws and dissipation number in physical scales.** Kolmogorov K41 theory is a statistical theory of *homogeneous isotropic turbulence*. Homogeneity and isotropy imply that the representation of the key turbulence phenomena in the Fourier space is a faithful representation of the corresponding phenomena in the physical space. They are also essential in formulating various universal scaling laws including Kolmogorov power law for the energy spectrum. In particular, the concept of the energy spectrum is canonical. The theory of turbulent cascades in physical scales presented in [DaGr11-1, DaGr11-2, DaGr11-3] does not assume any homogeneity; the homogeneity is replaced by a suitable ensemble averaging process. As a result, defining a sensible analogue of the K41 spectrum requires some care. Preliminary investigations suggest that the following construction may prove fruitful.

Define the local energy associated with a shell $S(x_0,R) = \{x : R < |x - x_0| < 2R\}$ around a point x_0 belonging to the integral domain $B(0,R_0)$ (with suitable modifications in the case the shell intersects the boundary of the domain) by $e_{S(x_0,R)} = \frac{1}{T}\frac{1}{R_0^3}\int\limits_0^T \int\limits_{S(x_0,R)} |u|^2\phi^\delta_{S(x_0,R)}dx\,dt$, for some δ in $(0,1)$; the refined cut-offs ϕ can be chosen such that $\phi_{B(x_0,R)} + \sum\limits_{j=1}^{m_R} \phi_{S(x_0,2^jR)} = \phi_0$, where $m_R = \left[\log_2\left(\frac{R_0}{R}\right)\right]$, and $\phi_{B(x_0,R)}$ is a refined cut-off corresponding to $B(x_0,R)$. Then – around any point x_0 – the following dyadic decomposition of the total (integral scale-) energy e_0 transpires, $e_0 = e_{B(x_0,R)} + \sum\limits_{j=1}^{m_R} e_{S(x_0,2^jR)}$ where $e_{B(x_0,R)} = \frac{1}{T}\frac{1}{R_0^3}\int\limits_0^T \int\limits_{B(x_0,2R)} |u|^2\phi^\delta_{B(x_0,R)}dx\,dt$. Next, introduce *$(K_1,K_2)$-averaged dyadic decomposition of e_0 seeded at scale R_d* as follows. Let $\{B(x_i,R_d)\}_{i=1}^{n_d}$ be a (K_1,K_2)-cover at scale R_d. (At this point, R_d is simply a 'small scale'.) Then, for each x_i there exists a local dyadic decomposition of e_0 as described above. For simplicity, denote

$$B_i = B(x_i,R_d), \quad \phi_i = \phi_{B(x_i,R_d)}, \quad e_i = e_{B(x_i,R_d)},$$
$$S_{i,j} = S(x_i,2^jR_d), \quad \phi_{ij} = \phi_{S_{ij}} \quad e_{ij} = e_{S_{ij}} \quad \text{and } m_d = \left[\log_2\left(\frac{R_0}{R_d}\right)\right].$$

Note that, for individual j, the shells $\{S_{i,j}\}_{i=1}^{n_d}$ do not necessarily cover the integral domain. Defining by $\mathcal{E}_0 = \frac{1}{n_d}\sum\limits_{i=1}^{n_d} e_i$ the average energy associated with scales below R_d, and by $\mathcal{E}_j = \frac{1}{n_d}\sum\limits_{i=1}^{n_d} e_{ij}$ the average energy associated with the scale 2^jR_d, the following identity holds,

$$e_0 = \mathcal{E}_0 + \sum_{j=1}^{m_d}\mathcal{E}_j; \text{ this is the sought after decomposition of } e_0. \tag{4.1}$$

