数理解析研究所講究録
第 1798 巻 2012 年 188-197

Blow-up analysis and optimal Trudinger-Moser inequalities for some mean field equations in statistical hydrodynamics

Tonia Ricciardi
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli Federico II
Via Cintia - 80126 Napoli - Italy
E-mail: tonia.ricciardi@unina.it

Abstract

We outline some mathematical results concerning mean field equations derived within Onsager's statistical hydrodynamics theory. Such equations contain a probability measure describing the circulations of the vortex points. Our analysis shows that in the deterministic vs. stochastic approach we obtain similar blow-up properties of bubbling solutions, whereas the corresponding optimal Trudinger-Moser constants, corresponding to the critical temperatures, may be substantially different.

1 Some mean field equations from Onsager's vortex theory

In recent years, several mean field equations have been derived in order to describe two-dimensional turbulence, following Onsager's celebrated statistical mechanics approach [21], see also [10, 28]. By well-known work of Caglioti, Lions, Marchioro and Pulvirenti [2, 3] and Kiessling [14] it is well-known and rigorously established that under the assumption that all vortex points have identical vorticity and orientation, the mean field limit, in the case of a compact two-dimensional surface without boundary Ω, is described by the semilinear elliptic equation with exponential nonlinearity:

$$\begin{cases} -\Delta v=\lambda\left(\dfrac{e^v}{\int_\Omega e^v\,dx}-\dfrac{1}{|\Omega|}\right) & \text{on } \Omega,\\ \displaystyle\int_\Omega v\,dx=0. \end{cases} \tag{1}$$

Here, v is the stream function, $\lambda>0$ is a constant related to the statistical temperature and dx denotes the surface element on Ω. The normalization $\int_\Omega v\,dx=0$ is chosen in order to rule out an additive constant. It is worth noticing that equation (1) is also relevant in other contexts, including differential geometry (Nirenberg's problem), chemotaxis, Chern-Simons gauge theory. Consequently, equation (1) and its variations have been extensively studied, particularly in relation to the blow-up properties of concentrating sequences of solutions, topological degree properties, existence, uniqueness, symmetry of solutions, just to mention a few aspects. See, e.g., [15, 28]. Equation (1) is the Euler-Lagrange equation of the functional

$$\mathcal{I}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v \, dx \tag{2}$$

defined on the space

$$\mathcal{E} = \left\{ v \in H^1(\Omega) : \int_\Omega v \, dx = 0 \right\}.$$

In view of the classical Trudinger-Moser inequality, as established in [11]:

$$\sup \left\{ \int_\Omega e^{4\pi v^2} : v \in \mathcal{E}, \ \|\nabla v\|_2 \leqslant 1 \right\} < +\infty,$$

where the constant 4π is sharp, we derive that

$$\inf_{v \in \mathcal{E}} \mathcal{I}_\lambda(v) > -\infty \Longleftrightarrow \lambda \leq 8\pi. \tag{3}$$

In the context of the statistical mechanics of vortices the optimal constant $\lambda = 8\pi$ is related to the critical temperature. An alternative proof of (3) was derived in [2].

Here, we are interested in some generalizations of (1) which were recently derived in the above mentioned statistical mechanics context, with the aim of considering vortex points with arbitrary circulation and orientation. Assuming that the distribution of circulations is determined by a general Borel probability measure $\mathcal{P} \in \mathcal{M}(I)$, where $I = [-1, 1]$, by extending the methods introduced in Joyce and Montgomery [13], Pointin and Lundgren [22], the following "continuous" equation was derived in [26]:

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v} \, dx} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) & \text{on } \Omega \\ \displaystyle\int_\Omega v \, dx = 0. \end{cases} \tag{4}$$

