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Propagation of the real analyticity for the solution to the Euler equations in the Besov space

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Abstract

We consider the initial value problems for the incompressible Euler equations with non-decaying initial velocity like a trigonometric function. We prove that if the initial velocity is real analytic then the solution is also real analytic with respect to spatial variables. Furthermore, we shall establish the lower bound for the size of the radius of convergence of Taylor's expansion.

1 Introduction

In this note, we consider the initial value problems for the Euler equations in the whole space $\mathbb{R}^n$ with $n \geqslant 2$, describing the motion of perfect incompressible fluids,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{E}$$

where $u = u(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ denotes the unknown velocity fields, and $p = p(x,t)$ denotes the unknown pressure of the fluids, while $u_0 = u_0(x) = (u_0^1(x), \ldots, u_0^n(x))$ denotes the given initial velocity field satisfying the compatibility condition $\operatorname{div} u_0 = 0$.

This note is a survey of our paper [14], and the main purpose of this note is to prove the propagation properties of the real analyticity with respect to spatial variables for the solution to (E) with non-decaying initial velocity. For the local-in-time existence and uniqueness of smooth solutions to (E), Kato [8] proved that for the given initial velocity $u_0 \in H^m(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$ and $m > n/2 + 1$, there exists a $T = T(\|u_0\|_{H^m}) > 0$ such that the Euler equation (E) possesses a unique solution u in the class $C([0,T]; H^m(\mathbb{R}^n))^n$. Kato and Ponce [9] extended this result to the Sobolev spaces of the fractional order $W^{s,p}(\mathbb{R}^n) := (1-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ for $s > n/p + 1$ with $1 < p < \infty$. Later, Chae [5] [6] obtained a local-in-time well-posedness for (E) in the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ for $s > n/p+1$ with $1 < p, q < \infty$, and in the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ for $s > n/p + 1, 1 < p < \infty, 1 \leqslant q \leqslant \infty$ or $s = n/p + 1, 1 < p < \infty, q = 1$, respectively. Pak and Park [13] proved the local well-posedness for (E) in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$.

For the real analyticity of the solution to (E) in the framework of the Sobolev spaces $H^m(\mathbb{R}^n)$, Alinhac and Métivier [2] proved that Kato's solution is real analytic in $\mathbb{R}^n$ if the initial velocity is real analytic. See also Bardos, Benachour and Zerner [3], Le Bail [11] and Levermore and Oliver [12]. Kukavica and Vicol [10] considered the vorticity equations for (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$ and proved the propagation properties of the real analyticity. In particular, they improved the estimate for the size of the radius of the convergence of the Taylor expansion for the solution to the vorticity equations.

In this note, we prove the propagation of the analyticity for the solution to (E) constructed by Pak and Park [13] in the framework of the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Note that the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ contains some non-decaying functions at space infinity, for example, the trigonometric function $e^{ix\cdot a}$ with the wave vector $a \in \mathbb{R}^n$. In particular, we give an improvement for the estimate for the size of the radius of convergence of Taylor's expansion.

Before stating our result about the analyticity, we set some notation and function spaces. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing functions, and let $\mathscr{S}'(\mathbb{R}^n)$ be the space of all tempered distributions. We first recall the definition of the Littlewood-Paley operators. Let Φ and φ be the functions in $\mathscr{S}(\mathbb{R}^n)$ satisfying the following properties :

$$\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant 5/6\}, \quad \operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid 3/5 \leqslant |\xi| \leqslant 5/3\},$$

$$\widehat{\Phi}(\xi) + \sum_{j=0}^{\infty}\widehat{\varphi_j}(\xi) = 1 \quad \xi \in \mathbb{R}^n,$$

where $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ and $\widehat{f}$ denotes the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$. Given $f \in \mathscr{S}'(\mathbb{R}^n)$, we denote

$$\Delta_j f := \begin{cases} \Phi * f & j = -1, \\ \varphi_j * f & j \geqslant 0, \\ 0 & j \leqslant -2, \end{cases} \qquad S_k f := \sum_{j\leqslant k} \Delta_j f \quad k \in \mathbb{Z},$$

where $*$ denotes the convolution operator. Then, we define the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ by the following definition.

