数理解析研究所講究録
第 1800 巻 2012 年 130-144

多成分変形非線形シュレーディンガー方程式の多重ソリトン解

Multisoliton solution of a multi-component modified nonlinear Schrödinger equation

山口大学大学院理工学研究科　松野　好雅 (Yoshimasa Matsuno)
Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University

Abstract
We develop a systematic method for constructing the bright N-soliton solution of a multi-component modified nonlinear Schrödinger equation. We present the two different expressions of the solution both of which are expressed as a ratio of determinants. We find a simple relation between them by employing the properties of the Cauchy matrix. Last, we propose a (2+1)-dimensional nonlocal modified nonlinear Schrödinger equation arising from the multi-component system as the number of dependent variable tends to infinity and then obtain its bright N-soliton solution. In this paper, we describe only the main results. The detail has been published in Matsuno (2011) [1].

1. Introduction

We consider the following multi-component system of nonlinear PDEs which is a hybrid of the coupled nonlinear Schrödinger (NLS) equation and coupled derivative NLS equation

$$\mathrm{i}\, q_{j,t} + q_{j,xx} + \mu \left(\sum_{k=1}^{n} |q_k|^2 \right) q_j + \mathrm{i}\gamma \left[\left(\sum_{k=1}^{n} |q_k|^2 \right) q_j \right]_x = 0, \quad (j = 1, 2, ..., n), \quad (1.1)$$

where $q_j = q_j(x,t)$ $(j = 1, 2, ..., n)$ are complex-valued functions of x and t, $\mu(\geq 0)$ and γ are real constants, n is an arbitrary positive integer and subscripts x and t appended to q_j denote partial differentiations.

- **Integrability of Eq. (1.1)**: Hisakado and Wadati (1995) [2]
- **Special cases**:
  1) $n = 1, \mu \neq 0, \gamma = 0$: NLS equation, Zakharov and Shabat (1972) [3]
  2) $n = 1, \mu = 0, \gamma \neq 0$: Derivative NLS equation, Kaup and Newell (1978) [4]
  3) $n = 2, \mu \neq 0, \gamma = 0$: Manakov system, Manakov (1974) [5]
  4) $n = 2, \mu \neq 0, \gamma \neq 0$: A model equation describing the propagation of short pulses in birefringent optical fiber, Hisakado, Iizuka and Wadati (1994) [6]
  5) Bright N-soliton solution of Eq. (1.1) with $n = 2$: Matsuno (2011) [7]

2. Bilinearization and bright N-soliton solution

2.1. Bilinearization

We first apply the gauge transformations

$$q_j = u_j \exp\left[-\frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{x}\sum_{k=1}^{n}|u_k|^2 dx\right], \quad (j=1,2,...,n), \tag{2.1}$$

to the system (1.1) subjected to the the boundary conditions $q_j \to 0, u_j \to 0$ $(j=1,2,...,n)$ as $|x| \to \infty$, where $u_j = u_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t. Then, we obtain the system of nonlinear PDEs for u_j

$$\mathrm{i}\, u_{j,t} + u_{j,xx} + \mu\left(\sum_{k=1}^{n}|u_k|^2\right)u_j + \mathrm{i}\gamma\left(\sum_{k=1}^{n}u_k^* u_{k,x}\right)u_j = 0, \quad (j=1,2,...,n), \tag{2.2}$$

where the asterisk appended to u_k denotes complex conjugate.

Proposition 2.1. *By means of the dependent variable transformations*

$$u_j = \frac{g_j}{f}, \quad (j=1,2,...,n), \tag{2.3}$$

the system of nonlinear PDEs (2.2) can be decoupled into the following system of bilinear equations for f and g_j

$$(\mathrm{i}D_t + D_x^2) g_j \cdot f = 0, \quad (j=1,2,...,n), \tag{2.4}$$

$$D_x f\cdot f^* = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}|g_k|^2, \tag{2.5}$$

$$D_x^2 f\cdot f^* = \mu\sum_{k=1}^{n}|g_k|^2 + \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}D_x g_k\cdot g_k^*. \tag{2.6}$$

Here, $f = f(x,t)$ and $g_j = g_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f\cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,\, t'=t}, \tag{2.7}$$

where m and n are nonnegative integers.

Proof. Substituting (2.3) into (2.2) and rewriting the resultant equation in terms of the bilinear operators, equations (2.2) can be rewritten as

$$\frac{1}{f^2}(\mathrm{i}D_t g_j\cdot f + D_x^2 g_j\cdot f) + \frac{g_j}{f^3 f^*}\left(-f^* D_x^2 f\cdot f + \mu f\sum_{k=1}^{n}|g_k|^2 + \mathrm{i}\gamma\sum_{k=1}^{n}g_k^* D_x g_k\cdot f\right) = 0,$$

$$(j = 1, 2, ..., n). \tag{2.8}$$

Insert the identity

$$f^* D_x^2 f \cdot f = f D_x^2 f \cdot f^* - 2 f_x D_x f \cdot f^* + f(D_x f \cdot f^*)_x, \tag{2.9}$$

into the second term on the left-hand side of (2.8). Then, equations (2.8) become

$$\frac{1}{f^2}(\mathrm{i}D_t g_j \cdot f + D_x^2 g_j \cdot f) + \frac{g_j}{f^3 f^*}\Big[f\Big\{-D_x^2 f \cdot f^* + \mu \sum_{k=1}^{n} |g_k|^2 - (D_x f \cdot f^*)_x + \mathrm{i}\gamma \sum_{k=1}^{n} g_k^* g_{k,x}\Big\}$$

$$+ f_x\Big\{2D_x f \cdot f^* - \mathrm{i}\gamma \sum_{k=1}^{n} |g_k|^2\Big\}\Big] = 0, \quad (j = 1, 2, ..., n). \tag{2.10}$$