The variational functional for (4) is given by

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_I \log \left(\int_\Omega e^{\alpha v} \, dx \right) \mathcal{P}(d\alpha), \tag{5}$$

with $v \in \mathcal{E}$. The first mathematical results concerning the general equation (4) are rather recent, see [18]. However, some special cases of $\mathcal{P}$ have been considered. In particular, the "hyperbolic sine case"

$$\mathcal{P} = t\delta_{\alpha=-1} + (1-t)\delta_{\alpha=1}, \qquad t \in [0, 1], \tag{6}$$

which corresponds to the physics models considered in [13, 22], was considered in [12, 19, 20, 9]. When $\mathcal{P}$ has the atomic form

$$\mathcal{P} = \sum_{i=1}^{N} a_i \delta_{\alpha_i}, \tag{7}$$

where $\alpha_i \in I$ for $i = 1, 2, \ldots, N$, $a_i \geq 0$, $\sum_{i=1}^{N} a_i = 1$, equation (4) is equivalent to a Liouville system of the form

$$\begin{cases} -\Delta u_i = \lambda \left(V_i e^{\sum_{j=1}^{N} a_{ij} u_{ij}} - \kappa_i \right), \ 1 \leq i \leq N, & \text{in } \Omega \\ \displaystyle\int_\Omega u_i \, dx = 0, \end{cases}$$

where V_i is a continuous function and κ_i is a suitable constant ensuring that the right hand side in the equation above has zero mean. Systems of this form have been extensively analyzed by Chanillo and Kiessling [4], Chipot, Shafrir, Wolansky [7]. The related optimal Trudinger-Moser inequalities, in their equivalent dual logarithmic Hardy-Littlewood-Sobolev form, were obtained by Shafrir and Wolansky [27].

On the other hand, an equation similar to (4) was derived by Neri [17], under the assumption that the circulations of the vortex points are independent identically distributed random variables with probability distribution $\mathcal{P}$:

$$\begin{cases} -\Delta v = \lambda \dfrac{\int_I \alpha (e^{\alpha v} - \frac{1}{|\Omega|} \int_\Omega e^{\alpha v} dx) \, \mathcal{P}(d\alpha)}{\iint_{I \times \Omega} e^{\alpha v} \, \mathcal{P}(d\alpha) dx} & \text{on } \Omega \\ \displaystyle\int_\Omega v \, dx = 0. \end{cases} \tag{8}$$

The variational functional for (8) is given by

$$\mathcal{K}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left(\int_I \int_\Omega e^{\alpha v} \, dx \mathcal{P}(d\alpha) \right), \tag{9}$$

with $v \in \mathcal{E}$. Although similar, the "deterministic" equation (4) and the "stochastic" equation (8) are distinct unless $\mathcal{P} = \delta_{\alpha=1}$, in which case they both reduce to the standard mean field equation (1). It is therefore natural to seek similarities and differences between (4)–(8). Our studies recently carried out in the articles [18, 23, 24, 25] show that:

(i) Equation (4) and equation (8) share analogous blow-up properties;

(ii) Functional $\mathcal{J}_\lambda$ and functional $\mathcal{K}_\lambda$ have substantially different Trudinger-Moser optimal constants.

We summarize such results in the remaining part of this note. More precisely, in Section 2 we describe blow-up properties for a general equation containing (4) and equation (8) as special cases, thus emphasizing analogous behaviours of bubbling solutions. In Section 3 we provide the optimal Trudinger-Moser constants for both models. In particular, we show that such a best constant for the "stochastic" functional (9) coincides with the standard constant $\lambda = 8\pi$ in (3), whereas it is greater than 8π for the "deterministic" functional (5), see Theorem 3.1 below.

2 Blow-up properties

As already mentioned above, a blow-up analysis for equation (4) is provided in [18], extending techniques from [1, 19]. One difficulty in carrying out such an extension is due to the general form of the Borel measure $\mathcal{P}$, which in particular does not allow to assume satisfactory convergences of quantities indexed in $\alpha \in I$ by simply extracting subsequences. For this reason, a new point of view of considering concentrating measures on the *product space* $I \times \Omega$ was taken. It is not difficult to see that the main blow-up results from [18] can be extended to equation (8). Thus, in [24] we were motivated to prove blow-up properties for a class of equations including (4) and (8) as special cases.

