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On the base-line location problem for the maximum weight region decomposable into base-monotone shapes

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1 Introduction

Let P be an $n \times n$ pixel grid. A *pixel* (i, j) of P is the unit square whose top-right corner is the grid point $(i, j) \in \mathbb{Z}^2$. For example the bottom-left cell of P is $(1, 1)$ and the top-right cell is (n, n). Each pixel $p = (i, j)$, where $1 \leq i, j \leq n$, has its *weight* $w(p) \in \mathbb{Z}$. Now we define the following general problem.

Problem: MAXIMUM WEIGHT REGION PROBLEM (MWRP)

Instance: An $n \times n$ pixel grid P.

Objective: Find a region $R \in \mathcal{F}$ maximizing the weight $w(R) = \sum_{p \in R} w(p)$, where $\mathcal{F} \subseteq 2^P$ be a fixed family of pixel regions.

The general problem MWRP has been studied for several families $\mathcal{F}$. Observe that if $\mathcal{F} = 2^P$, then R can be arbitrarily chosen, and thus the answer is the set of all positive cells. On the other hand, if $\mathcal{F}$ is the family of connected regions (in the usual 4-neighbor topology), then the problem becomes NP-hard [1]. For the complexity of MWRP for other families, see the paper by Chun, Kasai, Korman, and Tokuyama [2] and the references therein.

Motivated by the image segmentation problem, Chun et al. [2] studied more complicated family of pixel regions for MWRP (see Figure 1). A *baseline* of an $n \times n$ pixel grid P is a vertical line $x = b$ or horizontal line $y = b$, where $0 \leq b \leq n$. A pixel region R is a *based x-monotone region* if there is a horizontal baseline $y = b$ such that $(i, j) \in R$ implies $(i, j - 1) \in R$ for $j \geq b + 1$, and $(i, j) \in R$ implies $(i, j + 1) \in R$ for $j < b$ (see Figure 2). *Based y-monotone regions* are analogously defined. Based x- and y-monotone regions are *base-monotone regions*. Given a set of k baselines, a region R is *base-monotone feasible* if it can be decomposed into pairwise disjoint base-monotone regions with respect to the baselines. The *k baseline MWRP* is MWRP in which we are given k (vertical or horizontal) baselines, and we find a maximum-weight base-monotone feasible region respect to the baselines. Chun et al. showed that the k baseline MWRP can be solved in polynomial time. They also studied the *k base-segment MWRP*, in which we are given k segments and find a region decomposable into base-monotone regions respect to the given base-segments. (We will define this problem more precisely in the next section.) They showed some partial results on this problem. For other formulations, as optimization problems, of the image segmentation problem, see the recent work by Gibson, Han, Sonka, and Wu [5].

In the setting of the k baseline MWRP, we are given k baselines. Thus a natural question would be "What if baselines are not given?" In other words, "How can we divide the pixel grid into subgrids with vertical and horizontal lines?" We study this problem and show that the problem of optimally locating k baselines is NP-hard but can be approximated within factor 2. Next we propose another way to divide the pixel grid into subgrid, and show that this variant can be solved in polynomial time. Finally, we study the k base-segment MWRP and present sharp contrasts of its computational com-

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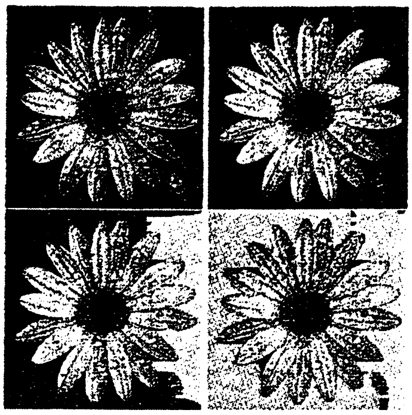

Figure 1: Image segmentation via k baseline MWRP. We first convert a picture to a gray scale image. Next, with some suitable function, we construct a pixel grid in which each dark pixel has positive weight and each light pixel has negative weight. Finally we solve the k baseline MWRP to segment the back ground from the objects. In this example, the edges of the picture is used as baselines ($k = 4$). For example, the red region in the third figure (from left) uses the top edge as its baseline.

