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Pricing of the Bermudan Swaption under the Generalized Ho–Lee Model

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1 Introduction

The Generalized Ho–Lee model is proposed by Ho and Lee (2007). It is an arbitrage–free binomial lattice interest rate model, which is an extension of their previous Ho–Lee Model proposed in Ho and Lee (1986). The previous Ho–Lee model is the first arbitrage–free term structure model of interest rates, while the model assumes that the forward volatility of interest rate movements is a constant, independently of state and time. To improve this point, the Generalized Ho–Lee model uses a state– and time– dependent implied volatility function.

As an interest rate derivative we consider a Bermudan swaption. A Bermudan swaption is an exotic interest rate derivative that the underlying asset is an interest rate swap. The feature of the Bermudan swaption is that its holder has a right to choose an exercise time from a set of prespecified multiple exercise opportunities over a prescribed exercise period. Generally, pricing of the Bermudan swaption is more difficult than that of plain vanilla European swaption.

In this paper, we first specify the bond price by the Generalized Ho–Lee model. Then, we derive the optimality equation of the Bermudan swaption price via a dynamic programming approach to the induced optimal stopping problem, and we compute the Bermudan swaption price by solving it backwardly in time.

2 Description of the Generalized Ho–Lee model

2.1 The Generalized Ho–Lee model

The Generalized Ho–Lee model is a discrete–time model, which uses a recombining binomial lattice model to represent an uncertainty of an interest rate. In the model we consider an economy that all agents can trade securities without any market friction. We assume that the risk–neutral probabilities of the up and down states are equivalent and equal to 0.5. A node on the binomial lattice is represented by (n, i) where n denotes the time and i the state ($0 \leq n \leq N^{*}$, $0 \leq i \leq n$, N^{*} is the time horizon). Let $P(n, i\,; T)$ be the zero–coupon bond price at node (n, i) with remaining maturity of T period ($0 \leq T \leq N^{*}$, $P(n, i\,; 0) = 1$ for any n and i). $P(0, 0\,; T)$ denotes the zero–coupon bond price observed at node $(0, 0)$, which gives the discount function at the initial time. As the Generalized Ho–Lee model is a term structure model of an interest rate, it has an interest rate term structure in all node (n, i).

The uncertainty on the binomial lattice is represented by the binomial volatilities $\delta(n, i\,; \cdot)$. $\delta(n, i\,; 1)$ denotes the binomial volatility of one–period at node (n, i), and it is represented by

$$\delta(n, i\,; 1) = \frac{P(n+1, i+1\,; 1)}{P(n+1, i\,; 1)}, \quad 0 \leq n \leq N^{*} - 1,\ 0 \leq i \leq n. \tag{1}$$

Similarly the binomial volatility of T–period at node (n, i) is represented by

$$\delta(n, i\,; T) = \frac{P(n+1, i+1\,; T)}{P(n+1, i\,; T)}, \quad 0 \le n \le N^* - 1,\ 0 \le i \le n,\ 0 \le T \le N^*, \tag{2}$$

where we define $\delta(n, i\,; 0) = 1$. When $\delta(n, i\,; \cdot) = 1$, there is no risk on the binomial lattice by definition. As the binomial volatilities is larger, the uncertainty also increases more.

The binomial volatilities are imposed on several requirement in Ho and Lee (2007). The first requirement is the mean reversion property of interest rate. The binomial volatility of one–period in the model is given by

$$\delta(n, i\,; 1) = \exp\Big(-2\sigma(n) \min\big(R(n, i\,; 1), R\big)\Delta t^{3/2}\Big), \tag{3}$$

where $R(n, i\,; 1)$ denotes the one–period yield at node $(n.i)$, R the threshold rate which is implied from the market prices, and Δt the time interval of one period. Equation (3) represents that the interest rate is proportional to interest rates level when interest rates are low, and it is constant when interest rates are high. Therefore, Equation (3) ensures that interest rates are non–negative and non–explosive. $\sigma(n)$ of Equation (3) is some function of time n, which represents the term structure of volatilities:

$$\sigma(n) = (\sigma_0 - \sigma_\infty + \alpha_0 n) \exp(-\alpha_\infty n) + \alpha_1 n + \sigma_\infty. \tag{4}$$