Definition 1.1. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that the following norm is finite :

$$\|f\|_{B^s_{p,q}} := \left\| \{2^{sj}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \right\|_{\ell^q}.$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all positive integers. For $k \in \mathbb{N}_0$, put

$$m_k := c\frac{k!}{(k+1)^2},$$

where c is a positive constant such that one has

$$\sum_{0\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} \leqslant m_{|\alpha|} \quad \alpha \in \mathbb{N}_0^n,$$

$$\sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} \leqslant |\alpha| m_{|\alpha|} \quad \alpha \in \mathbb{N}_0^n \setminus \{0\}^n.$$

For example, it suffices to take $c \leqslant 1/16$. For the detail, see Kahane [7] and Alinhac and Métivier [1].

Our result on the propagation of the analyticity now reads:

Theorem 1.2. *Let* $u_0 \in B^1_{\infty,1}(\mathbb{R}^n)^n$ *be an initial velocity field satisfying* $\operatorname{div} u_0 = 0$*, and let* $u \in C([0,T]; B^1_{\infty,1}(\mathbb{R}^n))^n$ *be the solution of* (E)*. Suppose that* u_0 *is real analytic in the following sense : there exist positive constants* K_0 *and* ρ_0 *such that*

$$\|\partial_x^\alpha u_0\|_{B^1_{\infty,1}} \leqslant K_0 \rho_0^{-|\alpha|} m_{|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$*. Then,* $u(\cdot,t)$ *is also real analytic for all* $t \in [0,T]$ *and satisfies the following estimate : there exist positive constants* $K = K(n, K_0), L = L(n, K_0)$ *and* $\lambda = \lambda(n)$ *such that*

$$\|\partial_x^\alpha u(\cdot,t)\|_{B^1_{\infty,1}} \leqslant K \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|}(1+t)^{\max\{|\alpha|-1,0\}} \exp\left\{\lambda|\alpha| \int_0^t \|u(\cdot,\tau)\|_{B^1_{\infty,1}} d\tau\right\} \quad (1.1)$$

for all $\alpha \in \mathbb{N}_0^n$ *and* $t \in [0,T]$*.*

Remark 1.3. (i) Since K, L and λ do not depend on T, (1.1) gives a grow-rate estimate for large time behavior of the higher order derivatives of Pak-Park's solutions.

(ii) From (1.1), one can derive the estimate for the size of the uniform analyticity radius of the solutions as follows :

$$\liminf_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha u(t)\|_{L^\infty}}{\alpha!}\right)^{-\frac{1}{|\alpha|}} \geqslant \frac{\rho_0}{L}(1+t)^{-1} \exp\left\{-\lambda \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau\right\}.$$

Moreover, since $B^1_{\infty,1}(\mathbb{R}^n)$ is continuously embedded in $C^1(\mathbb{R}^n)$ (see Triebel [15]), we have by (1.1) that

$$\liminf_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!}\right)^{-\frac{1}{|\alpha|}} \geqslant \frac{\rho_0}{L}(1+t)^{-1} \exp\left\{-\lambda \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau\right\}.$$

Recently, Kukavica and Vicol [10] considered the vorticity equations of (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$, and obtained the following estimate for uniform analyticity radius :

$$\liminf_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!}\right)^{-\frac{1}{|\alpha|}} \geqslant \rho(1+t^2)^{-1} \exp\left\{-\lambda \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right\}$$

with some $\rho := \rho(r, \operatorname{rot} u_0)$ and $\lambda = \lambda(r)$. Hence our result is an improvement of the previous analyticity-rate in the sense that $(1+t^2)^{-1}$ is replaced by $(1+t)^{-1}$, and clarifies that $\rho = \rho_0/L$.

This note is organized as follows. In Section 2, we recall the key lemmas which play important roles in our proof. In Sections 3, we present the proof of Theorems 1.2.

2 Key Lemmas

Throughout this note, we shall denote by C the constants which may change from line to line. In particular, $C = C(\cdot, \ldots, \cdot)$ will denote the constants which depend only on the quantities appearing in parentheses.

In this section, we recall some key lemmas and prove a bilinear estimate in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. We first prepare the commutator type estimates and the bilinear estimates in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ for nonlinear terms of (E).

Lemma 2.1 (Pak-Park [13]). *There exists a positive constant $C = C(n)$ such that*

$$\sum_{j\in\mathbb{Z}} 2^j \left\|(S_{j-2}u\cdot\nabla)\Delta_j f - \Delta_j((u\cdot\nabla)f)\right\|_{L^\infty} \leqslant C\|u\|_{B^1_{\infty,1}}\|f\|_{B^1_{\infty,1}}$$

holds for all $(u,f)\in B^1_{\infty,1}(\mathbb{R}^n)^{n+1}$ *with* $\operatorname{div} u = 0$.