As easily confirmed by a direct calculation, the left-hand side of (2.10) becomes zero by virtue of equations (2.4)-(2.6). □

It now follows from (2.3) and (2.5) that

$$-\frac{\mathrm{i}\gamma}{2} \sum_{k=1}^{n} |u_k|^2 = \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \tag{2.11}$$

which, substituted into (2.1), yields the solution of the system (1.1) in the form

$$q_j = \frac{g_j f^*}{f^2}, \qquad (j = 1, 2, ..., n). \tag{2.12}$$

Note that for the n-component NLS equation (the system (1.1) with $\gamma = 0$), the solution (2.12) simplifies to $q_j = g_j/f$. Indeed, if $\gamma = 0$, then the bilinear equation (2.5) reduces to $D_x f \cdot f^* = 0$. Thus, the ratio f^*/f turns out to be an arbitrary function of t which can be set to 1 under appropriate boundary condition.

2.2. Bright N-soliton solution

We now state the main result:

Theorem 2.1. *The bright N-soliton solution of the system of bilinear equations (2.4)-(2.6) is given by the determinants f and g_j $(j = 1, 2, ..., n)$ where*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, ..., n). \tag{2.13}$$

Here, A, B and I are $N \times N$ matrices and $\mathbf{z}, \mathbf{a}_j$ and $\mathbf{0}$ are N-component row vectors defined below and the symbol T denotes the transpose:

$$A = (a_{jk})_{1 \le j,k \le N}, \quad a_{jk} = \frac{1}{2}\frac{z_j z_k^*}{p_j + p_k^*}, \quad z_j = \exp(p_j x + \mathrm{i}p_j^2 t), \tag{2.14a}$$

$$B = (b_{jk})_{1\le j,k\le N}, \quad b_{jk} = \frac{(\mu + \mathrm{i}\gamma p_k)c_{jk}}{p_j^* + p_k}, \quad c_{jk} = \sum_{s=1}^{n} \alpha_{sj}\alpha_{sk}^*, \tag{2.14b}$$

$$I = (\delta_{jk})_{1\le j,k\le N}, : N \times N \textit{ unit matrix}, \tag{2.14c}$$

$$\mathbf{z} = (z_1, z_2, ..., z_N), \quad \mathbf{a}_j = (\alpha_{j1}, \alpha_{j2}, ..., \alpha_{jN}), \quad \mathbf{0} = (0, 0, ..., 0). \tag{2.14d}$$

The above bright N-soliton solution is characterized by $(n+1)N$ complex parameters $p_j(j = 1, 2, ..., n)$ and $\alpha_{sj}(s = 1, 2, .., n;\ j = 1, 2, ..., N)$. The former parameters determine the amplitude and velocity of the solitons whereas the latter ones determine the polarizations and the envelope phases of the solitons.

To simplify the proof of theorem 2.1, the following observation is useful:

Proposition 2.2. *If we introduce the gauge transformations*

$$f = \tilde{f}, \quad g_j = \exp\left[\mathrm{i}\left\{\frac{\mu}{\gamma}\tilde{x} + \left(\frac{\mu}{\gamma}\right)^2 \tilde{t}\right\}\right]\tilde{g}_j, \quad (j = 1, 2, ..., n), \tag{2.15a}$$

$$x = \tilde{x} + \frac{2\mu}{\gamma}\tilde{t}, \quad t = \tilde{t}, \tag{2.15b}$$

then the bilinear equations (2.4)-(2.6) recast to

$$(\mathrm{i}D_{\tilde{t}} + D_{\tilde{x}}^2)\tilde{g}_j \cdot \tilde{f} = 0, \quad (j = 1, 2, ..., n), \tag{2.16}$$

$$D_{\tilde{x}}\tilde{f} \cdot \tilde{f}^* = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n} |\tilde{g}_k|^2, \tag{2.17}$$

$$D_{\tilde{x}}^2\tilde{f} \cdot \tilde{f}^* = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n} D_{\tilde{x}}\tilde{g}_k \cdot \tilde{g}_k^*, \tag{2.18}$$

respectively.

Thus, the form of equations (2.4) and (2.5) is unchanged whereas equation (2.6) becomes a simplified equation with $\mu = 0$. Consequently, the proof of the N-soliton solution may be performed for the corresponding solution with $\mu = 0$. Hence, in the analysis developed in the following sections, we put $\mu = 0$ without loss of generality.

3. Notation and some basic formulas for determinants

In this section, we first introduce the notation for matrices and then provide some basic formulas for determinants.