Each parameter would be estimated so that the observed swaption prices and the model prices would fit. In Ho and Lee (2007), these parameter are interpreted as follows: The parameter σ_0 denotes the short–rate volatility over the first period, the parameter α_∞ the exponential decay which represents the speed of mean reversion in the interest rate process, the parameter α_0 the size of the hump of the volatility curve, and the expression $\alpha_1 n + \sigma_\infty$ is approximately the short–rate forward volatility at time n when n is sufficiently large. Equation (4) ensures a mean–reversion behavior in interest rate movements.

The next requirement for the binomial volatility is the arbitrage–free condition. As the Generalized Ho–Lee model is an arbitrage–free interest rate model, the bond price for all different maturities at each node (n, i) is modeled to simultaneously satisfy risk–neutral valuation formula under the risk–neutral probability $\mathbb{Q}$. In the Generalized Ho–Lee model, the arbitrage–free condition requires the following:

$$\delta(n, i\,; T) = \delta(n, i\,; 1)\delta(n+1, i\,; T-1) \left(\frac{1 + \delta(n+1, i+1\,; T-1)}{1 + \delta(n+1, i\,; T-1)}\right). \tag{5}$$

Therefore, the Generalized Ho–Lee model defines the arbitrage–free condition by Equation (5) as the relationship of the binomial volatility for T–period. Given the above conditions, we define the arbitrage–free bond pricing model for the one–period bond price as

$$P(n, i\,; 1) = \frac{P(0, 0\,; n+1)}{P(0, 0\,; n)} \prod_{k=1}^{n} \left(\frac{1 + \delta(k-1, 0\,; n-k)}{1 + \delta(k-1, 0\,; n-k+1)}\right) \prod_{j=0}^{i-1} \delta(n-1, j\,; 1). \tag{6}$$

Similarly, we can derive the T–period bond price as

$$P(n, i\,; T) = \frac{P(0, 0\,; n+T)}{P(0, 0\,; n)} \prod_{k=1}^{n} \left(\frac{1 + \delta(k-1, 0\,; n-k)}{1 + \delta(k-1, 0\,; n-k+T)}\right) \prod_{j=0}^{i-1} \delta(n-1, j\,; T), \tag{7}$$

where the three parts of Equation (7) denote the forward price, the convexity adjustment term and the stochastic movement term, respectively.

2.2 A Recursive Algorithm for Bond Pricing

In this section, we propose a recursive algorithm for generating the arbitrage–free one–period bond prices $P(n,i\,;1)$ in the Generalized Ho–Lee model. Although its essential idea is explained in Ho and Lee (2007), it is not sufficiently clear for readers to understand its details. We now clearly explain its details in an algorithmic form.

In order to construct the one–period bond prices, we use Equations (3), (5), and (6). We assume that the followings are initially given:

(i). The bond prices with remaining maturity of T–period ($0 \leq T \leq N^*$) observed at initial time: $P(0,0\,;T)$;

(ii). The threshold rate: R;

(iii). The term structure of volatilities: $\sigma(n)$.

In order to generate the one–period bond pricing algorithm, we begin with initial time $n = 0$. Then, we iterate the algorithm forwardly in time.

To generalize more, now, we consider the algorithm at time m.

- Step 1. Derive the one–period bond price for $n = m$ and $i = 0$ by using Equation (6):

$$P(m,0\,;1) = \frac{P(0,0\,;m+1)}{P(0,0\,;m)} \prod_{k=1}^{m} \frac{(1+\delta(k-1,0\,;m-k))}{(1+\delta(k-1,0\,;m-k+1))}. \tag{8}$$

- Step 2. Derive the one–period bond prices for $n = m$ and $i = 1, 2, \ldots, m$ by using Equation (6):

$$P(m,i\,;1) = P(m,0\,;1) \prod_{j=0}^{i-1} \delta(m-1,j\,;1). \tag{9}$$

By Steps 1 and 2, we can construct arbitrage–free one–period bond prices at time m and state i ($i = 0, 1, \ldots, m$). Then, we derive the one–period yields by using the computed one–period bond prices.