Lemma 2.2. *There exists a positive constant $C = C(n)$ such that*

$$\|fg\|_{B^1_{\infty,1}} \leqslant C(\|f\|_{L^\infty}\|g\|_{B^1_{\infty,1}} + \|g\|_{L^\infty}\|f\|_{B^1_{\infty,1}})$$

holds for all $f,g\in B^1_{\infty,1}(\mathbb{R}^n)$.

Proof. For the proof, we use the Bony paraproduct formula [4]. Let us decompose fg as

$$fg = \sum_{j=2}^{\infty} S_{j-3}f\Delta_j g + \sum_{j=2}^{\infty} S_{j-3}g\Delta_j f + \sum_{j=1}^{\infty}\sum_{k=j-2}^{j+2} \Delta_j f\Delta_k g.$$

Since $\operatorname{supp}\mathscr{F}[\varphi_j]\cap\operatorname{supp}\mathscr{F}[\varphi_{j'}]=\emptyset$ if $|j-j'|\geqslant 2$, we see that

$$\operatorname{supp}\mathscr{F}\left[S_{j-3}f\Delta_j g\right]\subset\left\{\xi\in\mathbb{R}^n \;\middle|\; 2^{j-2}\leqslant|\xi|\leqslant 2^{j+2}\right\}$$

and

$$\operatorname{supp}\mathscr{F}\left[\Delta_j f\Delta_k g\right]\subset\left\{\xi\in\mathbb{R}^n \;\middle|\; |\xi|\leqslant 2^{\max\{j,k\}+2}\right\},$$

which yield that

$$\begin{aligned}\Delta_j(fg) &= \sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}} \Delta_j\left(S_{j'-3}f\Delta_{j'}g\right) + \sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}} \Delta_j\left(S_{j'-3}g\Delta_{j'}f\right)\\ &\quad + \sum_{\max\{j',j''\}\geqslant j-2}\ \sum_{|j''-j'|\leqslant 2} \Delta_j\left(\Delta_{j'}f\Delta_{j''}g\right)\\ &=: I_1+I_2+I_3. \end{aligned} \tag{2.1}$$

By the Hausdorff-Young inequality and the Hölder inequality, we have that

$$\begin{aligned}\|I_1\|_{L^\infty} &\leqslant C\sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}} \left\|S_{j'-3}f\right\|_{L^\infty}\left\|\Delta_{j'}g\right\|_{L^\infty}\\ &\leqslant C\|f\|_{L^\infty}\sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}}\left\|\Delta_{j'}g\right\|_{L^\infty}.\end{aligned} \tag{2.2}$$

Similarly, it holds that

$$\|I_2\|_{L^\infty}\leqslant C\|g\|_{L^\infty}\sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}}\left\|\Delta_{j'}f\right\|_{L^\infty}. \tag{2.3}$$

Moreover, we see that

$$\|I_3\|_{L^\infty}\leqslant C\sum_{\max\{j',j''\}\geqslant j-2}\ \sum_{|j''-j'|\leqslant 2}\left\|\Delta_{j'}f\right\|_{L^\infty}\left\|\Delta_{j''}g\right\|_{L^\infty}$$

$$\leqslant C\|g\|_{L^\infty} \sum_{j'\geqslant j-4} \|\Delta_{j'} f\|_{L^\infty} \,. \tag{2.4}$$

Hence it follows from (2.1), (2.2), (2.3) and (2.4) that

$$\begin{aligned}
\|fg\|_{B^1_{\infty,1}} &= \sum_{j\in\mathbb{Z}} 2^j \, \|\Delta_j(fg)\|_{L^\infty} \\
&\leqslant C\|f\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}} 2^j \, \|\Delta_{j'} g\|_{L^\infty} + C\|g\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{\substack{j'\geqslant 2\\ |j'-j|\leqslant 3}} 2^j \, \|\Delta_{j'} f\|_{L^\infty} \\
&\quad + C\|g\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{j'\geqslant j-4} 2^j \, \|\Delta_{j'} f\|_{L^\infty} \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{2.5}$$

For the estimate of J_1, we have that

$$\begin{aligned}
I_1 &\leqslant C\|f\|_{L^\infty} \sum_{|k|\leqslant 3} 2^{-k} \sum_{j=-1}^{\infty} 2^{j+k} \, \|\Delta_{j+k} g\|_{L^\infty} \\
&\leqslant C\|f\|_{L^\infty} \|g\|_{B^1_{\infty,1}}.
\end{aligned} \tag{2.6}$$

