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Absence of Phase Transitions in Two-dimensional $O(N)$ Spin Models with Large N –Through the Renormalization Group Flow –

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Abstract

We Fourier-transform the classical $O(N)$ spin models in two dimensions to obtain a Gaussian system perturbed by a functional determinant. We analyze the system by renormalization group type arguments, and show that there exist no phase transitions if N is sufficiently large, no matter how large β is.

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1. Introduction. The existence of the phase transition in two dimensional (2D) Ising model was established by Onsager [1] in the middle of the last century, and the existence of the Kosterlitz-Thouless transition in 2D XY-model was rigorously established by Fröhlich and Spencer [2] three decades ago.

As for non-abelian systems in lower dimensions, however, our knowledge is very poor. Spontaneous mass generations in 2D non-Abelian sigma models (Heisenberg model) and quark confinement in 4D non-Abelian lattice gauge theories have been widely believed [3–5] since the last century, but their proofs still remain to be seen. These models exhibit no phase transitions in the hierarchical model approximations of Wilson-Dyson type or Migdal-Kadanov type [6, 7].

One of the main difficulties in these models is that the field variables are non-abelian objects and block spin transformations break the structures. In some cases, this can be avoided by introducing an auxiliary field ψ [9]. Using this idea, together with the help of the cluster expansion [10], we showed [13, 14] in the 2D $O(N)$ sigma model with large N that

$$\beta_c \geq \text{const} N \log N \tag{1}$$

where $\beta_c(N)$ be the lower bound for the critical inverse temperature of 2d $O(N)$ spin model.

In this Letter, we show our new analysis [16] based on the duality arguments type, and announce some partial results:

Theorem *There exist no phase transitions in the two-dimensional $O(N)$ classical spin model if N is sufficiently large.*

We scale the inverse temperature β by N. The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$\langle f \rangle \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - N\beta) d\phi_i \tag{2}$$

Here Λ is an arbitrarily large square with center at the origin. Moreover $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $< 1 >= 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|=1} \phi(x)\phi(y), \tag{3}$$

First substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ into (2) with the condition that $\mathrm{Im} a_i < -\nu$ [9], we set

$$\mathrm{Im}\, a_i = -(\nu + \frac{m^2}{2}), \quad \mathrm{Re}\, a_i = \frac{1}{\sqrt{N}}\psi_i \tag{4}$$

where $m > 0$ is an arbitrary constant. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|} \int \cdots \int \exp[-W_0(\phi,\psi)] \prod \frac{d\phi_j d\psi_j}{2\pi} \\ &= c^{|\Lambda|} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \end{aligned} \tag{5}$$

where

$$\begin{aligned} W_0(\phi,\psi) &= \frac{1}{2}\langle \phi, (m^2 - \Delta + i\alpha\psi)\phi\rangle - \sum_j i\sqrt{N}\beta\psi_j \\ &= \frac{1}{2}\langle \phi, (m^2 - \Delta)\phi\rangle + \frac{i}{\sqrt{N}}\langle \phi^2 - N\beta, \psi\rangle \end{aligned} \tag{6a}$$

$$F(\psi) = \det(1 + i\alpha G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j] \tag{6b}$$

$\alpha \equiv 2/\sqrt{N}$, c's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian and $G = (m^2 - \Delta)^{-1}$. Note that $F(\psi)$ is integrable with respect to ψ if and only if $N \geq 3$.

In the same way, the two-point function is given by

$$\langle \phi_0 \phi_x \rangle = \frac{1}{Z} \int \cdots \int (m^2 - \Delta + i\alpha\psi)^{-1}_{0x} F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{7}$$

Set $\nu = 2$ below. Then we can choose m so that $G(0) = \beta$ ($m^2 \sim \exp[-4\pi\beta]$) and

$$F(\psi) = {\det}_3^{-N/2}(1 + i\alpha G\psi) \exp[-\langle \psi, G^{\circ 2}\psi\rangle], \tag{8}$$

$${\det}_3(1 + A) \equiv \det[(1 + A)e^{-A + A^2/2}] \tag{9}$$

where $G^{\circ 2}(x,y) = G(x,y)^2$ so that $\mathrm{Tr}(G\psi)^2 = \langle \psi, G^{\circ 2}\psi\rangle$. Then we expect that the subtracted determinant $\det_3(1 + i\alpha \cdots) \sim 1$ and that exponential decay follows from (7) since $\tilde{Z} = \int F(\psi) \prod d\psi_i/2\pi \sim \int |F(\psi)| \prod d\psi_i/2\pi$.

