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Optimal Stopping Rules for the Random Horizon Duration Problems

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1 Introduction

In the *classical best choice problem*, a version of the secretary problem studied by Gilbert and Mosteller (1966) extensively, a fixed known number n of rankable objects (1 being the best and n the worst) appear one at a time in random order with all $n!$ permutations equally likely. Each time an object appears, we observe only the relative rank of the current object with respect to its predecessors. We must select one object and find a stopping rule that maximizes the probability of selecting the best one. The optimal rule passes over the first $s_c^*(n)-1$ objects and stops with the first relatively best object if any, where $s_c^*(n)=\min\left\{k\geq 1:\sum_{j=k+1}^{n}\frac{1}{j-1}\leq 1\right\}$. As $n\to\infty$, $s_c^*(n)/n\to e^{-1}\approx 0.3679$ and the optimal probability of selecting the best overall also converges to e^{-1}.

As a different version of the secretary problem, Ferguson, Hardwick and Tamaki (1992) considered an optimal stopping problem called the *duration problem*. Among other models, the basic one is the *classical duration problem* (see their Section 2.2, in which this problem is called the finite horizon duration problem), where we must find a stopping rule that maximizes the expected duration of holding a relatively best object. Clearly, we only select a relatively best object, receiving a payoff of 1 plus the number of future observations before a new relatively best object appears or until the final stage n is reached. The optimal rule passes over the first $t_c^*(n)-1$ objects and stops with the first relatively best object if any, where $t_c^*(n)=\min\left\{k\geq 1:\sum_{j=k+2}^{n}\frac{1}{j-1}\sum_{i=j}^{n}\frac{1}{i}\leq\sum_{i=k+1}^{n}\frac{1}{i}\right\}$. As $n\to\infty$, $t_c^*(n)/n\to e^{-2}\approx 0.1353$ and the optimal proportional payoff (= payoff/n) converges to $2e^{-2}\approx 0.2707$.

The optimal stopping rule in each of the above two classical problems is simple in a sense that it stops on the first relatively best object appearing after a given stage. Such a simple rule is often referred to as a *threshold rule*. $s_c^*(n)$ and $t_c^*(n)$ are called threshold values.

In this paper, we attempt to generalize the classical duration problem into two directions by allowing the number of objects to be random and also allowing the objects to appear in accordance with Bernoulli trials. More specifically this can be stated as follows, if we call an object *candidate* when it is relatively best. Each object observed is immediately judged either to be a candidate or not. Let $X_j, j \geq 1$, be the indicator of the event that the jth object is a candidate and suppose that $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables with $P\{X_j = 1\} = a_j, j \geq 1$. Let also N be a bounded random variable representing the number of available objects, i.e. length of random horizon. It is assumed that N is independent of the sequence $X_1, X_2, \ldots$ and has a prior distribution $p_k = P\{N = k\}$ such that $\sum_{k=1}^{n} p_k = 1$ and $p_n > 0$ for a known upper bound n and that the payoff is zero if no object is chosen. This problem, referred to as the *random horizon duration problem*, is completely specified by two sequences $\{a_j\}_{j=1}^{n}$ and $\{p_k\}_{k=1}^{n}$, and aims to find a stopping rule that maximizes the expected duration of holding a candidate based on available information $X_1, X_2, \ldots, X_N$.

The classical duration problem occurs as a special case of the random horizon duration problem if N is degenerated to n, i.e. $P\{N = n\} = 1$ and $a_j = 1/j, 1 \leq j \leq n$, because the relative ranks $R_1, R_2, \ldots, R_n$ of n rankable objects presented one by one in random order have a property that the $R_j^{'}$s are independent with $P\{R_j = i\} = 1/j, 1 \leq i \leq j$, and the only relevant information about R_j is whether R_j takes the value 1 or not. We are said to be in the *secretary case* if $a_j = 1/j, 1 \leq j \leq n$.

In Section 2, we formulate the random horizon duration problem. This problem can be distinguished into two models, MODEL 1 and MODEL 2, according to whether the final stage of the planning horizon is N or n. This distinction is related to the last candidate. That is, it is assumed that if the chosen object is the last candidate, we can hold it until stage N in MODEL 1, whereas until stage n in MODEL 2. MODELs 1 and 2 will be considered in Sections 3 and 4 respectively. It is easy to see that the optimal rule is always a threshold rule for $n \leq 3$. However, for $n \geq 4$, the form of the optimal rule heavily depends on $\{a_j\}_{j=1}^{n}$ and $\{p_k\}_{k=1}^{n}$, implying that the optimal rule is not necessarily a threshold rule. Hence, our main concern is to give a simple sufficient condition for the optimal rule to be a threshold rule. An interesting application of this condition appears in the secretary case. For N having a uniform, generalized uniform or curtailed geometric distribution, the optimal rule is shown to be a threshold rule. The asymptotic results, as $n \to \infty$, will be also obtained for these prior distributions.