3.1. Notation

We define the following matrices associated with the N-soliton solution (2.13) with (2.14):

$$D = \begin{pmatrix} A & I \\ -I & B \end{pmatrix}, \tag{3.1}$$

$$D(\mathbf{a}^*; \mathbf{b}) = \begin{pmatrix} A & I & \mathbf{0}^T \\ -I & B & \mathbf{b}^T \\ \mathbf{0} & \mathbf{a}^* & 0 \end{pmatrix}, \quad D(\mathbf{a}^*; \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 \end{pmatrix},$$

$$D(\mathbf{z}^*; \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{z}^* & \mathbf{0} & 0 \end{pmatrix}. \tag{3.2}$$

Note the position of the vectors $\mathbf{a}^*$, $\mathbf{b}, \mathbf{z}$ and $\mathbf{z}^*$ in the above expressions. The matrices which include more than two vectors will be introduced as well. For example,

$$D(\mathbf{a}^*, \mathbf{z}^*; \mathbf{b}, \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{0} & \mathbf{z}^T \\ -I & B & \mathbf{b}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 & 0 \\ \mathbf{z}^* & \mathbf{0} & 0 & 0 \end{pmatrix}, \quad D(\mathbf{a}^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}') = \begin{pmatrix} A & I & \mathbf{z}^T & \mathbf{z}'^T \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 & 0 \\ \mathbf{z}^* & \mathbf{0} & 0 & 0 \end{pmatrix}. \tag{3.3}$$

3.2. Formulas for determinants

Let $A = (a_{jk})_{1\leq j,k\leq M}$ be an $M \times M$ matrix with M being an arbitrary positive integer, A_{jk} be the cofactor of the element a_{jk} and $\mathbf{a}, \mathbf{b}, \mathbf{a}_j$ and $\mathbf{b}_j$ $(j = 1, 2, ..., n)$ be M-component row vectors. The following well-known formulas are used frequently in our analysis:

$$\frac{\partial}{\partial x}|A| = \sum_{j,k=1}^{M} \frac{\partial a_{jk}}{\partial x} A_{jk}, \tag{3.4}$$

$$\begin{vmatrix} A & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |A|z - \sum_{j,k=1}^{M} A_{jk} a_j b_k, \tag{3.5}$$

$$|A(\mathbf{a}_1, \mathbf{a}_2; \mathbf{b}_1, \mathbf{b}_2)||A| = |A(\mathbf{a}_1; \mathbf{b}_1)||A(\mathbf{a}_2; \mathbf{b}_2)| - |A(\mathbf{a}_1; \mathbf{b}_2)||A(\mathbf{a}_2; \mathbf{b}_1)|. \tag{3.6}$$

The formula (3.4) is the differentiation rule of the determinant and (3.5) is the expansion formula for a bordered determinant with respect to the last row and last column. The formula (3.6) is Jacobi's identity.

The following two formulas may not be popular but are very important in our analysis. In particular, formula (3.7) gives rise to the expansion formulas for the bordered determinant (see (3.9) and (3.10) below):

$$|A(\mathbf{a}_1,...,\mathbf{a}_n;\mathbf{b}_1,...,\mathbf{b}_n)||A|^{n-1}=\begin{vmatrix}|A(\mathbf{a}_1;\mathbf{b}_1)| & \cdots & |A(\mathbf{a}_1;\mathbf{b}_n)|\\ \vdots & \ddots & \vdots \\ |A(\mathbf{a}_n;\mathbf{b}_1)| & \cdots & |A(\mathbf{a}_n;\mathbf{b}_n)|\end{vmatrix}, \quad (n\geq 2), \quad (3.7)$$

$$\left|A+\epsilon\sum_{s=1}^{n}\mathbf{b}_s^T\mathbf{a}_s\right| = |A|+\sum_{m=1}^{n'}(-\epsilon)^m \sum_{1\leq s_1<...<s_m\leq n}|A(\mathbf{a}_{s_1},...,\mathbf{a}_{s_m};\mathbf{b}_{s_1},...,\mathbf{b}_{s_m})|$$
$$=|A|+\sum_{m=1}^{n'}\frac{(-\epsilon)^m}{m!}\sum_{s_1,...,s_m=1}^{n}|A(\mathbf{a}_{s_1},...,\mathbf{a}_{s_m};\mathbf{b}_{s_1},...,\mathbf{b}_{s_m})|. \quad (3.8)$$

Here, ϵ is an arbitrary parameter, the notation $\mathbf{b}_s^T\mathbf{a}_s$ on the left-hand side of (3.8) represents an $M\times M$ matrix whose (j,k) element is given by $\beta_{sj}\alpha_{sk}$ and $n'=\min(n,M)$. The formula (3.7) is a variant of the Sylvester theorem in the theory of determinants.