We justify this argument by renormalization group methods. The cancelation between the first term of the expanstion of the determinant and the phase factor $\exp[i\sqrt{N}\beta\psi]$, and the change of effective mass m^2 are carried out recursively.

2. Proof of the Theorem. We use the block spin transformation [4] to justify the previous idea. Intuitively speaking, we set

$$\phi(x) = \phi_<\left(\left[\frac{x}{L}\right]\right) + \tilde{\phi}(x) \tag{10}$$

$$\psi(x) = \frac{1}{L^2}\psi_<\left(\left[\frac{x}{L}\right]\right) + \tilde{\psi}(x) \tag{11}$$

where $\phi(x)$, $\phi_<$ and $\tilde{\phi}$ have the momentum $|p_i| \le \pi$, $|p_i| \le \pi/L$ and $\pi(1-1/L) \le |p_i| \le \pi$ $(i = 1, 2)$ respectively. The same is true for $\psi(x)$. The point $[x/L] \in Z^2$ means the lattice point nearest to $x/L \in R^2$, then $\phi_<(x)$ and $\psi_<(x)$ again have the momentum $|p_i| \le \pi$ and living on the scaled lattice points.

Starting with $\phi_0 = \phi$ and $\psi_0 = \psi$, we recursively define

$$\exp[-W_{n+1}(\phi_{n+1}, \psi_{n+1})] = \int \exp[-W_n(\phi_{n+1} + \tilde{\phi}_n, L^{-2}\psi_{n+1} + \tilde{\psi}_n)] \prod d\tilde{\phi}_n d\tilde{\psi}_n \tag{12}$$

Our theorem follows from the main term of the n'th action W_n:

$$\begin{aligned} W_n(\phi_n, \psi_n) &= \frac{1}{2}\langle \phi_n, (-\Delta + m_n^2)\phi_n\rangle + \frac{\gamma_n}{2}\sum(\nabla_\mu \phi_n^2(x))^2 \\ &\quad + \langle \psi_n, H_n^{-1}\psi_n\rangle + \frac{i}{\sqrt{N}}\langle(\phi_n^2 - N\beta_n), \psi_n\rangle \end{aligned} \tag{13}$$

where

$$\begin{aligned} m_n^2 &= L^{2n} m_0^2, \quad \gamma_n = \frac{n}{N} \\ \beta_n &= \beta - O(n), \quad H_n^{-1} = O(1) > 0 \end{aligned}$$

Therefore the integration over ψ_n yields the potential

$$V_n(\phi_n) = \frac{1}{2}\langle \phi_n, (-\Delta + m_n^2)\phi_n\rangle + \frac{\gamma_n}{2}\sum(\nabla_\mu \phi_n^2(x))^2 \frac{1}{N}\sum_x (\phi_n^2(x) - N\beta_n)^2 \tag{14}$$

where $\beta_n \to 0$ for large n. The term after γ_n is of the form of

$$\sum(\phi_n(x+e_\mu)^2 - \phi_n(x)^2)^2$$

This means that $\phi_n(x) \in R^N$ and $\phi_n(x + e_\mu) \in R^N$ have the same radius and has no effects on the non-existence of phase transition no matter how large γ_n is. Thus the system is the $O(N)$ symmetric Heisenberg model of inverse temperature $N\beta_n = O(N)$ which is in massive phase, see eq. (1).

3. Block Spin Transformation and Stability Bounds. To obtain the flow $\{W_n\}$, we use the mathematically controllable block spin transformation introduced by Kupiainen and Gawedzki [12] some decades ago, and integrate $\exp[-W_0]$ recursively from high momentum parts. This is done by decomposing ϕ_n and ψ_n into the next order block spins ϕ_{n+1} and ψ_{n+1} and zero-average fluctuations $Q\xi_n$ and $Q\tilde{\psi}_n$ as

$$\begin{aligned} \phi_n &= A_{n+1}\phi_{n+1} + Q\xi_n \\ \psi_n &= \tilde{A}_{n+1}\psi_{n+1} + Q\tilde{\psi}_n \end{aligned}$$

and by integrating over ξ_n and $\tilde{\psi}_n$ after the substitution. Here A_{n+1} and $\tilde{A}_{n+1}$ are chosen so that

$$\begin{aligned} \langle \phi_n, G_n^{-1}\phi_n\rangle &= \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1}\rangle + \langle \xi_n, Q^+G_n^{-1}Q\xi_n\rangle \\ \langle \psi_n, H_n^{-1}\psi_n\rangle &= \langle \psi_{n+1}, \tilde{H}_{n+1}^{-1}\psi_{n+1}\rangle + \langle \tilde{\psi}_n, Q^+H_n^{-1}Q\tilde{\psi}_n\rangle \end{aligned}$$