Similarly, we have for I_2 that

$$I_2 \leqslant C\|g\|_{L^\infty} \|f\|_{B^1_{\infty,1}}. \tag{2.7}$$

Concerning the estimate of I_3, we have

$$\begin{aligned}
I_3 &\leqslant C\|g\|_{L^\infty} \sum_{k\geqslant -4} 2^{-k} \sum_{j=-1}^{\infty} 2^{j+k} \, \|\Delta_{j+k} f\|_{L^\infty} \\
&\leqslant C\|g\|_{L^\infty} \|f\|_{B^1_{\infty,1}}.
\end{aligned} \tag{2.8}$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain that

$$\|fg\|_{B^1_{\infty,1}} \leqslant C\big(\|f\|_{L^\infty} \|g\|_{B^1_{\infty,1}} + \|g\|_{L^\infty} \|f\|_{B^1_{\infty,1}}\big).$$

This completes the proof of Lemma 2.2. □

Next, we give the estimate for the gradient of pressure $\pi = \nabla p$.

Lemma 2.3 (Pak-Park [13]). *There exists a positive constant $C = C(n)$ such that*

$$\|\pi(u,v)\|_{B^1_{\infty,1}} \leqslant C\|u\|_{B^1_{\infty,1}} \|v\|_{B^1_{\infty,1}}$$

holds for all $u, v \in B^1_{\infty,1}(\mathbb{R}^n)^n$ with $\operatorname{div} u = \operatorname{div} v = 0$, where

$$\pi(u,v) = \sum_{j,k=1}^{n} \nabla(-\Delta)^{-1} \partial_{x_j} u^k \partial_{x_k} v^j = \nabla(-\Delta)^{-1} \operatorname{div}\{(u\cdot\nabla)v\}\,.$$

Finally, we recall the Gronwall inequality.

Lemma 2.4 (The Gronwall inequality). *Let $A \geqslant 0$, and let f, g and h be non-negative, continuous functions on $[0,T]$ satisfying*

$$f(t) \leqslant A + \int_0^t g(s)ds + \int_0^t h(s)f(s)ds$$

for all $t \in [0,T]$. Then it holds that

$$f(t) \leqslant Ae^{\int_0^t h(\tau)d\tau} + \int_0^t e^{\int_s^t h(\tau)d\tau} g(s)ds$$

for all $t \in [0,T]$.

3 Proof of Theorem 1.2

Proof of Theorem 1.2. Let u_0 satisfy the assumption of Theorem 1.2. We first remark that $u \in C([0,T]; B^s_{\infty,1}(\mathbb{R}^n)^n)$ for all $s \geqslant 1$ if $u_0 \in B^s_{\infty,1}(\mathbb{R}^n)^n$ for all $s \geqslant 1$. Hence $u(\cdot,t) \in C^\infty(\mathbb{R}^n)^n$ for all $t \in [0,T]$ by our assumption on the initial velocity u_0 and the embedding theorem. Moreover, the time-interval in which the solution exists does not depend on s. Indeed, we can choose T such that $T \geqslant C/\|u_0\|_{B^1_{\infty,1}}$ with some positive constant C depending only on n by the blow-up criterion, and the solution u satisfies

$$\sup_{t\in[0,T]} \|u(t)\|_{B^1_{\infty,1}} \leqslant C_0\|u_0\|_{B^1_{\infty,1}} \tag{3.1}$$

with some positive constant C_0 depending only on n.

Now we discuss with the induction argument. In the case $\alpha = 0$, (1.1) follows from (3.1) with $K = C_0K_0$. Next, we consider the case $|\alpha| \geqslant 1$. We first introduce some notation. For $l \in \mathbb{N}$ and $\lambda, L > 0$, we put

$$X_l(t) := \max_{|\alpha|=l} \|\partial_x^\alpha u(t)\|_{B^1_{\infty,1}}, \quad t \in [0,T],$$

$$Y_l = Y_l^{\lambda,L} := \max_{1\leqslant k\leqslant l} \sup_{t\in[0,T]} \left\{ \frac{M_k(t)}{m_k} X_k(t) \right\},$$

where

$$M_k(t) = M_k^{\lambda,L}(t) := \rho_0^k L^{-(k-1)}(1+t)^{-(k-1)} e^{-\lambda k \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau}.$$