Assume that $\mathcal{E}_j$ follow Kolmogorov-type power law $\mathcal{E}_j \sim \varepsilon^{2/3}(2^jR_d)^{2/3}$ (ε is the total energy dissipation rate corresponding to the integral domain). The *local energy inequality* written on each shell S_{ij} together with the estimates on derivatives of ϕ_{ij} yields

$$\varepsilon_{ij} - c\left(\frac{1}{T} + \frac{\nu}{(2^jR_d)^2}\right)e_{ij} \le \Phi_{ij} \le \varepsilon_{ij} + c\left(\frac{1}{T} + \frac{\nu}{(2^jR_d)^2}\right)e_{ij},$$

where ε_{ij} and Φ_{ij} are the total energy dissipation rate and the inward energy flux associated with $S(x_i,2^jR_d)$, respectively; averaging in i and summing up in j, we obtain

$$\varepsilon - c\sum_{j=1}^{m_d}\left(\frac{1}{T} + \frac{\nu}{(2^jR_d)^2}\right)\mathcal{E}_j \le \Phi_0 \le \varepsilon + c\sum_{j=1}^{m_d}\left(\frac{1}{T} + \frac{\nu}{(2^jR_d)^2}\right)\mathcal{E}_j,$$

where Φ_0 is the space-time average of the flux into $B(0,R_0)$. The sufficient condition for the energy cascade presented in [DaGr11-1] implies $\Phi_0 \sim \varepsilon$; this together with the above Kolmogorov-type power law implies

$$\frac{\epsilon_0^{1/3}}{\nu} \sim \sum_{j=1}^{m_d}\frac{1}{(2^jR_d)^{4/3}} \sim \left(\frac{1}{R_d^{4/3}} - \frac{1}{R_0^{4/3}}\right) \sim \frac{1}{R_d^{4/3}},$$

or equivalently,

$$R_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \tag{4.2}$$

which is precisely the definition of Kolmogorov dissipation scale in the empirical theory of homogeneous isotropic turbulence.

The main goal here is to investigate the conditions under which such scaling power laws are possible. In particular, unraveling the power law $\mathcal{E}_j \sim \varepsilon^{2/3}(2^j R_d)^{2/3}$ – in the absence of anomalous dissipation – reveals that it is essentially a requirement on saturation of a suitable inequality involving Morrey-type quantities of u and ∇u; hence, Morrey-type analysis of weak solution to the 3D NSE will be a main mathematical device (cf. [Ku08, OL03, CP01]).

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REFERENCES

[AK98] V. Arnold, B. Khesin, Topological methods in hydrodynamics. Springer, 1998.

[BG95] G. I. Barenblatt and N. Goldenfeld, *Does fully developed turbulence exist? Reynolds number independence versus asymptotic covariance*, Phys. Fluids, **7** (1995), 3078–3082.

[BT10] C. Bardos and E. Titi, *Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations*, Discrete Contin. Dyn. Syst. S, **3** (2010), 185–197.

[Bat53] G. K. Batchelor, The theory of homogeneous turbulence, Cambridge University Press, 1953.

[daVeigaBe02] H. Beirao da Veiga and L.C. Berselli, *On the regularizing effect of the vorticity direction in incompressible viscous flows*, Differential Integral Equations, **15** (2002), 345–356

[BCFM95] H. Bercovici, P. Constantin, C. Foias, and O. P. Manley, *Exponential decay of the power spectrum of turbulence*, J. Stat. Phys, **80** (1995), 579–602.

[CKN82] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations.* Comm. Pure Appl. Math, **35** (1982), 771–831.

[CBBL10] A. Cahuzac, J. Boudet, P. Borgnat, and E. Léveque, *Smoothing algorithms for mean-flow extraction in large-eddy simulation of complex turbulent flows*, Phys. Fluids, **22**, (2010), 125104–125118

[ChKaLe07] D. Chae, K. Kang and J. Lee, *On the interior regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Partial Differential Equations, **32** (2007), 1189–1207.

[CP01] Z.-M. Chen and W.G. Price, *Morrey space techniques applied to the interior regularity problem of the Navier-Stokes equations*, Nonlinearity, **14** (2001), 1453–1472.

[CCFS08] A. Cheskidov, P. Constantin, S. Friedlander and R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations*, Nonlinearity, **21** (2008), 1233–1252.

[CF09] A. Cheskidov and S. Friedlander *The vanishing viscosity limit for a dyadic model*, Phys. D, **238** (2009), 783–787.