We briefly discuss about matrices A_n, $\tilde{A}_n$ and Q. Let $G_0 = (-\Delta + m_0^2)^{-1}$ and define G_n and $C : R^{\Lambda_n} \to R^{\Lambda_{n+1}}$ by

$$G_{n+1}(x,y) = (CG_nC^+)(x,y), \quad (Cf)(x) = \frac{1}{L^2}\sum_{z\in\Delta_0} f(Lx+z) \tag{15}$$

where L is a positive integer (e.g. 2,3, etc.) and Δ_0 is the box of size $L \times L$ centered at the origin. The operator C takes averages of spins over boxes with centers $Lx \in LZ^2$ and scales down the coordinates by L^{-1}. $\Lambda_n = Z^2 \cap L^{-1}\Lambda_{n-1}$ is the lattice space shrinked by L. Let A^+ mean the adjoint of A with respect to the real inner product. The following choice of A_n and Q satisfies our requirement:

$$A_n = G_{n-1}C^+G_n^{-1} \tag{16}$$

$$Q(x,y) = \begin{cases} 1 & \text{if } x = y \notin L\Lambda_n \\ -1 & \text{if } x \in L\Lambda_n \text{ and } y \in \Delta_x \\ 0 & \text{if otherwise} \end{cases} \tag{17}$$

The matrix $Q : R^{\Lambda_n \backslash L\Lambda_{n+1}} \to R^{\Lambda_n}$ is block-wise diagonal and constructs zero-average fluctuation field $Q\xi$. We then see

$$CA_n = 1,\ C(Q\xi)(x) = \frac{1}{L^2}\sum_{\zeta\in\Delta_0}(Q\xi)(Lx+\zeta) = 0 \tag{18}$$

The covariance of the fluctuation field $\{\xi_n(x); x \in \Lambda_n \backslash L\Lambda_{n+1}\}$ is given by

$$\Gamma_n = [Q^+ G_n^{-1} Q]^{-1} \tag{19}$$

and we see that $\Gamma_n(x,y)$ decays exponentially fast uniformly in β. Put

$$\mathcal{A}_n = A_1 A_2 \cdots A_n = G_0 (C^+)^n G_n^{-1},\ G_0 = \mathcal{G}_0 \tag{20}$$

and define

$$\mathcal{G}_n = \mathcal{A}_n G_n \mathcal{A}_n^+, \quad \mathcal{T}_n = \mathcal{A}_n Q \Gamma_n Q^+ \mathcal{A}_n^+ \tag{21}$$

so that

$$\mathcal{G}_n = \mathcal{G}_{n+1} + \mathcal{T}_n \tag{22}$$

By putting $\phi_0 = A_1\phi_1 + Q\xi_0$ and integrating over ξ_0, we obtain the determinants

$$\det{}^{-N/2}(1 + i\alpha \mathcal{T}_0 \psi)$$

and the Gaussian term of ψ:

$$\exp\left[-\left\langle \psi, (\frac{2}{N}(\varphi_1\varphi_1) \circ \left(Q\frac{1}{P}Q^+\right)\psi\right\rangle\right] \sim \exp\left[-\left\langle \psi, (\frac{2}{N}(\varphi_1\varphi_1) \circ \mathcal{T}_0)\psi\right\rangle\right] \tag{23}$$

where

$$P(\psi) = \Gamma_0^{-1} + i\alpha Q^+ \psi Q, \tag{24}$$

and $A \circ B$ stands for the Hadamard product of A and B, i.e. $(A \circ B)_{xy} = A_{xy}B_{xy}$, and $A^{\circ 2} = A \circ A$. Remark $\mathrm{Tr}(A\psi)(B\psi) = \langle \psi, (A^t \circ B)\psi \rangle$ for any matrices A and B. We approximate $\varphi_1(x)\varphi_1(y) = N\mathcal{G}_1(x,y) + :\varphi_1(x)\varphi_1(y):$ by $N\mathcal{G}_1(x,y)$ assuming that the Wick product term is small. There exist configurations which violate this approximation:

$$\begin{aligned} &D_w(\varphi_1) = \text{minimal paved set such that} \\ &|\varphi_1(x)\varphi_1(y) - N\mathcal{G}_1(x,y)| < N^{1+\epsilon_1} \exp[\frac{c}{10}|x-y|],\ \forall x \in D_w, \forall y \in D_w^c \end{aligned}$$

where paved set is a collection of squares $\{\square\}$ each of which consists of squares $\Delta \subset \Lambda$ of size $L \times L$. We call $D_w(\varphi_1)$ domain wall regions. If all spins are in the same direction and their lengths are in $(N\beta_1)^{1/2}(1 \pm N^\epsilon/2\beta_1)$, then $D_w = \emptyset$ by the minimality. Similarly we define the large field region $D(\psi_1)$ of ψ_1 by the paved set such that

$$D(\psi_1) = \{\square \mid |\psi_1(x)| > N^\epsilon, \exists x \in \square\}$$

$\{D_\omega\}$ have small probabilities to exist because of the large energy $\langle\phi_1, (-\Delta)_D\phi_1\rangle$ of ϕ_1 and the factor $\exp[-i\langle: \varphi^2 :, \psi\rangle/\sqrt{N}]$, where $(-\Delta)_D$ is the restriction of $-\Delta$ on to the region $\{\phi_1(x); x \in D\}$. Similarly $D(\psi_1)$ have small probability to exist because of the determinants. D can be decomposed into connected components $\{D_i\}$. These regions are extracted as $g(D_i, \psi_1, \phi_1)$ from the Gibbs measure as large field regions. (This definition applies for $\beta >> N$.) These factors satisfy

$$|g(D_i, \varphi_1, \psi_1)| \leq \exp[-\text{const.}\, N^{1+\varepsilon}|D|]$$

In other regions, the fields are small and smooth, we can extract a Gaussian factor:

$$\begin{aligned} & \det{}^{-N/2}(1 + i\alpha\mathcal{T}_0\psi) \\ = & \det{}_3^{-N/2}(1 + i\alpha\mathcal{T}_0\psi) \times \exp[-i\sqrt{N}\langle\mathcal{T}_0, \psi\rangle - \langle\psi, \mathcal{T}_0^{\circ 2}\psi\rangle] \end{aligned} \tag{25}$$

This and the previous factor yield a new Gaussian term of ψ:

$$\begin{aligned} & \exp\left[-\frac{i}{\sqrt{N}}\langle(\varphi_1^2 - N\beta_1), \psi\rangle - \langle\psi, \tilde{H}_1^{-1}\psi\rangle\right] \\ & \tilde{H}_1^{-1} = \mathcal{T}_0^{\circ 2} + 2\mathcal{G}_1 \circ \mathcal{T}_0 \end{aligned}$$

Here $\beta_1 = \beta_0 - \mathcal{T}_0(x, x)$, $(\beta_0 = \beta)$. Since $\beta_0 >> 1$, $\tilde{H}_1^{-1} \sim 2\beta_1\mathcal{T}_0$ is again a Laplacian with small mass term. But we see that $\tilde{H}_n^{-1}$ becomes soon massive.

We need another block spin transformation of the auxiliary field ψ to decompose the bilinear form of ψ. Since the field ψ has the dimension $(\text{length})^{-2}$, we define the block spin operator $C' = L^2 C$ of ψ by

$$(C'\psi)(x) = L^2(C\psi)(x) = \sum_{\zeta\in\Delta_0}\psi(Lx + \zeta) \tag{26}$$