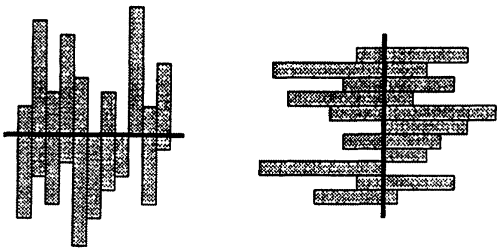

Figure 2: A based x-monotone region (left) and a based y-monotone region (right).

plexity.

Due to space limitation all proofs are omitted.

2 Preliminaries

In this paper we study three different but well related problems. This section introduces these three problems.

2.1 Baseline Location

Chun et al. [2] presented the following positive result.

Theorem 2.1 ([2]). *The k baseline MWRP can be solved in $O(n^3)$ time.*

To complement this result, we study the computational complexity of the following problem.

Problem: Baseline Location

Instance: An $n \times n$ pixel grid P and positive integers k and w.

Question: Is there k baselines in P such that a maximum-weight base-monotone feasible region has weight at least w?

It is easy to see that there are only $\binom{2n+2}{k}$ possible allocations of k baselines. Therefore, Baseline Location can be solved in $O(2^k n^{k+3})$ time. However, this is far from efficient if k is a part of the input. We would like to solve this problem in $O(f(k) \cdot \mathrm{poly}(n))$ time or even in $O(\mathrm{poly}(k \cdot n))$ time. Unfortunately, the latter case very unlikely happens as we will prove the problem is NP-hard if k is a part of the input. The possibility of the former case remains unsettled in this paper.

2.2 The k base-segment MWRP

Consider a segment s in a baseline ℓ. If a monotone region R with baseline ℓ intersects ℓ only in s, then R has s as its *base-segment*. Chun et al. [2] also studied *k base-segment MWRP*, in which k base-segments are given, and one wants to find a region that can be decomposed into disjoint monotone regions with respect to the given base-segments. They also studied two-directional version of this problem in which the region can be built only on the right side of each vertical base-segment and on the upper side of each horizontal base-segment. They showed the following results.

Theorem 2.2 ([2]). *The two-directional k base-segment MWRP can be solved in $O(k^{O(k)} n^4)$ time. The original k base-segment MWRP can be solved in $O(n^{O(k)})$ time.*

The first statement says that the two-directional version is fixed parameter tractable when parameterized by k. The second statement says that the original problem can be solved in polynomial time if k is not a part of the input. It was not known whether the two-directional version can be solved in polynomial time with both n and k, and whether the original version is NP-hard when k is a part of the input. We will answer these two questions: the original problem is NP-hard and the two-directional version is polynomial-time solvable, when k is a part of the input.

3 NP-hardness of BASELINE LOCATION

As the first step of the study on BASELINE LOCATION, we prove the following theorem.

Theorem 3.1. BASELINE LOCATION *is NP-complete in the strong sense.*

The problem is clearly in NP. We prove its NP-hardness by reducing INDEPENDENT SET to this problem. An *independent set* of a graph is a set of pairwise non-adjacent vertices. It is known that the following decision problem is NP-complete [3].

Problem: INDEPENDENT SET

Instance: A graph G and a positive integer s.

Question: Does G have an independent set of size at least s?

Note that INDEPENDENT SET is NP-complete even with the restriction $|V(G)| = |E(G)|$.

Proposition 3.2. INDEPENDENT SET *is NP-complete for the instances with* $|V(G)| = |E(G)|$.

3.1 Gadgets

We first define two small gadgets for forcing baselines into restricted zones. Throughout this paper, each red × in a pixel grid represents a huge negative weight whose absolute value is equal to the sum of all the positive weights in the grid. Also, each blue • represents a (not necessarily large) positive weight. All the other cells have weight 0.

Our first gadget is the 3×3 grid depicted in Figure 3. If we want to take the positive cell at the center, we need one baseline as in the figure. Since we cannot take any huge negative cell, the possible locations of the baselines are restricted to the four in the figure. We call this gadget a *baseline forcer*. The *weight* of a baseline forcer is the weight of the positive cell, and the *position* of a baseline forcer is the position of its bottom-left cell.