More precisely, in [24] we study concentrating sequences of solutions to the following equation:

$$\begin{cases} -\Delta v = \lambda \int_I V(\alpha, x, v) e^{\alpha v} \mathcal{P}(d\alpha) - \frac{\lambda}{|\Omega|} \iint_{I\times\Omega} V(\alpha, x, v) e^{\alpha v} \mathcal{P}(d\alpha) dx, & \text{in } \Omega \\ \int_\Omega v \, dx = 0, \end{cases} \tag{10}$$

where $V(\alpha, x, v)$ is a *functional* satisfying the condition $\alpha V(\alpha, x, v) \geq 0$, as well as suitable bounds which will be specified below. Clearly, when

$$V(\alpha, x, v) = V_1(\alpha, v) = \frac{\alpha}{\int_\Omega e^{\alpha v} \, dx}, \tag{11}$$

equation (10) reduces to (4). On the other hand, when

$$V(\alpha, x, v) = V_2(\alpha, v) = \frac{\alpha}{\iint_{I\times\Omega} e^{\alpha v} \, \mathcal{P}(d\alpha)}, \tag{12}$$

equation (10) reduces to (8). We make the following assumptions on the functional V.

(V1) $(\operatorname{sign} \alpha) \, V(\alpha, x, v) \geq 0$ for all $(\alpha, x, v) \in I \times \Omega \times \mathcal{E}$;

(V2) $\sup_{\mathcal{E}} \|V(\alpha, x, v(x))\|_{L^\infty(I\times\Omega)} \leqslant C_1$ for some constant $C_1 > 0$;

(V3) $\iint_{I\times\Omega} |V(\alpha, x, v)| e^{\alpha v} \mathcal{P}(d\alpha) dx \leqslant C_2$ for some constant $C_2 > 0$.

We consider solution sequences $\{v_n\}$, $\lambda_n \to \lambda_0$ to

$$\begin{cases} -\Delta v_n = \lambda_n \int_I \left(V(\alpha, x, v_n) e^{\alpha v_n} - \frac{1}{|\Omega|} \int_\Omega V(\alpha, x, v_n) e^{\alpha v_n} \, dx \right) \mathcal{P}(d\alpha) \\ \int_\Omega v_n = 0. \end{cases} \tag{13}$$

Following the approach of Brezis and Merle [1], see also Nagasaki and Suzuki [16], we first show that the blow-up set for concentrating solutions is finite and that a "minimum mass" is necessary for blow-up to occur. Namely, we define the blow-up sets:

$$\mathcal{S}_\pm = \{p \in \Omega : \exists p_{\pm,n} \to p : v_n(p_{\pm,n}) \to \pm\infty)\}$$

and denote $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. We define the measures $\nu_{\pm,n} \in \mathcal{M}(\Omega)$ by setting

$$\nu_{\pm,n} = \lambda_n \int_{I_\pm} |V(\alpha, x, v_n)| e^{\alpha v_n} \mathcal{P}(d\alpha)$$

where $I_+ = [0,1]$ and $I_- = [-1,0)$. Since in view of (V3) we have $\int_\Omega \nu_{\pm,n} \leqslant C_2\lambda_n$, we may assume that $\nu_{\pm,n} \overset{*}{\rightharpoonup} \nu_\pm$ for some measure $\nu_\pm \in \mathcal{M}(\Omega)$.

Theorem 2.1 ([24], Brezis-Merle alternative). *Assume (V1)-(V2)-(V3). Let v_n be a solution sequence to* (13) *with $\lambda_n \to \lambda_0$. Then, the following alternative holds.*

i) Compactness: $\limsup_{n\to\infty} \|v_n\|_\infty < +\infty$. There exist a solution $v \in \mathcal{E}$ to (10) *with $\lambda = \lambda_0$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$.*

ii) Concentration: $\limsup_{n\to\infty} \|v_n\|_{L^\infty} = +\infty$. The sets $\mathcal{S}_\pm$ are finite and $\mathcal{S} = \mathcal{S}_- \cup \mathcal{S}_+ \neq \emptyset$. For some $s_\pm \geqslant 0$, $s_\pm \in L^1(\Omega)$ we have