Since $\mathcal{T}_0(x, y)$ decreases exponentially fast in $|x - y|$, and $\mathcal{G}_1(x, y)$ is a slowly decreasing function such that $\mathcal{G}_1(x, y) \sim \beta_1$ for $|x - y| < O(1)$, $\mathcal{T}_0^{\circ 2} + 2\mathcal{G}_1 \circ \mathcal{T}_0$ has two types of eigenvectors. The first one is (almost) a block-wise constant vector corresponding to the eigenvalue $O(1)$ and the second ones are the zero-average eigenvectors corresponding to the eigenvalues of order $O(\beta_1)$. Put

$$\psi(x) = \tilde{A}_1\psi_1 + Q\tilde{\psi}_0 \tag{27}$$

$$\psi_1(x) = (C'\psi)(x) = \sum_{\zeta\in\Delta_0}\psi(Lx + \zeta) \tag{28}$$

so that

$$\langle\psi,\tilde{H}_1^{-1}\psi\rangle = \langle\psi_1,H_1^{-1}\psi_1\rangle+\langle\tilde{\psi}_0,Q^+\tilde{H}_1^{-1}Q\tilde{\psi}_0\rangle \tag{29}$$

$$\langle:\varphi_1^2:,\psi\rangle = \langle:\varphi_1^2:,\tilde{A}_1\psi_1\rangle+\langle:\varphi_1^2:,Q\tilde{\psi}_0\rangle \tag{30}$$

where $:\varphi_1^2: := \varphi_1^2 - N\beta_1$ and $H_1^{-1} = \tilde{A}_1^+\tilde{H}_1^{-1}\tilde{A}_1$. Contrary to $CA_n = 1$, it holds that $C\tilde{A}_n \doteq L^{-2}C'\tilde{A}_n = L^{-2}$. Thus the Gaussian integration over $\tilde{\psi}_0$ yields

$$\mathcal{F}_1 = \frac{1}{4N} <:\varphi_1^2:, Qf_1Q^+ :\varphi_1^2:>$$

where $f_1 = [Q^+\tilde{H}_1^{-1}Q]^{-1}$. Then we have

$$\begin{aligned}\mathcal{F}_1 &= \frac{1}{32N\beta_1}\sum_\mu\sum_x(\nabla_\mu\varphi^2(x))^2\\ &\sim \frac{1}{8N\beta_1}\langle\nabla_\mu\varphi_1,(\varphi_1\otimes\varphi_1)\nabla_\mu\varphi_1\rangle\end{aligned}$$

This is a reminiscence of $(\varphi_1^2-N\beta_1)^2$ which shows that the fluctuation field of φ_1 is perpendicular with φ_1 itself. The RG flow of this term is different from that of $(\varphi_n^2-N\beta_n)^2$ since $\mathcal{F}$ is made at each step and the latter term keeps its form with a slight change of β_n.

The origin of this term is found in the hierarchical approximation of Dyson-Wilson type [7, 8] and rediscovered in [11]. This is a part of the probability that two spins $\phi_\pm \equiv \phi\pm\xi$ form the block spin ϕ such that $\phi^2 = x$. In fact put $\phi = (\varphi,0)\in R_+\times R^{N-1}$ and $\xi = (s,u)\in R\times R^{N-1}$. Then

$$\begin{aligned}&\int f((\phi+\xi)^2)f((\phi-\xi)^2)ds d^{N-1}u\\ &= \int f((\varphi+s)^2+u^2)f((\varphi-s)^2+u^2)dsd^{N-1}u\\ &= \frac{1}{\sqrt{x}}\int_0^{N\beta}\int_0^{N\beta} f(p)f(q)\left(\frac{p+q}{2}-x-\frac{(p-q)^2}{16x}\right)^{(N-3)/2} dpdq\end{aligned}$$

where

$$\frac{(p-q)^2}{16x} = \frac{(\phi_+^2-\phi_-^2)^2}{16\phi^2} = \frac{\langle\phi,\xi\rangle^2}{\phi^2} \tag{31}$$

corresponds to $\mathcal{F}_1$. In the hierarchical model, this is restricted to each block, and does not enter the next step. In the real systems, however, this enters the next step since $(\phi_+^2-\phi_-^2)^2$ is replaced by $\sum_\mu(\nabla_\mu\phi^2)^2$, and this term increases slowly in n.