The similar notaion were used in [1] and [2]. In what follows, we shall show that $Y_{|\alpha|} \leqslant 2K_0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 1$ when λ and L are sufficiently large. We now consider the case $|\alpha| = 1$. Let k be an integer with $1 \leqslant k \leqslant n$. Taking the differential operation ∂_{x_k} to the first equation of (E), we have

$$\partial_t(\partial_{x_k}u) + (\partial_{x_k}u \cdot \nabla)u + (u\cdot\nabla)\partial_{x_k}u + \partial_{x_k}\pi(u,u) = 0, \tag{3.2}$$

where

$$\nabla p = \pi(u,u) = \sum_{j,k=1}^n \nabla(-\Delta)^{-1}\partial_{x_j}u^k\partial_{x_k}u^j = \nabla(-\Delta)^{-1}\operatorname{div}\{(u\cdot\nabla)u\}.$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)$ to the both sides of (3.2), we have

$$\begin{aligned}&\partial_t\Delta_j(\partial_{x_k}u)+(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)\\&\quad=(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)-\Delta_j((u\cdot\nabla)\partial_{x_k}u)-\Delta_j((\partial_{x_k}u\cdot\nabla)u)-\Delta_j(\partial_{x_k}\pi(u,u)).\end{aligned}\tag{3.3}$$

Here we consider the family of trajectory flows $\{Z_j(y,t)\}$ defined by the solution of the ordinary differential equations

$$\begin{cases}\dfrac{\partial}{\partial t}Z_j(y,t)=S_{j-2}u(Z_j(y,t),t),\\ Z_j(y,0)=y.\end{cases}\tag{3.4}$$

Note that $Z_j\in C^1(\mathbb{R}^n\times[0,T])^n$, and $\operatorname{div} S_{j-2}u=0$ implies that each $y\mapsto Z_j(y,t)$ is a volume preserving mapping from $\mathbb{R}^n$ onto itself. From (3.3) and (3.4), we see that

$$\partial_t\Delta_j(\partial_{x_k}u)+(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)\Big|_{(x,t)=(Z_j(y,t),t)}=\frac{\partial}{\partial t}\left\{\Delta_j(\partial_{x_k}u)(Z_j(y,t),t)\right\},$$

which yields that

$$\begin{aligned}\Delta_j(\partial_{x_k}u)(Z_j(y,t),t)&=\Delta_j(\partial_{x_k}u_0)(y)-\int_0^t\Delta_j((\partial_{x_k}u\cdot\nabla)u)(Z_j(y,s),s)ds\\&\quad+\int_0^t\left\{(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)-\Delta_j((u\cdot\nabla)\partial_{x_k}u)\right\}(Z_j(y,s),s)ds\\&\quad-\int_0^t\Delta_j(\partial_{x_k}\pi(u,u))(Z_j(y,s),s)ds.\end{aligned}\tag{3.5}$$

Since the map $y\mapsto Z_j(y,t)$ is bijective and volume-preserving for all $t\in[0,T]$, by taking the L^∞-norm with respect to y to both sides of (3.5), we have

$$\begin{aligned}\|\Delta_j(\partial_{x_k}u)(t)\|_{L^\infty}&\leqslant\|\Delta_j(\partial_{x_k}u_0)\|_{L^\infty}+\int_0^t\|\Delta_j((\partial_{x_k}u\cdot\nabla)u)(s)\|_{L^\infty}ds\\&\quad+\int_0^t\left\|\left\{(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)-\Delta_j((u\cdot\nabla)\partial_{x_k}u)\right\}(s)\right\|_{L^\infty}ds\\&\quad+\int_0^t\|\Delta_j(\partial_{x_k}\pi(u,u))(s)\|_{L^\infty}ds.\end{aligned}\tag{3.6}$$

Multiplying both sides of (3.6) by 2^j and then taking the ℓ^1-norm in j, we obtain that

$$\begin{aligned}\|\partial_{x_k}u(t)\|_{B^1_{\infty,1}}&\leqslant\|\partial_{x_k}u_0\|_{B^1_{\infty,1}}+\int_0^t\|(\partial_{x_k}u\cdot\nabla)u(s)\|_{B^1_{\infty,1}}ds+\int_0^t\|\partial_{x_k}\pi(u,u)(s)\|_{B^1_{\infty,1}}ds\\&\quad+\int_0^t\sum_{j\in\mathbb{Z}}2^j\left\|\left\{(S_{j-2}u\cdot\nabla)\Delta_j(\partial_{x_k}u)-\Delta_j((u\cdot\nabla)\partial_{x_k}u)\right\}(s)\right\|_{L^\infty}ds\\&=:I_1+I_2+I_3+I_4.\end{aligned}\tag{3.7}$$