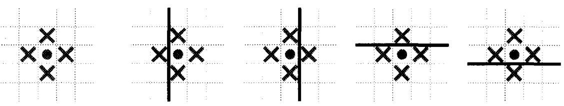

Figure 3: A baseline forcer: forcing one baseline.

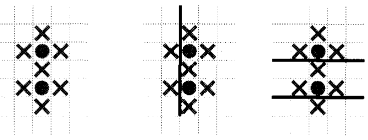

Figure 4: A vertical baseline forcer: forcing one vertical baseline or two horizontal baselines.

Next we consider a similar gadget depicted in Figure 4. To take all positive cells and not to take any negative cell, we need either one vertical baseline or two horizontal baselines. Therefore, if we need to minimize the number of baselines, then we have to use one vertical baseline. We call this gadget a *vertical baseline forcer*. By rotating this gadget, we can also obtain a gadget for forcing two vertical baselines or one horizontal baseline. We call it a *horizontal baseline forcer*. Two positive cells in this gadget have the same weight, and their weight is the *weight* of the vertical or horizontal baseline forcer. The *position* of a vertical or horizontal baseline forcer is the position of its bottom-left cell.

Vertical and horizontal baseline forcers work even if we insert some space between columns or rows as in Figure 5. The location of the baseline is restricted to the area depicted in the figure. We say that a vertical (horizontal) baseline forcer *intersects* a vertical (horizontal resp.) baseline if the baseline is in the restricted area; that is, a base monotone shape with the vertical or horizontal baseline can contain the positive cells in the vertical or horizontal baseline forcer. The number of the columns used by a vertical baseline forcer is its *width*, and the number of rows used by a horizontal baseline forcer is its *height*. For example, the original vertical baseline forcer in Figure 4 is of width 3.

3.2 Reduction

Given an instance (G, s) of INDEPENDENT SET, we construct an instance (P, k, w) of BASELINE LOCATION as follows. It is easy to see that the reduction below

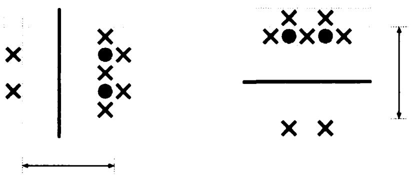

Figure 5: Forced baselines are restricted to the area indicated by double headed arrows.

can be done in polynomial time, and the absolute values of the weights are bounded by a polynomial of the input size.

By Proposition 3.2, we may assume $|V(G)| = |E(G)|$ for notational convenience. Let $V(G) = \{v_1, \ldots, v_m\}$ and $E(G) = \{e_1, \ldots, e_m\}$. We set the number of baselines $k = 2m$ and the required weight $w = 8m^3 + 8m^2 + s$. The grid P is the $(20m + 20) \times (20m + 20)$ pixel grid with the following entries. (see Figure 6).

Vertex gadgets

For each vertex v_i, we put a vertical baseline forcer of width 5 and weight $2m^2 + m$, denoted VF_i, at the position $(10i, 5i)$. We also put a baseline forcer of weight 1, denoted BF_i, at the position $(10i - 1, 20m + 15)$.

Edge gadgets

Let $e_h = \{v_i, v_j\} \in E(G)$ be an edge with $i < j$. We put a horizontal baseline forcer of height 10 and weight $2m^2 + m$, denoted HF_h, at the position $(10m + 5h, 5m + 15h)$. Next we put two horizontal baseline forcers $HF_{h,i}$ and $HF_{h,j}$ of height 3 and weight m at the positions $(10i - 3, 5m + 15h - 1)$ and $(10j - 3, 5m + 15h + 8)$, respectively. Also, we put two baseline forcers $BF_{h,i}$ and $BF_{h,j}$ of weight m at the positions $(10i + 3, 5m + 15h + 2)$ and $(10j + 3, 5m + 15h + 5)$, respectively.

The weight of negative cells

We have the following positive cells in the grids:

- $4m$ cells of weight $2m^2 + m$,
- $6m$ cells of weight m, and
- m cells of weight 1.