[CS11] A. Cheskidov and R. Shydkoy, *A unified approach to regularity problems for the 3D Navier-Stokes and Euler equations: the use of Kolmogorov's dissipation range*, arXiv:1102.1944, (2011).

[Ch94] A. Chorin, Vorticity and Turbulence. Applied Mathematics Sciences **103**, Springer-Verlag, New York, 1994.

[Co90] P. Constantin, *Navier-Stokes equations and area of interfaces*, Comm. Math. Phys. **129** (1990), 241–266.

[Co94] P. Constantin, *Geometric statistics in turbulence*, SIAM Rev. **36** (1994), 73–98.

[PC11] P. Constantin, private communication (2011).

[CD94] P. Constantin and C. R. Doering, *Variational bounds on energy dissipation in incompressible flows: shear flow*, Phys. Rev. E, **49** (1994), 4087–4099.

[CD95] P. Constantin and C. R. Doering, *Variational bounds on energy dissipation in incompressible flows II: channel flow*, Phys. Rev. E **51** (1995), 3192–3198.

[CET94] P. Constantin, W. E and E. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys, (165) (1994), 207–209.

[CoFe93] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 775–789.

[CPS95] P. Constantin, I. Procaccia and D. Segel, *Creation and dynamics of vortex tubes in three dimensonal turbulence*, Phys. Rev. E, **51** (1995), 3207–3222.

[CF01] D. Cordoba and C. Fefferman, *On the collapse of tubes carried by 3D incompressible flows*, Comm. Math. Phys. **222** (2001), 293–298.

[CFL04] D. Cordoba, C. Fefferman and R. De La LLave, *On squirt singularities in hydrodynamics*, Siam J. Math. Anal. **36** (2004), 204–213.

[DFJ05] R. Dascaliuc, C. Foias, and M. S. Jolly, *Relations between energy and enstrophy on the global attractor of the 2-D Navier-Stokes equations*, J. Dynam. Differential Equations, **17** (2005), 643–736.

[DFJ08] R. Dascaliuc, C. Foias, and M. S. Jolly, *Some specific mathematical constraints on 2D turbulence*, Phys. D, **237** (2008), 3020–3029.

[DFJ09] R. Dascaliuc, C. Foias, and M. S. Jolly, *On the asymptotic behavior of average energy and enstrophy in 3D turbulent flows*, Phys. D, **238** (2009), 725–736.

[DFJ10] R. Dascaliuc, C. Foias, and M. S. Jolly, *Estimates on enstrophy, palinstrophy, and invariant measures for 2-D turbulence*, J. Differential Equations, **248** (2010), 792–819.

[DaGr11-1] R. Dascaliuc and Z. Grujić, *Energy cascades and flux locality in physical scales of the 3D Navier-Stokes equations*, Comm. Math. Phys, **305** (2011), 199–220.

[DaGr11-2] R. Dascaliuc and Z. Grujić, *Anomalous dissipation and energy cascade in 3D inviscid flows*, Comm. Math. Phys. (to appear) (2011). arXiv:1102.1951

[DaGr11-3] R. Dascaliuc and Z. Grujić, *Dissipation anomaly and energy cascade in 3D incompressible flows*, (submitted) (2011). arXiv:1103.4158

[DaGr11-4] R. Dascaliuc and Z. Grujić, *2D turbulence in physical scales of the Navier-Stokes equations* (submitted) (2011). arXiv:1101.2209

[DaGr11-5] R. Dascaliuc and Z. Grujić, *Coherent vortex structures and 3D enstrophy cascade*, (submitted) (2011). arXiv:1107.0058

[Dav04] P. A. Davidson, Turbulence - An Introduction for Scientists and Engineers. Oxford University Press, Oxford, 2004.

[DLS09] C. De Lellis and L. Szekelyhidi Jr, *The Euler equations as a differential inclusion*, Ann. Math, (2) **170** (2009), 1417–1436.

[DLS10] C. De Lellis and L. Szekelyhidi Jr, *On admissibility criteria for weak solutions of the Euler equations*, Arch. Rational Mech. Anal, **195** (2010), 225–260.