Suppose that $|A|\neq 0$. Expanding the determinant on the right-hand side of (3.7) with respect to the first column and using (3.7) with n replaced by $n-1$, we then obtain an expansion formula

$$|A(\mathbf{a}_1,...,\mathbf{a}_n;\mathbf{b}_1,...,\mathbf{b}_n)|$$
$$=\frac{1}{|A|}\sum_{j=1}^{n}(-1)^{j-1}|A(\mathbf{a}_j,\mathbf{b}_1)||A(\mathbf{a}_1,...,\mathbf{a}_{j-1},\mathbf{a}_{j+1},...,\mathbf{a}_n;\mathbf{b}_2,...,\mathbf{b}_n)|. \quad (3.9)$$

Similarly, the expansion with respect to the first row gives

$$|A(\mathbf{a}_1,...,\mathbf{a}_n;\mathbf{b}_1,...,\mathbf{b}_n)|$$
$$=\frac{1}{|A|}\sum_{j=1}^{n}(-1)^{j-1}|A(\mathbf{a}_1,\mathbf{b}_j)||A(\mathbf{a}_2,...,\mathbf{a}_n;\mathbf{b}_1,...,\mathbf{b}_{j-1},\mathbf{b}_{j+1},...,\mathbf{b}_n)|. \quad (3.10)$$

4. Proof of the bright N-soliton solution

4.1. Formulas

In terms of the notation introduced in section 3.1 (see (3.1) and (3.2)), f and g_j are written in the form

$$f=|D|, \qquad g_j=-|D(\mathbf{a}_j^*;\mathbf{z})|, \quad (j=1,2,...,n). \quad (4.1)$$

The differentiation rules of f and g_j with respect to t and x are given by the following formulas:

Lemma 4.1.

$$f_t = -\frac{\mathrm{i}}{2}\left\{|D(\mathbf{z}^*;\mathbf{z}_x)| - |D(\mathbf{z}_x^*;\mathbf{z})|\right\}, \tag{4.2}$$

$$f_x = -\frac{1}{2}|D(\mathbf{z}^*;\mathbf{z})|, \tag{4.3}$$

$$f_{xx} = -\frac{1}{2}\left\{|D(\mathbf{z}^*;\mathbf{z}_x)| + |D(\mathbf{z}_x^*;\mathbf{z})|\right\}, \tag{4.4}$$

$$g_{j,t} = -|D(\mathbf{a}_j^*;\mathbf{z}_t)| + \frac{\mathrm{i}}{2}|D(\mathbf{a}_j^*,\mathbf{z}^*;\mathbf{z},\mathbf{z}_x)|, \tag{4.5}$$

$$g_{j,x} = -|D(\mathbf{a}_j^*;\mathbf{z}_x)|, \tag{4.6}$$

$$g_{j,xx} = -|D(\mathbf{a}_j^*;\mathbf{z}_{xx})| + \frac{1}{2}|D(\mathbf{a}_j^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|. \tag{4.7}$$

Here, $\mathbf{z}_t$, $\mathbf{z}_x$ and $\mathbf{z}_{xx}$ are N-component row vectors given by $\mathbf{z}_t = (\mathrm{i}p_1^2 z_1, \mathrm{i}p_2^2 z_2, ..., \mathrm{i}p_N^2 z_N)$, $\mathbf{z}_x = (p_1 z_1, p_2 z_2, ..., p_N z_N)$ and $\mathbf{z}_{xx} = (p_1^2 z_1, p_2^2 z_2, ..., p_N^2 z_N)$, respectively.

Lemma 4.2.

$$f^* = |\bar{D}|, \qquad \bar{D} \equiv \begin{pmatrix} A & I \\ -I & B - \mathrm{i}\gamma C \end{pmatrix}, \tag{4.8}$$

$$f_x^* = -\frac{1}{2}|\bar{D}(\mathbf{z}^*;\mathbf{z})|, \tag{4.9}$$

$$g_j^* = |\bar{D}(\mathbf{z}^*;\mathbf{a}_j)|. \tag{4.10}$$

Lemma 4.3.

$$|\bar{D}| = |D| + \frac{1}{2}|D(\mathbf{z}^*;\tilde{\mathbf{z}})|, \tag{4.11}$$

$$|D(\mathbf{b}_k^*;\tilde{\mathbf{z}})| = |\bar{D}(\mathbf{a}_k^*;\mathbf{z})|, \tag{4.12}$$

$$|\bar{D}(\mathbf{a}_k^*;\mathbf{b}_k) = -|D(\mathbf{b}_k^*;\mathbf{a}_k)| - \frac{1}{2}|D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{a}_k,\tilde{\mathbf{z}})|, \tag{4.13}$$

$$|\bar{D}(\mathbf{a}_k^*;\mathbf{z}_x)| = |D(\mathbf{b}_k^*;\mathbf{z}) + \frac{1}{2}|D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{z},\tilde{\mathbf{z}})|. \tag{4.14}$$

$$|D(\mathbf{z}^*;\mathbf{z})| = 2\mathrm{i}\gamma\sum_{k=1}^{n}|D(\mathbf{b}_k^*;\mathbf{a}_k)|, \tag{4.15}$$

$$|\bar{D}(\mathbf{z}^*;\mathbf{z})| = -2\mathrm{i}\gamma\sum_{k=1}^{n}|\bar{D}(\mathbf{a}_k^*;\mathbf{b}_k)|, \tag{4.16}$$

where $\tilde{\mathbf{z}}$ and $\mathbf{b}_k$ are N-component row vectors given respectively by $\tilde{\mathbf{z}} = (z_1/p_1, z_2/p_2, ..., z_N/p_N)$ and $\mathbf{b}_k = (\alpha_{k1}p_1^*, \alpha_{k2}p_2^*, ..., \alpha_{kN}p_N^*)$.