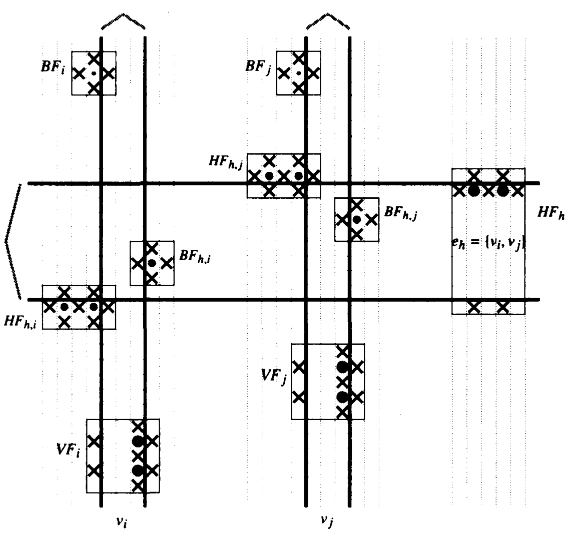

Figure 6: Gadgets for an edge $\{v_i, v_j\}$: black thick lines are the candidates of required baselines, two vertical and one horizontal.

The total weight of the positive cells is $W = 4m(2m^2 + n) + 6m^2 + n = 8m^3 + 10m^2 + m$. We set the weight of the negative cells to $-W$ so that these cells cannot be taken in any solution with a positive total weight.

3.3 Equivalence

Lemma 3.3. *(G, s) is a yes-instance of* Independent Set *if and only if (P, k, w) is a yes-instance of* Baseline Location.

4 A 2-approximation algorithm for Baseline Location

Our approximability result is based on the polynomial-time solvability of the following problem.

Problem: Vertical Baseline Location

Instance: An $n \times n$ pixel grid P and a positive integer k.

Objective: Find k vertical baselines in P that maximize the weight of an optimal base-monotone feasible region respect to these baselines.

The problem Horizontal Baseline Location is defined analogously.

Theorem 4.1. Vertical Baseline Location *and* Horizontal Baseline Location *can be solved in* $O(n^3)$ *time.*

Theorem 4.2. *There is an* $O(n^3)$*-time 2-approximation algorithm for locating k baselines to maximize the weight of optimum base-monotone feasible region.*

5 The k base-segment MWRP

We extend the results of Chun et al. [2] (Theorem 2.2). We first reduce the two-directional version to Weighted Independent Set in bipartite graphs, which can be solve in polynomial time [6]. We next reduce the Independent Set in planar graphs to the original problem. This implies the NP-hardness of the original problem, since Independent Set is NP-hard for planar graphs [4].

In what follows, we may assume without loss of generality that no base-monotone shape with respect to a base-segment contains another base-segment properly (in such a case, we can just partition the base-monotone shape). We may also assume that two parallel base-segments may have intersection only at their end-points.

5.1 Two-directional version

We first divide each base-segment of length ℓ into ℓ base-segments of length 1. This refinement does not change the optimum value of the k base-segment MWRP. Now we have $O(kn)$ base-segments of length 1. We identify a base-segment s with (i, j) if s is the left or bottom edge of a pixel (i, j).

For each vertical base-segment $s = (i, j)$, we define its *range* as follows: if there is no vertical base-segment $s' = (i', j)$ with $i' > i$, then the range of s is $[i, n]$; otherwise the range of s is $[i, i']$, where i' is the smallest index for which such a segment exists. (see Figure 7). We define the *range* of a horizontal base-segment analogously.

Let $s = (i, j)$ be a vertical base-segment with range $[i, i']$. Let $a_s(0) = i - 1$, and for $p \geq 1$, let $a_s(p)$ be the minimum index h such that $a_s(p-1) < h \leq i'$ and $\sum_{a_s(p-1)<q\leq h} w(q, j)$ is positive. If there is no such index, then $a_s(p)$ is undefined. If $a_s(p)$ is defined for some $p \geq 1$, then let

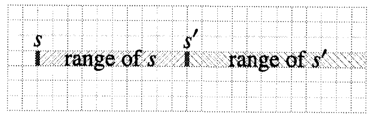

Figure 7: The ranges of vertical base-segments s and s'.