Let us see the role of ψ integration. We observe

$$\int\frac{e^{-i:\varphi^2:\psi}}{(1+i\psi)^{N/2}}d\psi = \text{const.}\, e^{:\varphi^2:}\times\begin{cases}(-:\varphi^2:)^{N/2-1} & \text{if } :\varphi^2:<0\\ 0 & \text{otherwise}\end{cases}$$

Applying this discussion to the block Δ in which $| : \varphi^2 : | = |\varphi^2 - N\beta|$ is large, one finds a constant $\beta_1 = \beta - O(1)$ such that for $\varphi^2 < N\beta$

$$\begin{aligned} &\exp[L^2(\varphi^2 - N\beta) + \frac{L^2}{2}(N-2)\log(N\beta - \varphi^2)] \\ &\sim \exp\left[-\frac{L^2}{N}(\varphi^2 - N\beta_1)^2\right] \end{aligned} \tag{32}$$

This is the probability density that the arithmetic average of L^2 balls takes its value at φ. This is a rediscovery of the facts found in the hierarchical model approximation which goes back some decades [7, 8]. This means that the fluctuation of $: \varphi^2 :$ is considerably small.

For very large ψ ($|\psi| \geq O(N^{1/2})$) where $1/P(\psi)$ is small and the Gaussian factor is small, we have the stability of the determinant which comes from the determinant inequality

$$|\det{}^2(1 + i\alpha ABA^*)| \geq \det(1 + k_0^2\alpha^2 B^2) \tag{33}$$

where $k_0 = \inf \mathrm{spec} AA^*$ and $B = B^*$. (We put $A = (\Gamma_0)^{1/2}$, $B = Q^+\psi Q$). This is the reason why we need $N \geq 3$.

4. Renormalization Group Flow. We combine two types of block transformations to $W_n(\phi_n, \psi_n)$. One is the block spin transformation of the N component boson model of mass m_n^2, and the other is the block spin transformation of the auxiliary field ψ_n which has the dimension $(\text{length})^{-2}$.

The induction assumption is that the main part of $W_n(\phi_n, \psi_n)$ is given by (13), and we have to prove that the change of W_n is absorbed by the parameters m_n^2 in G_n^{-1}, γ_n and $u_n = N\beta_n$. Moreover $H_0^{-1} = 0$, $\gamma_0 = 0$, $\beta_0 = \beta$ and we discarded irrelevant terms. Compared with W_0, the most strange term is

$$\gamma_n \sum (\nabla_\mu \phi_n^2(x))^2 \sim 4\gamma_n \langle \nabla_\mu \phi_n, (\phi_n \otimes \phi_n) \nabla_\mu \phi_n \rangle$$

which means that the fluctuation field $\xi_n \sim \nabla_\mu \phi_n$ is almost orthogonal to the block spin field ϕ_n since $\gamma_n \geq 0$ increases as $n \to \infty$. This term is a reminiscence of $\langle (\phi_k^2 - u_k), \psi_k \rangle$, $k \leq n$ and they sum up to yield γ_n.

Let $\Lambda_n = L^{-n}\Lambda \cap Z^2$ and let ϕ_n be the nth block spin ($\phi_{n+1} = C\phi_n$):

1. Set $\phi_n = A_{n+1}\phi_{n+1} + Q\xi_n$ so that

$$< \phi_n, G_n^{-1}\phi_n > = < \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} > + < \xi_n, \Gamma_n^{-1}\xi_n >$$

where $G_{n+1}^{-1} = A_{n+1}^+ G_n^{-1} A_{n+1}$ and $Q^+ G_n^{-1} Q = \Gamma_n^{-1}$.

2. The Gaussian part of ξ also comes from γ_n and we have

$$\begin{aligned}\gamma_n\langle\nabla_\mu\phi_n,(\phi_n\otimes\phi_n)\nabla_\mu\phi_n\rangle &= \gamma_n\langle\nabla_\mu\phi_{n+1},(\phi_{n+1}\otimes\phi_{n+1})\nabla_\mu\phi_{n+1}\rangle \\ &\quad+\gamma_n\langle\nabla_\mu Q\xi_n,(\phi_{n+1}\otimes\phi_{n+1})\nabla_\mu Q\xi_n\rangle\end{aligned}$$

where we assume $\phi_n(x)$ changes slowly in x (i.e. outside of the domain wall region) . Moreover we have

$$\frac{i}{\sqrt{N}} < \phi_n^2,\psi_n >= \frac{i}{\sqrt{N}} < \phi_{n+1}^2+2\phi_{n+1}(Q\xi_n)+(Q\xi_n)^2,\psi_n >$$

3. The ξ_n integration is strongly affected by the block spin ϕ_{n+1}.

$$d\mu(\xi_n) = \left[-\frac{1}{2}\langle\xi_n,P_n\xi_n\rangle\right]\prod_x d\xi_n(x) \tag{34}$$

$$P_n = 1_N\otimes[\Gamma_n^{-1}+i\alpha Q^+\Psi_nQ]+\gamma_n[\phi_{n+1}\otimes\phi_{n+1}]\otimes_x\Gamma_n^{-1} \tag{35}$$

where $([\phi\otimes\phi]\otimes_x\Gamma_n^{-1})(x,y)=\phi(x)\otimes\phi(x)\Gamma_n^{-1}(x,y)$ is an $N\times N$ matrix.