4.2. Proof of (2.4)

Let P_1 be

$$P_1 = (\mathrm{i}D_t + D_x^2)g_j \cdot f. \tag{4.17}$$

Substituting (4.1)-(4.7) into (4.17), P_1 becomes

$$P_1 = -|D(\mathbf{a}_j^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)||D| + |D(\mathbf{a}_j^*; \mathbf{z})||D(\mathbf{z}^*; \mathbf{z}_x)| - |D(\mathbf{a}_j^*; \mathbf{z}_x)||D(\mathbf{z}^*; \mathbf{z})|$$
$$- \left\{\mathrm{i}|D(\mathbf{a}_j^*; \mathbf{z}_t)| + |D(\mathbf{a}_j^*; \mathbf{z}_{xx})|\right\}. \tag{4.18}$$

Referring to Jacobi's identity (3.6) and the fundamental formula $\alpha|D(\mathbf{a}; \mathbf{b}_1)| + \beta|D(\mathbf{a}; \mathbf{b}_2)| = |D(\mathbf{a}; \alpha\mathbf{b}_1 + \beta\mathbf{b}_2)|$ $(\alpha, \beta \in \mathbb{C})$, P_1 simplifies to $P_1 = -|D(\mathbf{a}_j^*; \mathrm{i}\mathbf{z}_t + \mathbf{z}_{xx})|$. Since $\mathrm{i}\,\mathbf{z}_t + \mathbf{z}_{xx} = \mathbf{0}$ by (2.14a), the last column of the determinant consists only of zero elements, implying that $P_1 = 0$. □

4.3. Proof of (2.5)

The equation to be proved is $P_2 = 0$, where

$$P_2 = D_x f \cdot f^* - \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n} |g_k|^2. \tag{4.19}$$

Substituting (4.1), (4.3) and (4.8)-(4.10) into (4.19), P_2 becomes

$$P_2 = -\frac{1}{2}|\bar{D}||D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{2}|D||\bar{D}(\mathbf{z}^*; \mathbf{z})| + \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n} |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_\mathbf{k})|. \tag{4.20}$$

Further simplication is possible with use of (4.11), (4.15) and (4.16) with (4.13), giving rise to

$$P_2 = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}\Big(-|D(\mathbf{b}_k^*; \mathbf{a}_k)||D(\mathbf{z}^*; \tilde{\mathbf{z}})| + |D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{a}_k, \tilde{\mathbf{z}})||D| + |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_\mathbf{k})|\Big). \tag{4.21}$$

Applying Jacobi's identity (3.6) to the middle term and replacing $|D(\mathbf{b}_k^*; \tilde{\mathbf{z}})|$ by the right-hand side of (4.12) in the resultant expression, P_2 reduces to

$$P_2 = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}\Big(-|\bar{D}(\mathbf{a}_k^*; \mathbf{z})||D(\mathbf{z}^*; \mathbf{a}_k)| + |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_\mathbf{k})|\Big). \tag{4.22}$$

It now follows from (3.8) that

$$|\bar{D}(\mathbf{a}_k^*; \mathbf{z})| = |D(\mathbf{a}_k^*; \mathbf{z})| + \sum_{m=1}^{n''}\frac{(\mathrm{i}\gamma)^m}{m!}\sum_{k_1,\ldots,k_m=1}^{n} |D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \ldots, \mathbf{a}_{k_m}^*; \mathbf{z}, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|, \tag{4.23a}$$

$$|\bar{D}(\mathbf{z}^*;\mathbf{a}_k)| = |D(\mathbf{z}^*;\mathbf{a}_k)| + \sum_{m=1}^{n''} \frac{(\mathrm{i}\gamma)^m}{m!} \sum_{k_1,\ldots,k_m=1}^{n} |D(\mathbf{z}^*, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_k, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|, \tag{4.23b}$$

where $n'' = \min(n-1, N-1)$. Referring to the expansion formulas (3.9) and (3.10), one has

$$|D(\mathbf{a}^*_k, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{z}, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})| = |D|^{-1}|D(\mathbf{a}^*_k;\mathbf{z})||D(\mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|$$

$$+|D|^{-1}\sum_{l=1}^{m}(-1)^l|D(\mathbf{a}^*_{k_l};\mathbf{z})||D(\mathbf{a}^*_k, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_{l-1}}, \mathbf{a}^*_{k_{l+1}}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|, \tag{4.24a}$$

$$|D(\mathbf{z}^*, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_k, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})| = |D|^{-1}|D(\mathbf{z}^*;\mathbf{a}_k)||D(\mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|$$

$$+|D|^{-1}\sum_{l=1}^{m}(-1)^l|D(\mathbf{z}^*;\mathbf{a}_{k_l})||D(\mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_k, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \ldots, \mathbf{a}_{k_m})|. \tag{4.24b}$$