- Step 3. Derive the one–period yields:

$$R(m,i\,;1) = -\frac{\log P(m,i\,;1)}{\Delta t}. \tag{10}$$

By Step 3, we can determine the one–period yields at time m and state i ($= 0, 1, \ldots, m$). Next, we derive the one–period binomial volatility by using one–period yields.

- Step 4. Derive the one–period binomial volatilities, by using Equation (3):

$$\delta(m,i\,;1) = \exp\Big(-2\sigma(m)\min\Big(R(m,i\,;1), R\Big)\Delta t^{3/2}\Big). \tag{11}$$

By Step 4, we can determine the one–period binomial volatilities at time m and state i ($= 0, 1, \ldots, m$). Given $\delta(m,i\,;1)$ for $i = 0, 1, \ldots, m$, we can determine the binomial volatilities with remaining maturity of T–period (for $2 \leq T$) by using Equation (5) backwardly in time.

- Step 5. Derive the higher order of the binomial volatilities:

$$\delta(m,i\,;T) = \delta(m,i\,;1)\delta(m+1,i\,;T-1)\left(\frac{1+\delta(m+1,i+1\,;T-1)}{1+\delta(m+1,i\,;T-1)}\right). \tag{12}$$

In Step 5, given $\delta(m,i\,;1)$ by Step 4, we can firstly derive $\delta(m-1,i\,;2)$ for $i = 0, \ldots, m-1$ by Equation (5). Next, we can derive $\delta(m-2,i\,;3)$ for $i = 0, \ldots, m-2$, and we repeatedly continue Step 5 until we reach to time 0 in which we have $\delta(0,0\,;m+1)$. This completes the recursive algorithm for computing the one–period bond prices at time m, then we can similarly determine the one–period bond prices at time $m+1$ and so on.

3 Pricing the Bermudan Swaption

The interest rate swap is a contract where two parties agree to exchange a fixed rate and a floating rate over a prespecified period, we usually use LIBOR (London Inter-Bank Offered Rate) as a floating rate. Let N be the contract agreement time of the swap, and $M_1, M_2, \ldots, M_L$ be the L coupon payment times starting after the contract agreement time N. The time sequence of coupon payments is

$$0 \le N < M_1 < M_2 < \cdots < M_L \le N^*.$$

Besides, we set

$$M_{h+1} - M_h = \kappa \Delta t \ (= \text{constant}), \quad h = 0, \ldots, L-1,$$

where $M_0 = N$, and let $\kappa = 1$ for convenience.

The swap rate is the par rate for an interest rate swap, that is, it is the fixed rate that makes the values of both receiver and payer sides of an interest rate swap equal at the contract agreement time N. The swap rate to be set at the time N is called as a spot swap rate, and we define it as:

$$S(N,i) = \frac{1 - P(N,i;L)}{\Delta t \sum_{l=1}^{L} P(N,i;l)}, \tag{13}$$

where $P(N,i;l)$ denotes the bond price with remaining maturity of l-period ($1 \le l \le L$) at the contract agreement time N.

A European interest rate swaption is an interest rate derivative whose underlying asset is an interest rate swap. The holder of European swaption has a right to enter at the exercise time N into an interest rate swap. We consider *a payer swaption* case in which the holder of swaption pays a fixed rate K (> 0) and receives a floating rate. The value of the cash flow of the payer swaption at exercise time N can be represented by $\left[S(N,i) - K\right]_+ := \max\{S(N,i) - K, 0\}$, and the value of the European swaption at the exercise time N with L payment times is

$$\Delta t \cdot \left[S(N,i) - K\right]_+ \sum_{l=1}^{L} P(N,i;l), \quad i = 0, \ldots, N. \tag{14}$$

A Bermudan swaption is a swaption having multiple exercise opportunities over the prescribed exercise time interval, and the holder of a Bermudan swaption can exercise the right only once from the set of allowed exercise opportunities. We deal with a Bermudan swaption with fixed tenor, which have a constant length in its payment periods after the exercise. We define the set (N_B) of multiple exercise opportunities and the prescribed exercise time interval as

$$N_B := \{N_1, N_2, \ldots, N_k\} \subset \{N_b, N_b + 1, \ldots, N_e\}, \tag{15}$$

respectively.