[DF02] C. R. Doering and C. Foias, *Energy dissipation in body-forced turbulence*, J. Fluid Mech, **467** (2002), 289–306.

[DG95] C. R. Doering, J. D. Gibbon, Applied Analysis of the Navier-Stokes Equations. Cambridge University Press, Cambridge, 1995.

[DG02] C. R. Doering, J.D. Gibbon, *Bounds on moments of the energy spectrum for weak solutions of the three-dimensional Navier-Stokes equations*, Phys. D, **165** (2002), 163–175.

[DYS08] D. A. Donzis, P. K. Yeung and K. R. Sreenivasan *Energy dissipation rate and enstrophy in isotropic turbulence: resolution effects and scaling in direct numerical simulations*, Phys. Fluids, **20**, 045108–045132.

[DP01] P. A. Durbin and B. A. Pettersson Reif, Statistical Theory and Modeling for Turbulent Flows. Johns Wiley & Sons, 2001.

[DR00] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations*, Nonlinearity, **13** (2000), 249–255.

[E94] G. L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer*, Phys. D, **78** (1994), 222–240.

[E05] G. L. Eyink, Physica D *Locality of Turbulent Cascades*, **207** (2005), 91–116.

[EA09] G. L. Eyink and H. Aluie, *Localness of energy cascade in hydrodynamic turbulence. I. Smooth coarse graining*, Phys. Fluids, **21** (2009), 115107–115116.

[ES06] G. L. Eyink and K. Sreenivasan *Onsager and the theory of hydrodynamic turbulence*, Rev. Mod. Phys, **78** (2006), 87–135.

[FS06] G Falkovich and K. R. Sreenivasan, *Lessons from hydrodynamic turbulence*, Physics Today, **59** (2006), 43–49

[FeTi98] A. Ferrari and E. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Differential Equations, **23** (1998), 1–16.

[Fo97] C. Foias, *What do the Navier-Stokes equations tell us about turbulence?*, Contemporary Mathematics **208** (1997), 151–180.

[FJMR02] C. Foias, M. Jolly, O. Manley, and R. Rosa. *Statistical Estimates for the Navier-Stokes Equations and the Kraichnan Theory of 2-D Fully Developed Turbulence* , J. Stat. Phys, **108** (2002), 591–645.

[FMRT01] C. Foias, O. Manley, R. Rosa and R. Temam, *Estimates for the energy cascade in three-dimensional turbulent flows*, C.R. Acad. Sci. Paris Sér. I Math, **333** (2001), 499-502.

[FMRT] C. Foias, O. Manley, R. Rosa and R. Temam, Navier-Stokes equations and turbulence. Vol. **83** of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2001.

[FT89] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. **97** (1989), 359–369.

[Fr95] U. Frisch, Turbulence. Cambridge University Press, 1995. The legacy of A.N. Kolmogorov.

[GGH97] B. Galanti, J.D. Gibbon and M. Heritage, *Vorticity alignment results for the three-dimensional Euler and Navier-Stokes equations*, Nonlinearity **10** (1997), 1675–1694.

[GM11] T. Gallay and Y. Maekawa, *Three-dimensional stability of Burgers vortices*, Comm. Math. Phys. **302** (2011), 477–511.

[Ga94] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Vol I: Linearized steady problems, Vol II: Nonlinear steady problems, Springer-Verlag, New York-Berlin-Heidelberg, 1994.

[GFD99] J.D. Gibbon, A.S. Fokas and C.R. Doering, *Dynamically stretched vortices as solutions of the 3D Navier-Stokes equations*, Phys. D **132** (1999), 497–510.

[GSM62] H. L. Grant, R. W. Stewart, and A. Moilliet, *Turbulence spectra in a tidal channel*, J. Fluid Mech, **12** (1962), 241–268.

[Gr01] Z. Grujić, *The geometric structure of the super-level sets and regularity for 3D Navier-Stokes equations*, Indiana Univ. Math. J, **50** (2001), 1309–1317.

[Gr09] Z. Grujić, *Localization and geometric depletion of vortex-stretching in the 3D NSE*, Comm. Math. Phys, **290** (2009), 861–870.