It follows from the assumption on u_0 that

$$I_1\leqslant K_0\rho_0^{-1}m_1.\tag{3.8}$$

From Lemma 2.2, we see that

$$
\begin{aligned}
I_2 &\leqslant C\int_0^t \|\nabla u(s)\|_{L^\infty}\|\nabla u(s)\|_{B^1_{\infty,1}}ds \\
&\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_1(s)ds,
\end{aligned}
\tag{3.9}
$$

where we used the continuous embedding $B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n)$. For the pressure term I_3, it follow from Lemma 2.3 that

$$
\begin{aligned}
I_3 &\leqslant 2\int_0^t \|\pi(\partial_{x_k}u, u)(s)\|_{B^1_{\infty,1}}ds \\
&\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_1(s)ds.
\end{aligned}
\tag{3.10}
$$

For the estimate of I_4, we have from Lemma 2.1 that

$$
\begin{aligned}
I_4 &\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}\|\partial_{x_k}u(s)\|_{B^1_{\infty,1}}ds \\
&\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_1(s)ds.
\end{aligned}
\tag{3.11}
$$

Substituting (3.8), (3.9), (3.10) and (3.11) into (3.7), we have

$$
\|\partial_{x_k}u(t)\|_{B^1_{\infty,1}} \leqslant K_0\rho_0^{-1}m_1 + C_1\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_1(s)ds \tag{3.12}
$$

with some positive constant C_1 depending only on n. Since $k \in \{1, \dots, n\}$ is arbitrary, it follows from (3.12) that

$$
X_1(t) \leqslant K_0\rho_0^{-1}m_1 + C_1\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_1(s)ds,
$$

which implies by Lemma 2.4 that

$$
X_1(t) \leqslant K_0\rho_0^{-1}m_1 e^{C_1\int_0^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau}. \tag{3.13}
$$

By choosing $\lambda \geqslant C_1$, we obtain from (3.13) that

$$
\frac{M_1(t)}{m_1}X_1(t) \leqslant K_0 e^{(C_1-\lambda)\int_0^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau} \leqslant K_0,
$$

which yields that

$$
Y_1 \leqslant K_0. \tag{3.14}
$$

Next, we consider the case $|\alpha| \geqslant 2$. Let α be a multi-index with $|\alpha| \geqslant 2$. Taking the differential operation ∂_x^α to the first equation of (E), we have

$$
\partial_t(\partial_x^\alpha u) + \sum_{0\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}(\partial_x^\beta u\cdot\nabla)\partial_x^{\alpha-\beta}u + \partial_x^\alpha\pi(u,u) = 0. \tag{3.15}
$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2}u\cdot\nabla)\Delta_j(\partial_x^\alpha u)$ to the both sides of (3.15), we have

$$\begin{aligned}\partial_t\Delta_j(\partial_x^\alpha u)&+(S_{j-2}u\cdot\nabla)\Delta_j(\partial_x^\alpha u)\\ &=(S_{j-2}u\cdot\nabla)\Delta_j(\partial_x^\alpha u)-\Delta_j((u\cdot\nabla)\partial_x^\alpha u)\\ &\quad-\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\Delta_j\left((\partial_x^\beta u\cdot\nabla)\partial_x^{\alpha-\beta}u\right)-\Delta_j(\partial_x^\alpha\pi(u,u))\end{aligned}\tag{3.16}$$

Similarly to the case of $|\alpha|=1$, we have from (3.16) that

$$\begin{aligned}\|\Delta_j(\partial_x^\alpha u)(t)\|_{L^\infty}\leqslant{}&\|\Delta_j(\partial_x^\alpha u_0)\|_{L^\infty}\\ &+\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_0^t\left\|\Delta_j\left((\partial_x^\beta u\cdot\nabla)\partial_x^{\alpha-\beta}u\right)(s)\right\|_{L^\infty}ds\\ &+\int_0^t\|\Delta_j(\partial_x^\alpha\pi(u,u))(s)\|_{L^\infty}ds\\ &+\int_0^t\|\{(S_{j-2}u\cdot\nabla)\Delta_j(\partial_x^\alpha u)-\Delta_j((u\cdot\nabla)\partial_x^\alpha u)\}(s)\|_{L^\infty}ds.\end{aligned}\tag{3.17}$$