$$\nu_\pm = s_\pm dx + \sum_{p\in\mathcal{S}_\pm} n_{\pm,p}\delta_p$$

with $n_{\pm,p} \geqslant 4\pi$ for all $p \in \mathcal{S}$. Moreover, there exist $v \in H^1_{\text{loc}}(\Omega \setminus \mathcal{S})$, $k \in L^\infty(I\times\Omega)$ and $c_0 \in \mathbb{R}$ such that $v_n \to v$ in $H^1_{\text{loc}}(\Omega\setminus\mathcal{S})$ and

$$\begin{cases} -\Delta v = \lambda_0 \displaystyle\int_I k(\alpha,x) e^{\alpha v}\mathcal{P}(d\alpha) + \sum_{p\in\mathcal{S}_+} n_{+,p}\delta_p - \sum_{p\in\mathcal{S}_-} n_{-,p}\delta_p - c_0 & \text{in } \Omega, \\ \displaystyle\int_\Omega v = 0. \end{cases} \tag{14}$$

Under stronger assumptions on V, which are satisfied in the physically relevant cases (11)–(12), the blow-up results may be refined. Following [18], we consider measures defined on the product space $I\times\Omega$. We assume that V does not depend on x, namely $V = V(\alpha, v)$ and

(V0) $\nabla_x V(\alpha, v) = 0$.

We also strengthen assumptions (V2)–(V3) above as follows:

(V2') $\sup_{\mathcal{E}} \|\alpha^{-1}V(\alpha,v)\|_{L^\infty(I)} \leqslant C_1'$ for some constant $C_1' > 0$;

(V3') $\iint_{I\times\Omega} |\alpha^{-1}V(\alpha,v)| e^{\alpha v}\mathcal{P}(d\alpha)dx \leqslant C_2'$ for some constant $C_2' > 0$.

For every fixed $\alpha \in I$ we define $\mu_\alpha^n(dx) \in \mathcal{M}(\Omega)$ by setting

$$\mu_\alpha^n(dx) = \lambda_n \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n}\, dx.$$

We consider the sequence of measures $\mu_n = \mu_n(d\alpha dx) \in \mathcal{M}(I\times\Omega)$ defined by

$$\mu_n(d\alpha dx) = \mu_\alpha^n(dx)\mathcal{P}(d\alpha) = \lambda_n \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n}\, dx\mathcal{P}(d\alpha).$$

In view of (V3'), for large values of n we have:

$$\mu_n(I\times\Omega) = \iint_{I\times\Omega} \mu_\alpha^n(dx)\mathcal{P}(d\alpha) \leqslant C_2'(\lambda_0 + 1).$$

Hence, upon extracting a subsequence, we may assume that

$$\mu_n \stackrel{*}{\rightharpoonup} \mu \text{ for some Borel measure } \mu \in \mathcal{M}(I \times \Omega).$$

In the next result we describe some properties of μ.

Theorem 2.2 ([24], quadratic relation for the mass measure). *Suppose that V satisfies (V0)-(V1)-(V2')-(V3'). Let v_n be a solution sequence to* (13) *with $\lambda_n \to \lambda_0$. The following properties hold.*

(i) The singular part of μ has a "separation of variables" form:

$$\mu(d\alpha dx) = \sum_{p \in \mathcal{S}} \zeta_p(d\alpha)\delta_p(dx) + r(\alpha, x)\mathcal{P}(d\alpha)dx.$$

Here, $\zeta_p \in \mathcal{M}(I)$ and $r \in L^1(I \times \Omega)$.

(ii) For every $p \in \mathcal{S}$ the following relation is satisfied

$$8\pi \int_I \zeta_p(d\alpha) = \left[\int_I \alpha\zeta_p(d\alpha)\right]^2.$$

(iii) For every $p \in \mathcal{S}$ it holds

$$\int_{I_\pm} |\alpha|\zeta_p(d\alpha) = n_{\pm,p} \qquad \int_{I_\pm} |\alpha| r(\alpha, x)\mathcal{P}(d\alpha) = s_\pm(x),$$

where $n_{\pm,p}$ and $s_\pm(x)$ are as in Theorem 2.1. Moreover, for every $p \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$