See Gnedin (2005), Samuels (2004), and Tamaki et al. (1998) for the duration problems and Presman and Sonin (1972), Petruccelli (1983), and Tamaki (2011) for the stopping problems with random horizon.

2 The random horizon duration problem

For the random horizon duration problem having the sequences $\{a_j\}_{j=1}^n$ and $\{p_k\}_{k=1}^n$, we write $b_j = 1 - a_j = P\{X_j = 0\}, 1 \le j \le n$ and introduce the notations

$$B_{k,i} = b_{k+1}b_{k+2}\cdots b_i, \qquad 0 \le k < i \le n$$

with $B_{k,k} = 1$ for convenience, and

$$\pi_k = p_k + p_{k+1} + \cdots + p_n, \qquad 1 \le k \le n$$

with $\pi_1 = 1$ and $\pi_n > 0$. Further define

$$\sigma_k = \pi_k + (n-k)p_k.$$

Denote by k the *state*, where we have just observed the *kth* object to be a candidate, $1 \le k \le n$. Let $S(k)$ and $C(k)$ represent the expected payoff earned by stopping with the current candidate in state k and by continuing observations in an optimal manner respectively. Then $V(k) = \max\{S(k), C(k)\}$ represents the optimal expected payoff, provided that we start from state k. The following lemma gives the explicit form of $S(k)$.

Lemma 2.1. *We have, for* $1 \le k \le n$,

$$(i) \quad S(k) = \frac{1}{\pi_k}\sum_{i=k}^{n} \pi_i B_{k,i} \quad \text{for MODEL 1.}$$

$$(ii) \quad S(k) = \frac{1}{\pi_k}\sum_{i=k}^{n} \sigma_i B_{k,i} \quad \text{for MODEL 2.}$$

On the other hand, we have

$$C(k) = \frac{1}{\pi_k} \sum_{j=k+1}^{n} \pi_j B_{k,j} r_j V(j),$$

where $r_j = a_j/b_j, 1 \leq j \leq n$. Hence, the optimality equation can be solved recursively to yield the optimal rule and the optimal payoff.

3 MODEL 1

Theorem 3.1. *Let*

$$G(k) = \pi_k - r_{k+1} \sum_{i=k+1}^{n} \pi_i B_{k,i}, \quad 1 \leq k < n.$$

Then a sufficient condition for the optimal rule to be a threshold rule is that $G(k)$ changes its sign from $-$ to $+$ at most once, that is,

once $G(k) \geq 0$ for some k, then $G(j) \geq 0$ for all $k \leq j < n$.

Corollary 3.1. *Let, for $1 \leq k < n-1$,*

$$H(k) = (k+1)\pi_k - (k+2)\pi_{k+1}.$$

Then, in the secretary case, a sufficient condition for the optimal rule to be threshold rule is that $H(k)$ changes its sign from $-$ to $+$ at most once.

Corollary 3.1 is applicable to the following distributions.
Example 3.1 (N is degenerated to n): $p_n = 1$ and $p_k = 0, 1 \leq k < n$.
Example 3.2 (uniform): $p_k = 1/n, 1 \leq k \leq n$.
Example 3.3 (generalized uniform):

$$p_k = \begin{cases} 0, & \text{if } 1 \leq k < T \\ \frac{1}{n-T+1}, & \text{if } T \leq k \leq n, \end{cases}$$

for a given parameter $T = 1, 2, \ldots, n$.
Example 3.4 (curtailed geometric) : $p_k = (1-q)q^{k-1}/(1-q^n), 1 \leq k \leq n$ for a given parameter $0 < q < 1$.

3.1 Asymptotic results in the secretary case

Let z^* and v^* be the limiting threshold value and the limiting optimal payoff respectively. We have the following asymptotic results.