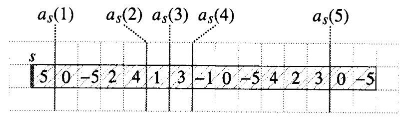

Figure 8: Example of $a_s(p)$. The corresponding weights $w_s(1), \ldots, w_s(5) = 5, 1, 1, 3, 3$.

$w_s(p) = \sum_{a_s(p-1)<q\leq a_s(p)} w(q, j)$. See Figure 8. For each horizontal base-segment s', we also define the sequence $a_{s'}(\cdot)$ analogously.

Now we construct a bipartite graph $G = (U, V; E)$. Let $s = (i, j)$ be a vertical base-segment. Assume that r is the largest index such that $a_s(r)$ is defined. Now all $a_s(0), \ldots, a_s(r)$ are defined from the definition. If $r = 0$, then this segment s is useless and just ignored. Otherwise, we put vertices $u_s(p)$, $1 \leq p \leq r$, with weight $w_s(p)$ into U. For each horizontal base-segment $s' = (i', j')$, we put vertices $v_{s'}(p')$ into V in the same way. Next we define the edge set E. Two vertices $u_s(p) \in U$ and $v_{s'}(p') \in V$ are adjacent if and only if two base-monotone regions with base-segments s and s' have nonzero area intersection if they contain $(a_s(p), j)$ and $(i', a_{s'}(p'))$, respectively. More precisely, this can be stated as: $i \leq i' \leq a_s(p)$ and $j' \leq j \leq a_{s'}(p')$. See Figure 9 for example.

Lemma 5.1. *An optimum solution of an instance of the two-directional k base-segment MWRP has weight at least W if and only if the corresponding*

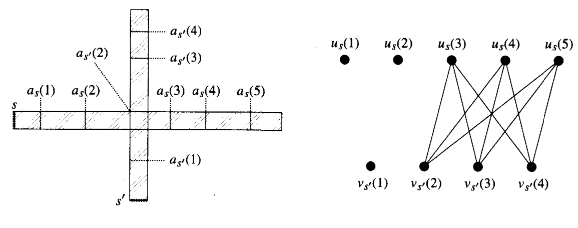

Figure 9: The bipartite graph construction. The vertices corresponding to the crossing ranges of two base-segments induce a disjoint union of an independent set and a complete bipartite graph.

bipartite graph G has an independent set of weight at least W.

Theorem 5.2. *The two-directional k base-segment MWRP can be solved in $O(k^3 n^6 \log kn)$ time.*

5.2 NP-hardness of the k base-segment MWRP

We now show the following theorem.

Theorem 5.3. *The k base-segment MWRP is NP-complete in the strong sense.*

The problem is clearly in NP, and thus it suffices to show the NP-hardness. To this end, we reduce the INDEPENDENT SET for planar graphs to the k base-segment MWRP. A graph is *planar* if it can be drawn in the plane without edge crossings. It is known that INDEPENDENT SET is NP-hard even for planar graphs [4].

5.2.1 Nice visibility representations

Planar graphs have several geometric representations. We use one of them here. A subset of plane $\{1, 2, \ldots, w\} \times \{1, 2, \ldots, h\}$ is a $w \times h$ *grid*. A *visibility representation* of a planar graph G maps each vertex of G to a horizontal segment with endpoints in a grid and each edge of G to a vertical segment with endpoints in a grid such that

1. no segments of two distinct vertices intersect,
2. segments of two distinct edges intersect only at their endpoints, and
3. the segment of an edge $\{u, v\}$ touches the segments of u and v.

Otten and van Wijk [7] showed that every planar graph has a visibility representation. It is known that a visibility representation of a planar graph in an $O(n) \times O(n)$ grid can be found in linear time [8, 9]. For the recent development on visibility representations, see the recent paper by Wang and He [10] and the references therein. For our purpose, we need the following additional conditions for visibility representations:

4. no two vertical segments have the same x-coordinate,
5. no two horizontal segments have the same y-coordinate, and
6. no two endpoints of segments have the same position.