By introducing (4.23) into (4.22) and then using (4.24), P_2 takes the form

$$P_2 = \frac{\mathrm{i}\gamma}{2|D|}\sum_{m=1}^{n''}\frac{(\mathrm{i}\gamma)^m}{m!}\sum_{l=1}^{m}(-1)^l \times$$

$$\times \sum_{k,k_1,\ldots,k_m=1}^{n}\Big[-|D(\mathbf{a}^*_{k_l};\mathbf{z})||D(\mathbf{z}^*;\mathbf{a}_k)||D(\mathbf{a}^*_k, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_{l-1}}, \mathbf{a}^*_{k_{l+1}}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|$$

$$+|D(\mathbf{a}^*_k;\mathbf{z})||D(\mathbf{z}^*;\mathbf{a}_{k_l})||D(\mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_k, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \ldots, \mathbf{a}_{k_m})|\Big]. \tag{4.25}$$

Interchange the indices k and k_l in the first term and then shift the row vector $\mathbf{a}^*_{k_l}$ in front of $\mathbf{a}_{k_{l+1}}$ and the column vector $\mathbf{a}_k$ in front of $\mathbf{a}_{k_1}$, respectively. This leads to the following relation

$$|D(\mathbf{a}^*_k, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_{l-1}}, \mathbf{a}^*_{k_{l+1}}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_m})|$$

$$\rightarrow |D(\mathbf{a}^*_{k_l}, \mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_{l-1}}, \mathbf{a}^*_{k_{l+1}}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_k, \mathbf{a}_{k_{l+1}}, \ldots, \mathbf{a}_{k_m})|$$
$$= |D(\mathbf{a}^*_{k_1}, \ldots, \mathbf{a}^*_{k_m}; \mathbf{a}_k, \mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \ldots, \mathbf{a}_{k_m})|.$$

Note that the value of the determinant is not altered since the total signature caused by the above manipulation is $(-1)^{2(l-1)} = 1$. Thus, the first term on the right-hand side of (4.25) coincides with the second term and cosequently, $P_2 = 0$. □

4.3. Proof of (2.6)

Instead of proving (2.6) directly, we differentiate (2.5) by x and add the resultant expression to (2.6) and then prove the equation $P_3 = 0$, where

$$P_3 = f_{xx}f^* - f_x f_x^* - \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n} g_{k,x}g_k^*. \tag{4.26}$$

This reduces the total amount of calculations considerably and the proof becomes transparent. It now follows from (4.1), (4.3), (4.4), (4.6) and (4.8)-(4.10) that

$$\begin{aligned} P_3 = -\frac{1}{2}\{|D(\mathbf{z}^*;\mathbf{z}_x)| + |D(\mathbf{z}_x^*;\mathbf{z})|\}\,|\bar{D}| - \frac{1}{4}|D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z})| \\ +\frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}|D(\mathbf{a}_k^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{a}_k)|. \end{aligned} \tag{4.27}$$

Differention of (4.15) with respect to x gives

$$|D(\mathbf{z}^*;\mathbf{z}_x)| + |D(\mathbf{z}_x^*;\mathbf{z})| = -\mathrm{i}\gamma\sum_{k=1}^{n}|D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{a}_k,\mathbf{z})|. \tag{4.28}$$

Inserting (4.15) and (4.28) into (4.27), P_3 can be put into the form

$$P_3 = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}\Big\{|\bar{D}||D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{a}_k,\mathbf{z})| + |D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{a}_k^*;\mathbf{b}_k)| + |D(\mathbf{a}_k^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{a}_k)|\Big\}. \tag{4.29}$$

Note from (4.11), (4.13), (4.14) and Jacobi's identity (3.6) that

$$\begin{aligned} &|\bar{D}||D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{a}_k,\mathbf{z})| + |D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{a}_k^*;\mathbf{b}_k)| \\ &= -|D(\mathbf{z}^*;\mathbf{a}_k)|\Big\{|D(\mathbf{b}_k^*;\mathbf{z}) + \frac{1}{2}|D(\mathbf{b}_k^*,\mathbf{z}^*;\mathbf{z},\tilde{\mathbf{z}})|\Big\} \\ &= -|D(\mathbf{z}^*;\mathbf{a}_k)||\bar{D}(\mathbf{a}_k^*;\mathbf{z}_x)|. \end{aligned} \tag{4.30}$$

After substituting (4.30) into (4.29), P_3 becomes

$$P_3 = \frac{\mathrm{i}\gamma}{2}\sum_{k=1}^{n}\Big\{-|\bar{D}(\mathbf{a}_k^*;\mathbf{z}_x)||D(\mathbf{z}^*;\mathbf{a}_k)| + |D(\mathbf{a}_k^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{a}_k)|\Big\}. \tag{4.31}$$

This expression reduces to (4.22) if one replaces $\mathbf{z}_x$ by $\mathbf{z}$. Hence, the proof of the relation $P_3 = 0$ completely parallels that of $P_2 = 0$ with P_2 from (4.22). □