$$\int_{I_\mp} |\alpha|\zeta_p(d\alpha) = 0.$$

We note that $\zeta_p \in \mathcal{M}(I)$ plays the role of the "mass measure" at the blow-up point $p \in \Omega$. In the deterministic case (4), we are able to show that ζ_p is absolutely continuous with respect to $\mathcal{P}(d\alpha)$, and more precisely that $\zeta_p(d\alpha) = m_p(\alpha)\,\mathcal{P}(d\alpha)$ for some $m_p \in L^\infty(I)$. We do not know whether such a property holds for the stochastic case (8). However, this is of course the case when $\mathcal{P}(d\alpha)$ is discrete, and in particular when $\mathcal{P}(d\alpha)$ has the hyperbolic sine form (6). Further analogies between (4) and (8) in the special case (6) have been emphasized in [25]. We refer to [25] for the details.

3 Optimal Trudinger-Moser inequalities

In this section we emphasize that the best constants in the corresponding Trudinger-Moser inequalities for the functionals (5) and (9) differ substantially. Indeed, in [23] we establish the following result.

Theorem 3.1 ([23], Best constant for $\mathcal{J}_\lambda$). *The functional $\mathcal{J}_\lambda$ is bounded below on $\mathcal{E}$ if $\lambda < \overline{\lambda}$, where $\overline{\lambda}$ is defined by*

$$\overline{\lambda} = \inf\left\{ \frac{8\pi\mathcal{P}(K_\pm)}{\left[\int_{K_\pm} \alpha\mathcal{P}(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \operatorname{supp}\mathcal{P} \right\}$$

where K is a Borel set and $I_+ = [0,1]$ and $I_- = [-1,0)$.

The constant $\overline{\lambda}$ is sharp in the sense that it may not be replaced by any larger constant. We note that $\overline{\lambda} \geq 8\pi$. In other words, the best constant for (5) is improved with respect to the best constant for the standard Trudinger-Moser functional (2), as stated in (3). We expect that the strict inequality is a technical limitation of the approximation argument employed in the proof.

In constrast to the above result for the functional (5), no improved Trudinger-Moser inequality holds for the functional (9). Indeed, it is not difficult to prove the following.

Theorem 3.2 ([17, 24], Best constant for $\mathcal{K}_\lambda$). *Let* $\operatorname{supp}\mathcal{P} \cap \{-1, 1\} \neq \emptyset$. *Then, the functional $\mathcal{K}_\lambda$ is bounded from below on $\mathcal{E}$ if and only if $\lambda \leqslant 8\pi$.*

The proof of Theorem 3.1 is more involved, see [23]. We first identify a duality principle for $\mathcal{J}_\lambda$. More precisely, we rigorously prove a Toland type non-convex duality principle for the following Lagrangian from [26, 28]:

$$\begin{aligned}\mathcal{L}(\oplus\rho_\alpha, v) = \iint_{I\times\Omega} \rho_\alpha(\log\rho_\alpha - 1)\,\mathcal{P}(d\alpha) &+ \frac{1}{2}\int_\Omega |\nabla v|^2 \\ &- \iint_{I\times\Omega} \alpha\rho_\alpha v\,\mathcal{P}(d\alpha).\end{aligned}$$

Here $(\oplus\rho_\alpha, v) \in \oplus\Gamma_\lambda \times \mathcal{E}$, where

$$\begin{aligned}\Gamma_\lambda &= \left\{\rho \in L\log L(\Omega) :\ \rho \geq 0,\ \int_\Omega \rho = \lambda\right\}, \\ \oplus\Gamma_\lambda &= \{\oplus\rho_\alpha :\ \rho_\alpha \in \Gamma_\lambda \text{ for } \mathcal{P} - a.e.\ \alpha \in I\}.\end{aligned}$$

We define the following free-energy functional of logarithmic Hardy-Littlewood-Sobolev type

$$\Psi(\oplus\rho_\alpha) = \iint_{I\times\Omega} \rho_\alpha(\log\rho_\alpha - 1) - \frac{1}{2}\iint_{I^2} \alpha\beta\int_\Omega \rho_\alpha G\star\rho_\beta\mathcal{P}(d\alpha)\mathcal{P}(d\beta)$$

for $\oplus\rho_\alpha \in \oplus\Gamma_\lambda$. The following duality principle implies that minimization of $\mathcal{J}_\lambda$ is equivalent to minimization of Ψ:

Theorem 3.3 ([23], Duality principle). *For any $\lambda > 0$ the following relation holds:*

$$\inf_{\oplus\Gamma_\lambda\times\mathcal{E}} \mathcal{L} = \inf_{\mathcal{E}} \mathcal{J}_\lambda + \lambda(\log\lambda - 1) = \inf_{\oplus\Gamma_\lambda} \Psi. \tag{15}$$

Theorem 3.3 may be viewed as a Toland non-convex duality principle for J_λ and Ψ. Identity (15) is stated without proof in [28]. The proof of Theorem 3.3 is achieved in [23] by a direct minimization argument which requires some care, since on one hand the space $L^1(I, \mathcal{P}; L\log L(\Omega))$ is not reflexive, and on the other hand the logarithmic nonlinearity is not differentiable at 0.

With the duality principle at hand, the study of $\mathcal{J}_\lambda$ is reduced to the study of functionals of the form

$$\begin{aligned}\widetilde{\Psi}_{\mathcal{P}}(\oplus\rho_\alpha) = &\int_I\int_\Omega \rho_\alpha \log\rho_\alpha \,\mathcal{P}(d\alpha) \\ &+ \iint_{I^2} A(\alpha,\beta) \iint_{\Omega^2} \rho_\alpha(x)\log d(x,y)\rho_\beta(y)\mathcal{P}(d\alpha)\mathcal{P}(d\beta),\end{aligned} \tag{16}$$

where $A(\alpha,\beta) \in C(I^2)$ is symmetric and satisfies the sign condition

$$\alpha\beta\, A(\alpha,\beta) \geq 0 \qquad \text{on } I^2,$$

and $\oplus\rho_\alpha \in \oplus\Gamma_{\lambda_\alpha}$, where

$$\oplus\Gamma_{\lambda_\alpha} = \{\oplus\rho_\alpha :\ \rho_\alpha \in \Gamma_{\lambda_\alpha} \text{ for } \mathcal{P} - a.e.\ \alpha \in I\}.$$

In the special case where $\mathcal{P}$ is the atomic measure (7), the free energy (16) takes the form

$$\widetilde{\Psi}(\rho_1,\ldots,\rho_n) = \sum_{i=1}^N a_i \int_\Omega \rho_i \log\rho_i + \sum_{i,j=1}^N a_{ij} \iint_{\Omega^2} \rho_i(x)\log d(x,y)\rho_j(y),$$

with $a_{ij} = A(\alpha_i,\alpha_j)a_i a_j$, $i,j = 1,2,\ldots,N$. Discrete functionals of the form above have been extensively investigated by Shafrir and Wolansky [27], who derived an optimal condition for boundedness below. We thus complete the proof of Theorem 3.1 by a careful approximation argument. We refer to [23] for the details.

Acknowledgements

I thank Professor H. Ohtsuka, Professor T. Yanagisawa and Professor Y. Yatsuyanagi for kindly inviting me to the RIMS Camp-Style Seminar *Modern approach and developments to Onsager's theory on statistical vortices*, Kyoto, August 28–31, 2011, as well as for support. I thank Professor P.-H. Chavanis for helpful discussions on the statistical mechanics underlying the models considered herein. This research is partially supported by FP7-MC-IRSES-2009-247486 *Mathematical studies on critical non-equilibrium phenomena via mean field theories and theories of nonlinear partial differential equations.*

References

[1] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Comm. Partial Differential Equations **16** (1991), no. 8–9, 1223–1253.

[2] E. Caglioti, P.L. Lions, C. Marchioro and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Comm. Math. Phys. **143** (1992), 501–525.

[3] E. Caglioti, P.L. Lions, C. Marchioro and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Part II, Comm. Math. Phys. **174** (1995), 229–260.