Originally we have $[(\phi\otimes\phi)\circ\Gamma_n^{-1}]_{(i,x),(j,y)}\equiv\phi_i(x)\phi_j(y)\Gamma_n^{-1}(x,y)$. This is approximated as above when $\Gamma_n^{-1}(x,y)$ decays fast in $|x-y|$.

4. The determinant $\det^{-1/2}(P_n)$ and P_n^{-1} depends on an approximate projection operator $\varphi_n\otimes\varphi_n$ and the fluctuations paralelle with ϕ_n are very much depressed and the fluctuations perpendicular with ϕ_n are not affected.

$$\begin{aligned}&\int\exp[-i\alpha<\xi_n,Q^+(\phi_{n+1}\psi_n)>]d\mu(\xi_n)\\ &=\det{}^{-1/2}(P_n)\exp\left[-\frac{1}{N}\langle\psi_n,\phi_{n+1}Q\frac{1}{P_n}Q^+\phi_{n+1}\psi_n\rangle\right]\end{aligned}$$

5. If γ_n is small, then

$$\phi_{n+1}QP_n^{-1}Q^+\phi_{n+1}\sim NG_{n+1}\circ T_n\sim N\beta_{n+1}T_n$$

where $T_n=Q\Gamma_nQ^+$. This is very large. If γ_n is large then

$$\phi_{n+1}QP_n^{-1}Q^+\phi_{n+1}=\frac{1}{\gamma_n}T_n\sim 0$$

In the same way, for small γ_n,

$$\det{}^{-1/2}(P_n)=\det{}^{-N/2}(1+i\alpha T_n\psi_n)$$

and for large γ_n

$$\det{}^{-1/2}(P_n)=\det{}^{-(N-1)/2}(1+i\alpha T_n\psi_n)$$

6. Thus we obtain the Gaussian term of ψ_n expanding the determinant up to the second order. The first term is used to decrease β_n by $T_n = O(1)$ and the second term is used to make the Hamiltonian of ψ_n. Remark that $\langle \varphi_n, \Psi_n \rangle \sim \langle \phi_n, \psi_n \rangle$, $\langle \varphi_n^2, \Psi_n \rangle \sim \langle \phi_n^2, \psi_n \rangle$ etc., thanks to the properties of $\mathcal{A}_n$ and $\tilde{\mathcal{A}}_n$.

   Then $H(\psi_n) = \langle \Psi_n(\mathcal{T}_n^{\circ 2} + 2\mathcal{T}_n \circ \mathcal{G}_{n+1})\Psi_n \rangle$ for small γ_n and $H(\psi_n) = \langle \Psi_n, \mathcal{T}_n^{\circ 2}\Psi_n \rangle$ for large γ_n. Put $\psi_n = \tilde{A}_{n+1}\psi_{n+1} + Q\tilde{\psi}_n$. The integral of $\langle \phi_{n+1}^2, Q\tilde{\psi}_n \rangle$ by $\tilde{\psi}_n$ yields a new factor of order $O(N^{-1})$ of form γ_n since Q^+ acts as a differential operator on ϕ_{n+1}^2.

7. As a conclusion, the term proportional to γ_n does not have strong effects on the flow. The flow of u_n is not affected by γ_n. The curvature of the potential at $\phi_n^2 = u_n = N(\beta_0 - O(n))$ is N^{-1} uniformly in n. This is what happens in the hierarchical model approximation of Dyson-Wilson type of the sigma model with large N.

8. Thus our iteration continues except for the regions of the large fields and domain walls which have a small probability to exist. Thus this transformation iterates well and we reach at the high-temperature region [16].

As we discussed, our conclusion follows from the form of W_n of large n such that $\beta_n = O(1)$. We are not sure if the idea used here can be applied to the study of the non-abelian lattice gauge theory.

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