We call a visibility representation satisfying the three additional conditions a *nice visibility representation*. Given a visibility representation of a planar graph, we can obtain a nice visibility representation of the graph in polynomial time by refining each cell of the grid to an $O(n) \times O(n)$ subgrid, slightly extending each horizontal segment, and slightly shifting each vertical segment.

5.2.2 Reduction

Let (G, s) be an instance of INDEPENDENT SET, where G is a planar graph with n vertices and m edges. Note that we do not assume $n = m$ here. We first construct a nice visibility representation $R = (A, B)$ of G in polynomial time, where A is the set of horizontal segments and B is the set of vertical segments. We construct a pixel grid P from R as follows (see Figure 11).

For each vertex $u \in V$ with the corresponding horizontal segment $a_u = [x_1, x_2] \times \{y\} \in A$, we put a vertical base-segment (x_1, y) and set the weight 1 to the cell (x_2, y). For each edge $e = \{v, w\} \in E$ with the corresponding vertical segment $b_u = \{x\} \times [y_1, y_2] \in B$, we put horizontal base-segments (x, y_1) and $(x, y_2 + 1)$ and set the weight n to the cell (x, y_e), where the y-coordinate y_e is not used by any vertical base-segment and $y_1 < y_e < y_2$. Such a coordinate can be chosen by the refinement of the grid. Note that the weight of a cell is at most n and there is no negative-weight cell.

5.2.3 Equivalence

We now show that (G, s) is a yes-instance if and only if the optimum value of k base-segment MWRP on P is at least $mn + s$. (The proof is omitted.)

5.2.4 The three-directional version

In the reduction above, we may assume without loss of generality that the region can be built only on the right side of each vertical base-segment, on the upper sides of some horizontal base-segments,

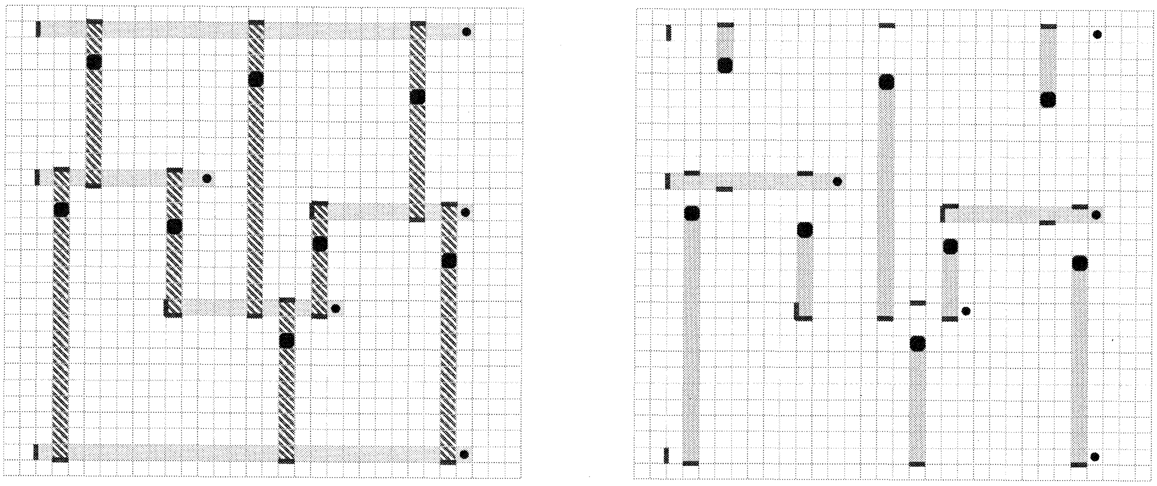

Figure 10: Reduction. Each green thick segment is a base-segment. In the right figure $S = \{b, d\}$. Yellow cells are taken by the vertical base-segments, and green cells are taken by the horizontal base-segments.

and on the lower sides of the remaining horizontal base-segments. We call this version the three-directional k base-segment MWRP.

Corollary 5.4. *The three-directional k base-segment MWRP is NP-complete in the strong sense.*

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