Let $V(n,i)$ ($n = 0, 1, \ldots, N_k, i = 0, 1, \ldots, n$) be the value of the Bermudan swaption at node (n,i), then $V(n,i)$ can be characterized as the optimal value of the following optimal stopping problem under the risk-neutral probability $\mathbb{Q}$:

$$V(n,i) = \sup_{\tau \in \mathcal{T}_{N_B \cap \{n,\ldots,N_e\}}} \mathbb{E}^{\mathbb{Q}} \left[\left(\prod_{k=n}^{\tau-1} P(k, I_k; 1) \right) \Delta t \cdot [S(\tau, I_\tau) - K]_+ \sum_{l=1}^{L} P(\tau, I_\tau; l) \,\middle|\, (n,i) \right], \tag{16}$$

where $\mathcal{T}_{N_B \cap \{n,n+1,\ldots,N_e\}}$ is the set of all stopping times which have values in $N_B \cap \{n, n+1, \ldots, N_e\}$, τ is a random exercise time, and I_τ is the state at the random exercise time τ.

Next, applying the dynamic programming approach, we derive the optimality equation to solve the above optimal stopping problem. For convenience, let us define the value of the European swaption as

$$U(n,i) := \Delta t \cdot [S(n,i) - K]_+ \sum_{l=1}^{L} P(n,i;l), \quad n \in N_B, \ i = 0, \ldots, n. \tag{17}$$

Since we consider an optimal stopping problem with a finite–horizon, the value of the Bermudan swaption must satisfy the following optimality equation:

$$V(n,i) = \max\left\{U(n,i), P(n,i;1)\mathbb{E}^{\mathbb{Q}}\left[V(n+1,I_{n+1})|\,(n,i)\right]\right\}, \quad n \in N_B,\ i=0,\ldots,n, \qquad (18)$$

where I_{n+1} denotes the state at time $n+1$. We can obtain the solution for the above optimal stopping problem by solving Equation (18) backwardly in time, and its algorithm is given by the followings:

- Step 0. [Initial Condition] For $n = N_k$,

$$V(N_k,i) = U(N_k,i), \quad i=0,\ldots,N_k. \qquad (19)$$

- Step 1–1. For $n \in N_B \setminus \{N_k\}$,

$$V(n,i) = \max\left\{U(n,i), P(n,i;1)\mathbb{E}^{\mathbb{Q}}\left[V(n+1,I_{n+1})|\,(n,i)\right]\right\}, \quad i=0,\ldots,n. \qquad (20)$$

- Step 1–2. For $n \notin N_B$,

$$V(n,i) = P(n,i;1)\mathbb{E}^{\mathbb{Q}}\left[V(n+1,I_{n+1})|\,(n,i)\right], \quad i=0,\ldots,n. \qquad (21)$$

Therefore, the Bermudan swaption price can be computed by solving Equations (20) and (21) backwardly in time n with the initial condition (19) at time N_k.

4 Computation of the Yield Curves and the Bermudan Swaption Prices

In this section, we present some numerical results of the yield curve movement and the Bermudan swaption prices based on the preceding arguments. The parameters in the Generalized Ho–Lee model are set as follows: $\Delta t = 0.25$, R (= the threshold rate) = 0.3, a flat yield curve of 5%, and the term structure of volatility is assumed so that its value starts from 0.3 and decreases 0.01 in each unit of time. Then, the arbitrage–free yield curve movements in the Generalized Ho–Lee model are given in the following table and figure.