[Gr11] Z. Grujić, *A geometric measure-type regularity criterion for solutions to the 3D Navier-Stokes equations*, (submitted) (2011). arXiv:1111.0217

[GrGu10-1] Z. Grujić and R. Guberović, *Localization of analytic regularity criteria on the vorticity and balance between the vorticity magnitude and coherence of the vorticity direction in the 3D NSE*, Comm. Math. Phys, **298** (2010), 407–418.

[GrGu10-2] Z. Grujić and R. Guberović, *A regularity criterion for the 3D NSE in a local version of the space of functions of bounded mean oscillations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **27** (2010), 773–778.

[GrKu98] Z. Grujić and I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in L^p*, J. Funct. Anal, **152** (1998), 447–466.

[GrKu99] Z. Grujić and I. Kukavica, *Space analyticity for the nonlinear heat equation in a bounded domain*. J. Differential Equations, **152** (1999), 42–54.

[GrZh06] Z. Grujić and Qi Zhang, *Space-time localization of a class of geometric criteria for preventing blow-up in the 3D NSE*, Comm. Math. Phys, **262** (2006), 555–561.

[Gu10] R. Guberović, *Smoothness of Koch-Tataru solutions to the Navier-Stokes equations revisited*, Discrete Cont. Dyn. Syst, **27** (2010), 231–236.

[ISS03] L. Iskauriaza, G. Seregin and V. Shverak, *$L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk, **58** (2003), 3–44.

[Ko41-1] A.N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*, Dokl. Akad. Nauk SSSR, **30** (1941), 299–303.

[Ko41-2] A.N. Kolmogorov, *On degeneration of isotropic turbulence in an incompressible viscous liquid*, Dokl. Akad. Nauk SSSR, **31** (1941) 538–540.

[Ko41-3] A.N. Kolmogorov, *Dissipation of energy in the locally isotropic turbulence*, Dokl. Akad. Nauk SSSR, **32** (1941), 16–18.

[Ko62] A.N. Kolmogorov, *A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*, J. of Fluid Mech, **13** (1962), 82Ð85.

[Ku99] I. Kukavica, *On the dissipative scale for the Navier-Stokes equation*, Indiana Univ. Math. J, **48** (1999), 1057–1081.

[Ku03] I. Kukavica, J. Differential Equations, *On local uniqueness of weak solutions of the Navier-Stokes system with bounded initial data*, **194** (2003), 39–50.

[Ku08] I. Kukavica, *Regularity for the Navier-Stokes equations with a solution in a Morrey space*, Indiana Univ. Math. J, **57** (2008), 2843-2860.

[La69] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, 2nd ed., Gordon and Breach, New York, 1969.

[LL95] L. D. Landau and E. M. Lifshitz, Fluid Mechanics. Butterworth-Heinemann, 1995.

[Le34] J. Leray, *Essai sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math, **63** (1934), 193–248.

[L-R02] P. G. Lemarie-Rieusset, Recent developments in the Navier-Stokes problem. CRC Press, 2002.

[PLL96] P.-L. Lions, Mathematical Topics in Fluid Mechanics vol 1 Incompressible Models. Oxford University Press: Clarendon, New York, 1996.

[LM00] P.-L. Lions and A. Majda, *Equilibrium statistical theory for nearly parallel vortex filaments*, Comm. Pure Appl. Math, **53** (2000), 76–142.

[LF92] V. L'vov and G. Falkovich, *Counterbalanced interaction locality of developed hydrodynamic turbulence*, Phys. Rev. A, **46** (1992), 4762–4772.

[MB02] A. Majda and A. Bertozzi, Vorticity and incompressible flow. Cambridge University Press, 2002.

[MKO94] H. K. Moffatt, S. Kida, K. Ohkitani, *Stretched vortices-the sinews of turbulence; large-Reynolds-number asymptotics*, J. Fluid Mech. **259** (1994), 241–264.

[MY71] A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2, Cambridge, MA: MIT Press, 1971.

[Ob41] A. Obukhoff. *On the energy distribution in the spectrum of a turbulent flow*, C. R. (Doklady) Acad. Sci. URSS (N.S.), **32** (1941), 19–21.