Multiplying both sides of (3.17) by 2^j and then taking the ℓ^1-norm in j, we obtain that

$$\begin{aligned}\|\partial_x^\alpha u(t)\|_{B^1_{\infty,1}}\leqslant{}&\|\partial_x^\alpha u_0\|_{B^1_{\infty,1}}\\ &+\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_0^t\|(\partial_x^\beta u\cdot\nabla)\partial_x^{\alpha-\beta}u(s)\|_{B^1_{\infty,1}}ds\\ &+\int_0^t\|\partial_x^\alpha\pi(u,u)(s)\|_{B^1_{\infty,1}}ds\\ &+\int_0^t\sum_{j\in\mathbb{Z}}2^j\|\{(S_{j-2}u\cdot\nabla)\Delta_j(\partial_x^\alpha u)-\Delta_j((u\cdot\nabla)\partial_x^\alpha u)\}(s)\|_{L^\infty}ds\\ =:{}&J_1+J_2+J_3+J_4.\end{aligned}\tag{3.18}$$

It follows from the assumption on u_0 that

$$J_1\leqslant K_0\rho_0^{-|\alpha|}m_{|\alpha|}.\tag{3.19}$$

For the estimate of J_2, we have from Lemma 2.2 and the continuous embedding that

$$\begin{aligned}J_2&\leqslant C\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_0^t\left(\|\partial_x^\beta u(s)\|_{L^\infty}\|\nabla\partial_x^{\alpha-\beta}u(s)\|_{B^1_{\infty,1}}+\|\nabla\partial_x^{\alpha-\beta}u(s)\|_{L^\infty}\|\partial_x^\beta u(s)\|_{B^1_{\infty,1}}\right)ds\\ &=C\sum_{j=1}^n\binom{\alpha}{e_j}\int_0^t\|\partial_{x_j}u(s)\|_{L^\infty}\|\nabla\partial_x^{\alpha-e_j}u(s)\|_{B^1_{\infty,1}}ds\\ &\quad+C\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant2}}\binom{\alpha}{\beta}\int_0^t\|\partial_x^\beta u(s)\|_{L^\infty}\|\nabla\partial_x^{\alpha-\beta}u(s)\|_{B^1_{\infty,1}}ds\\ &\quad+C\int_0^t\|\nabla u(s)\|_{L^\infty}\|\partial_x^\alpha u(s)\|_{B^1_{\infty,1}}ds\end{aligned}$$

$$+C\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}\int_0^t \|\nabla\partial_x^{\alpha-\beta}u(s)\|_{L^\infty}\|\partial_x^\beta u(s)\|_{B^1_{\infty,1}}ds$$
$$\leqslant C|\alpha|\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds+C\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant 2}}\binom{\alpha}{\beta}\int_0^t X_{|\beta|-1}(s)X_{|\alpha-\beta|+1}(s)ds$$
$$+C\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}\int_0^t X_{|\beta|}(s)X_{|\alpha-\beta|}(s)ds. \tag{3.20}$$

For the pressure term J_3, from Lemma 2.3, we have

$$J_3\leqslant \sum_{0\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_0^t \|\pi(\partial_x^\beta u,\partial_x^{\alpha-\beta}u)(s)\|_{B^1_{\infty,1}}ds$$
$$\leqslant C\sum_{0\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_0^t \|\partial_x^\beta u(s)\|_{B^1_{\infty,1}}\|\partial_x^{\alpha-\beta}u(s)\|_{B^1_{\infty,1}}ds$$
$$\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds+C\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}\int_0^t X_{|\beta|}(s)X_{|\alpha-\beta|}(s)ds. \tag{3.21}$$

For the estimate of J_4, it follows from Lemma 2.1 that

$$\begin{aligned}J_4&\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}\|\partial_x^\alpha u(s)\|_{B^1_{\infty,1}}ds\\&\leqslant C\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds.\end{aligned} \tag{3.22}$$

Substituting (3.19), (3.20), (3.21) and (3.22) to (3.18), we have

$$\begin{aligned}\|\partial_x^\alpha u(t)\|_{B^1_{\infty,1}}\leqslant{}& K_0\rho_0^{-|\alpha|}m_{|\alpha|}+C|\alpha|\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds\\&+C\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant 2}}\binom{\alpha}{\beta}\int_0^t X_{|\beta|-1}(s)X_{|\alpha-\beta|+1}(s)ds\\&+C\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}\int_0^t X_{|\beta|}(s)X_{|\alpha-\beta|}(s)ds.\end{aligned} \tag{3.23}$$