	Time to Maturity				
Node	1	2	3	4	5
(0,0)	0.05	0.05	0.05	0.05	0.05
(1,0)	0.042507	0.042639	0.042771	0.042903	0.043035
(1,1)	0.057507	0.057388	0.057268	0.057147	0.057026
(2,0)	0.036612	0.036837	0.037063	0.037289	0.037517
(2,1)	0.048939	0.048988	0.049035	0.049081	0.049124
(2,2)	0.065616	0.065343	0.065068	0.064792	0.064515

Table 1: The yield curve movement

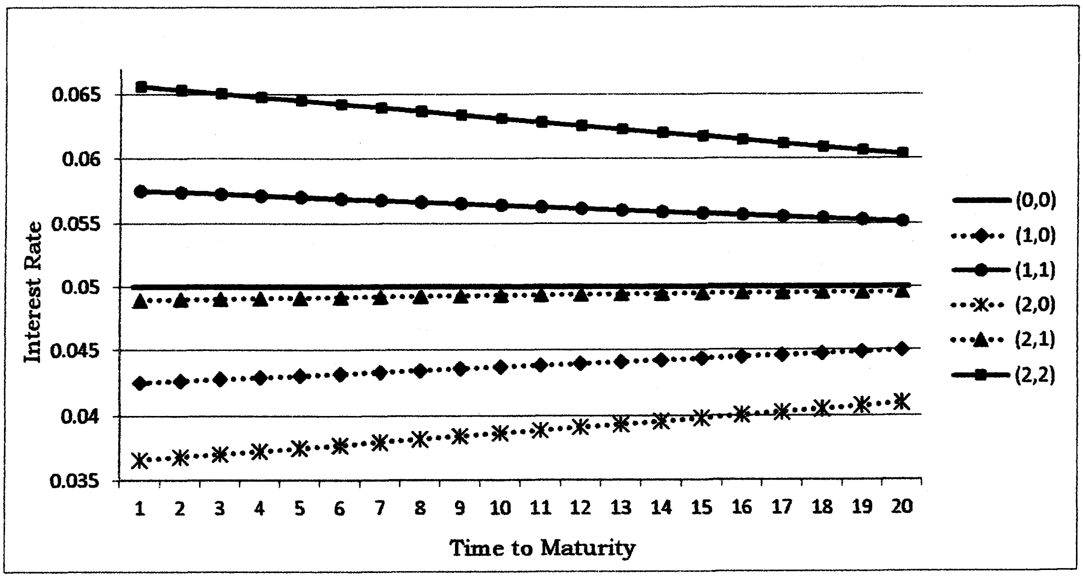

Figure 1: The yield curve movement

Table 1 and Figure 1 show the yield curve movement with time varying volatility in each possible node until time 2.

Next, we present the Bermudan swaption prices based on this yield curve movement in Table 2, where we assume a fixed rate of 10% and the prescribed set of multiple exercise times $N_B = \{4, 6, 8, 10, 12\}$. The surrounded area with the line represents the exercise nodes in the prescribed multiple exercise times.

												0.170
											0.146	0.132
										0.157	0.112	0.098
									0.132	0.116	0.081	0.068
								0.135	0.095	0.080	0.054	0.042
							0.110	0.092	0.063	0.048	0.030	0.019
						0.106	0.072	0.056	0.034	0.022	0.009	0
					0.083	0.063	0.039	0.025	0.013	0.005	0	0
				0.072	0.045	0.028	0.016	0.008	0.002	0	0	0
			0.051	0.032	0.019	0.010	0.004	0.001	0	0	0	0
		0.036	0.022	0.012	0.006	0.002	0.001	0	0	0	0	0
	0.025	0.015	0.008	0.004	0.001	0	0	0	0	0	0	0
0.017	0.010	0.005	0.002	0.001	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12

Time

Table 2: The Bermudan swaption prices and the exercise area

5 Conclusion

In this paper, we proposed a method for pricing the Bermudan swaption. We used the Generalized Ho–Lee model to represent bond price dynamics. A useful feature of the model comes from a state– and time– dependent implied volatility function. Then, we derived the optimality equation to solve the Bermudan swaption via a dynamic programming approach, and we proposed a dynamic programming algorithm to compute its prices by a backward induction technique.

As future research theme, we have to examine how each of parameters and initial conditions in the Generalized Ho–Lee model affects the prices of the Bermudan swaption. Further, we should analyze the effectiveness of the Generalized Ho–Lee model for pricing of the Bermudan swaption in details.

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