[Oh08] K. Ohkitani, *A geometrical study of 3D incompressible Euler flows with Clebsch potentials-a long-lived Euler flow and its power-law energy spectrum*, Phys. D, **237** (2008), 2020–2027.

[Oh09] K. Ohkitani, *Non-linearity depletion, elementary excitations and impulse formulation in vortex dynamics* Geophys. Astrophys. Fluid Dyn, **103** (2009), 113–133.

[OY89] K. Ohkitani and M. Yamada. *Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence*, Progr. Theoret. Phys, **81** (1989):329–341.

[OL03] M. O'Leary, *Conditions for the local boundedness of solutions of the Navier-Stokes system in three dimensions*, Comm. Partial Differential Equations, **28**(2003), 617–636.

[On45] Onsager, L. 1945. *The distribution of energy in turbulence*, Physical Review, **68** (1945) 286.

[On49] L. Onsager, Statistical hydrodynamics, Nouvo Cimento, supplement 6 no. 2 (1949), 279.

[Or77] S. A. Orszag, *Lectures on the statistical theory of turbulence*, Fluid Dynamics (Proceedings of the Summer School of Theoretical Physics, Les Houches, July 1973), Gordon and Breach, London, 1977, pp. 235–374.

[Ri22] L. F. Richardson, Weather Prediction by Numerical Process. Cambridge University Press, Cambridge, 1922.

[RDD11] B. Rollin, Y. Dubief, C. R. Doering, *Variations on Kolmogorov flow: turbulent energy dissipation and mean flow profiles*, J. Fluid Mech, **670** (2011), 204–213.

[RS78] H. A. Rose and P.-L. Sulem, *Fully developed turbulence and statistical mechanics*, Journ. de Physique, **39** (1978), 441-484.

[Ru83] D. Ruelle, *Large volume limit of the distribution of characteristic exponents in turbulence*, Comm. Math. Phys, **87** (1982/83), 287–302.

[RT71] D. Ruelle, F. Takens, *On the nature of turbulence*, Comm. Math. Phys, **20** (1971), 167–192.

[Sc77] V. Scheffer *Hausdorff measure and the Navier-Stokes equations*. Comm. Math. Phys, **55** (1977) 97–112.

[Sc93] V. Scheffer, *An inviscid flow with compact support in space-time*, J. Geom. Anal, **3** (1993), 343–401.

[SJO91] Z.-S. She, E, Jackson and S. Orszag, *Structure and dynamics of homogeneous turbulence: models and simulations*, Proc. R. Soc. Lond. A, **434** (1991), 101–124.

[Shn97] A. Shnirelman *On the nonuniqueness of weak solution of the Euler equation.*, Comm. Pure Appl. Math, **50** (1997), 1261–1286.

[Shn00] A. Shnirelman *Weak solutions with decreasing energy of incompressible Euler equations*, Comm. Math. Phys, **210** (2000), 541–603.

[Shvy09] R. Shvydkoy *On the energy of inviscid singular flows*, J. Math. Anal. Appl. **349** (2009), 583–595.

[Sr95] K. R. Sreenivasan, *On the universality of the Kolmogorov constant*, Phys. of Fluids, **7** (1995), 2778–2784.

[Ta35] G. I. Taylor, *Statistical theory of turbulence*, Proc. of the Royal Soc, **A151** (1935), 421Ð478.

[Te01] R. Temam, Navier-Stokes Equations. Theory and Numerical Analysis, Studies in Mathematics and its Applications, 3rd edition, North-Holland Publishing Co., Amsterdam-New York, 1984. Reedition in 2001 in the AMS Chelsea series, AMS, Providence.

[WK09] A. Willis, R. R. Kerswell, *Turbulent dynamics of pipe flow captured in a reduced model: puff relaminarisation and localized 'edge' state*, J. Fluid Mech, **619** (2009), 213Ð233.

[ZTMW94] G. Zocchi, P. Tabeling, J. Maurer, and H. Willaime, *Measurement of the scaling of the dissipation at high Reynolds number*, Phys. Rev. E, **50** (1994), 3693–3700.

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