Lemma 3.1.(Uniform prior.) *Let* $\alpha^*(\approx 0.0775)$ *be the unique root* $\alpha \in (0,1)$ *of the equation* $2(1+\sqrt{\alpha})+\log\alpha = 0$. *Then*

$$z^* = \alpha^*, \quad v^* = \alpha^*\left(1+\sqrt{\alpha^*}\right)^2.$$

Lemma 3.2.(Generalized uniform prior.) *Let T depend on n in such a way that $T/n \to \alpha$ as $n \to \infty$ for a fixed $0 < \alpha < 1$. Then the asymptotic results are distinguished into two cases according to whether $\alpha \le \alpha^*$ or $\alpha > \alpha^*$, where α^* is as defined in Lemma 3.1.(we use below the notations z^*_α and v^*_α for z^* and v^* respectively to make explicit the dependence on α).*

Case (i): $0 \le \alpha \le \alpha^*$

$$z^*_\alpha = \alpha^*, \quad v^*_\alpha = \frac{\alpha^*\left(1+\sqrt{\alpha^*}\right)^2}{1-\alpha}.$$

Case (ii): $\alpha^* < \alpha < 1$

$$z^*_\alpha = \alpha^{\frac{\sqrt{\alpha}-\alpha}{1-\alpha}}e^{-2}, \quad v^*_\alpha = \left(1-\frac{\sqrt{\alpha}}{1-\alpha}\log\alpha\right)z^*_\alpha.$$

Lemma 3.3.(Curtailed geometric prior.) *Let q depend on n through $q = 1 - c/n$ for a fixed positive value $c(< n)$. We use the notations z^*_c and v^*_c for z^* and v^* respectively to make explicit the dependence on c. Then z^*_c is a unique root $z \in (0,1)$ of the equation*

$$J_c(z) + \log z\left[I_c(z) + e^{-c}\left(1+\frac{1}{2}\log z\right)\right] = 0,$$

where

$$I_c(z) = \int_z^1 \frac{e^{-cx}}{x}dx, \quad J_c(z) = \int_z^1 \frac{e^{-cx}}{x}(1-\log x)\,dx.$$

Moreover,

$$v^*_c = \frac{z^*_c}{1-e^{-c}}\left[e^{-c}\log z^*_c + I_c(z^*_c)\right].$$

4 MODEL 2

Theorem 4.1. *Let*

$$G(k) = \sigma_k - r_{k+1} \sum_{i=k+1}^{n} \sigma_i B_{k,i}, \quad 1 \le k < n.$$

Then a sufficient condition for the optimal rule to be a threshold rule is that $G(k)$ changes its sign from $-$ to $+$ at most once, that is,

$$\text{once } G(k) \ge 0 \text{ for some } k, \text{ then } G(j) \ge 0 \text{ for all } k \le j < n.$$

For the applications to the secretary case, the following corollary is useful.

Corollary 4.1. *Let*

$$H(k) = (k+1)\sigma_k - (k+2)\sigma_{k+1}, \quad 1 \le k < n-1.$$

Then, in the secretary case, a sufficient condition for the optimal rule to be a threshold rule is that $H(k)$ changes its sign from $-$ to $+$ at most once

Corollary 4.1 is applicable to the distributions given in Section 3.

4.1 Asymptotic results in the secretary case

We have the following asymptotic results.

Lemma 4.1.(Uniform prior.) *Let $\alpha^*(\approx 0.0775)$ be as defined in Lemma 3.1. Then*

$$z^* = \alpha^*, \quad v^* = 2\alpha^*\left(1+\sqrt{\alpha^*}\right)^2.$$

Lemma 4.2.(Generalized uniform prior.) The two cases are distinguished according to whether $\alpha \le \alpha^*$ or $\alpha > \alpha^*$, where α^* is as defined in Lemma 3.1.(we use below the notations z^*_α and v^*_α for z^* and v^* respectively).

Case (i): $0 \le \alpha \le \alpha^*$;

$$z^*_\alpha = \alpha^*, \quad v^*_\alpha = \frac{2\alpha^*\left(1+\sqrt{\alpha^*}\right)^2}{1-\alpha}.$$

Case (ii): $\alpha^* < \alpha < 1$; Let

$$\rho = -\left(2 + \frac{1+\alpha}{1-\alpha}\log\alpha\right), \quad \sigma = 1 - \rho + \sqrt{1 + \frac{2\rho(2+\rho)}{1+\alpha}}.$$

Then

$$z_\alpha^* = e^{-\sigma}, \quad v_\alpha^* = (\rho+\sigma)e^{-\sigma}.$$

Lemma 4.3.(Curtailed geometric prior.) *We use the notations z_c^* and v_c^* for z^* and v^* respectively. Then z_c^* is a unique root $z \in (0,1)$ of the equation*

$$e^{-cz} - e^{-c} - e^{-c}\log z\left(2+\frac{1}{2}\log z\right) = (1+c)J_c(z) + \{1+(1+c)\log z\}\, I_c(z).$$

Moreover,

$$v_c^* = \frac{z_c^*}{1-e^{-c}}\left[e^{-c} - e^{-cz_c^*} + e^{-c}\log z_c^* + (1+c)I_c(z_c^*)\right].$$

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