5. Alternative expression of the bright N-soliton solution

Theorem 5.1. *The determinants f' and g'_j $(j=1,2,...,n)$ given below satisfy the system of bilinear equations (2.4)-(2.6):*

$$f' = |A' + B'|, \quad g'_j = \begin{vmatrix} A' + B' & \mathbf{y}^T \\ -\mathbf{a}_j'^* & 0 \end{vmatrix}, \qquad (j=1,2,...,n), \tag{5.1}$$

where A' and B' are $N \times N$ matrices and $\mathbf{y}$ and $\mathbf{a}'_j$ are N-component row vectors defined below:

$$A' = (a'_{jk})_{1\le j,k\le N}, \quad a'_{jk} = \frac{1}{2}\frac{y_j y_k^*}{q_j + q_k^*}, \quad y_j = \exp(q_j x + \mathrm{i} q_j^2 t), \tag{5.2a}$$

$$B' = (b'_{jk})_{1\le j,k\le N}, \quad b'_{jk} = \frac{(\mu - \mathrm{i}\gamma q_k^*)c'_{jk}}{q_j + q_k^*}, \quad c'_{jk} = \sum_{s=1}^{n} \alpha'_{sj}\alpha'^*_{sk}, \tag{5.2b}$$

$$\mathbf{y} = (y_1, y_2, ..., y_N), \qquad \mathbf{a}'_j = (\alpha'_{j1}, \alpha'_{j2}, ..., \alpha'_{jN}). \tag{5.2c}$$

Here, q_j $(j=1,2,...,N)$ and α'_{sj} $(s=1,2,...,n; j=1,2,...,N)$ are complex parameters characterizing the solution.

Let us show that the determinants f and g_j from (2.13) are closely related to the determinants f' and g'_j given by (5.1). The following lemma is useful for this purpose:

Lemma 5.1. *The determinants f and g_j given by (2.13) can be rewritten in the form*

$$f = |I + AB|, \qquad g_j = \begin{vmatrix} I + AB & \mathbf{z}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \qquad (j=1,2,...,n). \tag{5.3}$$

We now establish the following theorem:

Theorem 5.2. *Under the parameterization $q_j = -p_j^*$ $(j=1,2,...,N)$ and $\alpha'_{sj} = -\alpha_{sj}/(2c_j^*)$ $(s=1,2,...,n; j=1,2,...,N)$, the determinants f, f', g_j and g'_j satisfy the relations*

$$f = c|A|f', \tag{5.4}$$

$$g_j = c|A|g'_j, \quad (j=1,2,...,n), \tag{5.5}$$

where

$$c = (-1)^N \prod_{l=1}^{N}(4c_l^* c_l), \qquad c_l = \frac{\prod_{m=1}^{N}(p_l + p_m^*)}{\prod_{\substack{m=1\\(m\neq l)}}^{N}(p_l - p_m)}, \qquad (l=1,2,...,N). \tag{5.6}$$

The parameters p_j $(j = 1, 2, ..., N)$ *are assumed to satisfy the conditions* $p_l + p_m^* \neq 0$ *for all* l *and* m *and* $p_l \neq p_m$ *for* $l \neq m$.

Thus, we have obtained the two different expressions for the bright N-soliton solution of the system of nonlinear PDEs (2.2). Explicitly, they read $u_j = g_j/f = g_j'/f'$ $(j = 1, 2, ..., n)$.

The following proposition provides an alternative proof of theorem 5.1:

Proposition 5.1. *If* f *and* g_j *given respectively by (5.4) and (5.5) satisfy the system of bilinear equations (2.4)-(2.6), then* f' *and* g_j' *satisfy the same system of equations, and vice versa.*

6. A continuum model

The n-component system (1.1) yields a continuum model when one takes a limit $n \to \infty$. It represents a (2+1)-dimensional nonlocal modified NLS equation of the form

$$\mathrm{i}\, q_t + q_{xx} + \mu \left(\int_{-\infty}^{\infty} |q|^2 dy \right) q + \mathrm{i}\gamma \left(\int_{-\infty}^{\infty} |q|^2 dy\, q \right)_x = 0, \quad q = q(x, y, t). \tag{6.1}$$

Recall that when $\gamma = 0$, this equation reduces to a (2+1)-dimensional nonlocal NLS equation proposed by Zakharov [8]. The exact method of solution for equation (6.1) can be developed following the same procedure as that for the system of nonlinear PDEs (1.1). Hence, we summarize the main results.

First, application of the gauge transformation

$$q = u \exp\left[-\frac{\mathrm{i}\gamma}{2} \int_{-\infty}^{x} \int_{-\infty}^{\infty} |u(x, y, t)|^2 dx dy \right], \qquad u = u(x, y, t), \tag{6.2}$$

to the system (6.1) subjected to the the boundary conditions $q \to 0, u \to 0$ $|x| \to \infty$ transforms (6.1) to a nonlocal nonlinear PDE for u

$$\mathrm{i}\, u_t + u_{xx} + \mu \left(\int_{-\infty}^{\infty} |u|^2 dy \right) u + \mathrm{i}\gamma \left(\int_{-\infty}^{\infty} u^* u_x dy \right) u = 0. \tag{6.3}$$

The proposition below is an analog of proposition 2.1:

Proposition 6.1 *By means of the dependent variable transformation*

$$u = \frac{g}{f}, \tag{6.4}$$

equation (6.3) can be decoupled into the following system of bilinear equations for $f = f(x, t)$ *and* $g = g(x, y, t)$

$$(\mathrm{i}D_t + D_x^2) g \cdot f = 0, \tag{6.5}$$

$$D_x f \cdot f^* = \frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{\infty} |g|^2 dy, \tag{6.6}$$

$$D_x^2 f \cdot f^* = \mu\int_{-\infty}^{\infty} |g|^2 dy + \frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{\infty} D_x g \cdot g^* dy. \tag{6.7}$$

Proof. The proof proceeds exactly as that of proposition 2.1. Formally, one may simply replace the sum $\sum_{k=1}^{n}$ by the integral $\int_{-\infty}^{\infty} dy$. □

It follows from (6.2), (6.4) and (6.6) that

$$q = \frac{gf^*}{f^2}, \tag{6.8}$$

which is just a continuum limit of (2.12).

The following theorem can be derived from a continuum limit of the bright N-soliton solution given by theorem 2.1 and theorem 5.1:

Theorem 6.1. *The system of bilinear equations (6.5)-(6.7) admits the following two different expressions f, g and f', g' for the bright N-soliton solution :*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g = \begin{vmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}^* & 0 \end{vmatrix}, \tag{6.9}$$

$$f' = |A' + B'|, \quad g' = \begin{vmatrix} A' + B' & \mathbf{y}^T \\ -\mathbf{a}'^* & 0 \end{vmatrix}. \tag{6.10}$$

Here, A and B are $N \times N$ matrices given respectively by (2.14a) and (2.14b) with c_{jk} being replaced by $\int_{-\infty}^{\infty} \alpha_j(y)\alpha_k^*(y)dy$, A' and B' are $N \times N$ matrices given respectively by (5.2a) and (5.2b) with c'_{jk} being replaced by $\int_{-\infty}^{\infty} \alpha'_j(y)\alpha'^*_k(y)dy$ and $\mathbf{a} = \mathbf{a}(y) = (\alpha_1, \alpha_2, ..., \alpha_N)$ and $\mathbf{a}' = \mathbf{a}'(y) = (\alpha'_1, \alpha'_2, ..., \alpha'_N)$ are N-component row vectors where α_j and α'_j $(j = 1, 2, ..., N)$ are continuous functions of y.

Proof. The proof can be done in the same way as that of theorem 2.1 and theorem 5.1. □

Theorem 6.2. *Under the parameterization $q_j = -p_j^*$ and $\alpha'_j = -\alpha_j/(2c_j^*)$ $(j = 1, 2, ..., N)$, the determinants f, f', g and g' satisfy the relations*

$$f = c|A|f', \tag{6.11}$$

$$g = c|A|g', \tag{6.12}$$

where c is defined by (5.6) and the parameters p_j $(j = 1, 2, ..., N)$ *are specified such that* $p_l + p_m^* \neq 0$ *for all* l *and* m *and* $p_l \neq p_m$ *for* $l \neq m$.

Proof. The proof parallels theorem 5.2. □

Proposition 6.2. *If* f *and* g *given by (6.9) satisfy the system of bilinear equations (6.5)-(6.7), then* f' *and* g' *given by (6.11) and (6.12) satisfy the same system of equations, and vice versa.*

Proof. The proof is completely parallel to that of proposition 5.1. □

7. Conclusion

1. We have obtained two different expressions of the bright N-soliton solution of a multi-component modified NLS equation in terms of determinants.

2. We have proposed a continuum model arising from the multi-component system as the number of dependent variables tends to infinity and presented its bright N-soliton solution.

3. Our solutions include existing bright N-soliton solutions of the multi-component NLS equation and its continuum model.

Acknowledgement

This work was partially supported by the Grant-in-Aid for Scientific Research (C) No. 22540228 from Japan Society for the Promotion of Science.

References

[1] Matsuno Y 2011 The bright N-soliton solution of a multi-component modified nonlinear Schrödinger equation *J. Phys. A: Mat. Theor.* **44** 495202 (18pp)

[2] Hisakado M and Wadati M 1995 Integrable multi-component hybrid nonlinear Schrödinger equations *J. Phys. Soc. Japan* **64** 408-413

[3] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys. - JETP* **34** 62-69

[4] Kaup D J and Newell A C 1978 An exact solution for a derivative nonlinear Schrödinger equation *J. Math. Phys.* **19** 798-801

[5] Manakov S V 1974 On the theory of two-dimensional stationary self-focusing of electromagnetic waves *Sov. Phys. - JETP* **38** 248-253

[6] Hisakado M, Iizuka T and Wadati M 1994 Coupled hybrid nonlinear Schrödinger equation and optical solitons *J. Phys. Soc. Japan* **63** 2887-2894

[7] Matsuno Y 2011 The N-soliton solution of a two-component modified nonlinear Schrödinger equation *Phys. Lett.* **A 375** 2090-2094

[8] Zakharov V E 1980 The inverse scattering method *Solitons (Topics in Current Physics)* ed R K Bullough and P J Caudrey (Berlin: Springer) pp 243-285