Furthermore, for the third term of the right-hand side of (3.23), we see that

$$\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant 2}}\binom{\alpha}{\beta}\int_0^t X_{|\beta|-1}(s)X_{|\alpha-\beta|+1}(s)ds$$
$$=\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant 2}}\binom{\alpha}{\beta}\int_0^t \frac{M_{|\beta|-1}(s)}{m_{|\beta|-1}}X_{|\beta|-1}(s)\frac{M_{|\alpha-\beta|+1}(s)}{m_{|\alpha-\beta|+1}}X_{|\alpha-\beta|+1}(s)\frac{m_{|\beta|-1}}{M_{|\beta|-1}(s)}\frac{m_{|\alpha-\beta|+1}}{M_{|\alpha-\beta|+1}(s)}ds$$
$$\leqslant\sum_{\substack{0<\beta\leqslant\alpha\\|\beta|\geqslant 2}}\binom{\alpha}{\beta}m_{|\beta|-1}m_{|\alpha-\beta|+1}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds$$

$$\leqslant |\alpha| m_{|\alpha|}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds. \tag{3.24}$$

Similarly, for the fourth term of the right hand side of (3.23), we have

$$\begin{aligned}&\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}\int_0^t X_{|\beta|}(s)X_{|\alpha-\beta|}(s)ds\\ &\quad\leqslant m_{|\alpha|}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds.\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) to (3.23), we have

$$\begin{aligned}\|\partial_x^\alpha u(t)\|_{B^1_{\infty,1}} &\leqslant K_0\rho_0^{-|\alpha|}m_{|\alpha|} + C|\alpha|\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds\\ &\quad + C|\alpha| m_{|\alpha|}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds,\end{aligned}$$

which implies that

$$\begin{aligned}X_{|\alpha|}(t) &\leqslant K_0\rho_0^{-|\alpha|}m_{|\alpha|} + C|\alpha|\int_0^t \|u(s)\|_{B^1_{\infty,1}}X_{|\alpha|}(s)ds\\ &\quad + C|\alpha| m_{|\alpha|}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds.\end{aligned} \tag{3.26}$$

By Lemma 2.4, we obtain from (3.26) that

$$\begin{aligned}X_{|\alpha|}(t) &\leqslant K_0\rho_0^{-|\alpha|}m_{|\alpha|}e^{C_2|\alpha|\int_0^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau} + C_2|\alpha| m_{|\alpha|}\rho_0^{-|\alpha|}L^{|\alpha|-2}(Y_{|\alpha|-1})^2\\ &\quad\times\int_0^t (1+s)^{|\alpha|-2}e^{C_2|\alpha|\int_s^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau+\lambda|\alpha|\int_0^s \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds\end{aligned}$$

with some positive constant C_2 depending only on n. By choosing $\lambda \geqslant C_2$ and $L \geqslant 1$, we thus have

$$\begin{aligned}\frac{M_{|\alpha|}(t)}{m_{|\alpha|}}X_{|\alpha|}(t) &\leqslant K_0L^{-(|\alpha|-1)}(1+t)^{-(|\alpha|-1)}e^{(C_2-\lambda)|\alpha|\int_0^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau}\\ &\quad + C_2|\alpha|L^{-1}(1+t)^{-(|\alpha|-1)}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}e^{(C_2-\lambda)|\alpha|\int_s^t \|u(\tau)\|_{B^1_{\infty,1}}d\tau}ds\\ &\leqslant K_0 + C_2|\alpha|L^{-1}(1+t)^{-(|\alpha|-1)}(Y_{|\alpha|-1})^2\int_0^t (1+s)^{|\alpha|-2}ds\\ &\leqslant K_0 + \frac{2C_2}{L}(Y_{|\alpha|-1})^2.\end{aligned}$$

The above estimate with (3.14) implies that

$$Y_{|\alpha|} \leqslant K_0 + \frac{2C_2}{L}(Y_{|\alpha|-1})^2 \tag{3.27}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 2$. From (3.14) and (3.27), we obtain by the standard inductive argument that

$$Y_{|\alpha|} \leqslant 2K_0 \tag{3.28}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 1$, provided $\lambda \geqslant \max\{C_1, C_2\}$ and $L \geqslant \max\{1, 8C_2K_0\}$. Therefore, it follows from (3.28) that

$$\|\partial_x^\alpha u(t)\|_{B^1_{\infty,1}} \leqslant \frac{2K_0}{L}\left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|}(1+t)^{|\alpha|-1} e^{\lambda|\alpha|\int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau} \tag{3.29}$$

for all $t \in [0, T]$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 1$. From (3.1) and (3.29) with $K = K_0 \max\{C_0, 2/L\}$, we complete the proof of Theorem 1.2. □

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