[4] S. Chanillo and M. Kiessling, Rotational symmetry of solutions of some nonlinear problems is statistical mechanics and in geometry, Comm. Math. Phys. **160** (1994), 217–238.

[5] P.H. Chavanis, Kinetic theory of point vortices in two dimensions: analytical results and numerical simulations, Eur. Phys. J. B **59** (2007), 217–247.

[6] P.H. Chavanis, Statistical mechanics of two-dimensional vortices and stellar systems, arXiv:cond-mat/0212223.

[7] M. Chipot, I. Shafrir and G. Wolansky, On the solutions of Liouville systems, J. Differential Equations **140** (1997), 59–105.

[8] E.E. Espejo, A. Stevens and T. Suzuki, Simultaneous blowup and collapse mass separation during collapse in an interacting system of chemotactic species, Differential and Integral Equations **25**, nos. 3–4 (2012), 251–288.

[9] P. Esposito and J. Wei, Non-simple blow-up solutions for the Neumann two-dimensional sinh-Gordon equation, Calc. Var. Partial Differential Equations **34** (2009), 341–375.

[10] G.L. Eyink and K.R. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, Reviews of Modern Physics **78** (2006), 87–135.

[11] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv., **68** (1993), 415–454.

[12] J. Jost, G. Wang, D. Ye and C. Zhou, The blow up analysis of solutions of the elliptic sinh-Gordon equation, Calc. Var. Partial Differential Equations **31** (2008), no. 2, 263-276.

[13] G. Joyce, D. Montgomery, Negative temperature states for the two-dimensional guiding centre plasma, J. Plasma Phys. **10** (1973), 107–121.

[14] M.K.H. Kiessling, Statistical mechanics of classical particles with logarithmic interaction, Comm. Pure Appl. Math. **46** (1993), 27–56.

[15] C.S. Lin, An expository survey on the recent development of mean field equations, Discrete Contin. Dyn. Syst. **19** (2007) no. 2, 387–410.

[16] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially-dominated nonlinearities, Asymptotic Analysis **3** (1990), 173–188.

[17] C. Neri, Statistical mechanics of the N-point vortex system with random intensities on a bounded domain, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 3, 381–399.

[18] H. Ohtsuka, T. Ricciardi and T. Suzuki *Blow-up analysis for an elliptic equation describing stationary vortex flows with variable intensities in 2D-turbulence*, J. Differential Equations **249** (2010), no. 6, 1436–1465.

[19] H. Ohtsuka and T. Suzuki, Mean field equation for the equilibrium turbulence and a related functional inequality, Adv. Differential Equations **11** (2006), 281-304.

[20] H. Ohtsuka and T. Suzuki, A blowup analysis of the mean field equation for arbitrarily signed vortices, Self-similar solutions of nonlinear PDE, 185–197, Banach Center Publ. **74**, Polish Acad. Sci., Warsaw, 2006.

[21] L. Onsager, *Statistical hydrodynamics,* Nuovo Cimento Suppl. No. 2 6 (9) (1949) 279–287.

[22] Y.B. Pointin and T.S. Lundgren, Statistical mechanics of two-dimensional vortices in a bounded container, Phys. Fluids **19** (10) (1976), 1459–1470.

[23] T. Ricciardi and T. Suzuki, *Duality and best constant for a Trudinger–Moser inequality involving probability measures,* submitted.

[24] T. Ricciardi and G. Zecca, Blow-up analysis for some mean field equations involving probability measures from statistical hydrodynamics, Differential and Integral Equations **25**, nos. 3–4 (2012), 201–222.

[25] T. Ricciardi and G. Zecca, On Neri's mean field equation with hyperbolic sine vorticity distribution, submitted.

[26] K. Sawada and T. Suzuki, Derivation of the equilibrium mean field equations of point vortex and vortex filament system, Theoret. Appl. Mech. Japan **56** (2008), 285–290.

[27] I. Shafrir and G. Wolansky, The logarithmic HLS inequality for systems on compact manifolds, J. Funct. Anal. **227** (2005), no. 1, 220–226.

[28] T. Suzuki, *Mean Field Theories and Dual Variation.* Atlantis Press, Paris, 2009.