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Exactly solvable models of heat conduction

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Abstract

We review a class of interacting diffusions which have recently been introduced to model transport of heat in a systems in contact with two reservoirs (C. Giardinà, J. Kurchan, F. Redig, *Duality and exact correlations for a model of heat conduction*, J. Math. Phys. **48**, 033301 (2007)). The models can be exactly solved by means of a dual stochastic process made of interacting particles which are absorbed at the boundaries. The construction of the dual process is related to the underlying structure of $SU(1,1)$ algebra. The class of interacting diffusions includes as a special case the KMP model.

1 Brownian Energy Process: BEP(m)

We start with the Brownian Momentum Process introduced in [10]. For a set $\Sigma_m = \{1, \ldots, m\}$ with $m \in \mathbb{N}$ and for a graph $G = (V, E)$ with vertex set V and edges set E, we consider the real variables $\{x_{i,\alpha}\}_{i \in V, \alpha \in \Sigma_m}$ and the generator

$$L^{BMP(m)} = \sum_{(i,j) \in E} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 . \tag{1.1}$$

The random variables $x_{i,\alpha}(t)$, evolving with the above generator, represent m momenta per site at time t, and, in the course of evolution, kinetic energy is exchanged between any two momenta in neighboring (i.e. connected by an edge) sites. The energies on each site

$$z_i(t) = \frac{1}{2} \sum_{\alpha=1}^{m} x_{i,\alpha}^2(t)$$

evolve with the generator

$$L^{BEP(m)} = \sum_{(i,j) \in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{m}{2}(z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) . \tag{1.2}$$

This is the Brownian Energy Process, BEP(m). The model (1.2) can actually be defined for a real number $m \in \mathbb{R}$ (i.e. m does not need to be an integer). It is easy to check that the BEP(m) model has stationary measures given by product measures with marginals Gamma distributions with shape parameter $m/2$ and scale parameter $1/\lambda$, i.e. the marginal stationary density at site i is

$$f_\lambda(z_i) = \frac{\lambda^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} z_i^{\frac{m}{2}-1} e^{-\lambda z_i} . \tag{1.3}$$

In particular, for $m = 2$ one has products of Exponential distributions with parameter λ.

2 Symmetric Inclusion process: SIP(m)

The Symmetric Inclusion Process, SIP(m), is an interacting particle systems defined by the generator

$$(L^{SIP(m)} f)(\xi) = \sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{m}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2} \right) [f(\xi^{j,i}) - f(\xi)] . \tag{2.4}$$

Here $\{\xi_i\}_{i\in V}$ are integer variables counting the number of particles at every site i of the graph. Given the configuration $\xi = (\xi_1, \ldots, \xi_{|V|})$, we denote by $\xi^{i,j}$ the configuration obtained from ξ by removing one particle at i and placing it at j.

One can easily check that product measures with marginals given by Negative Binomials, i.e. with marginal probability mass function at site i given by $(0 \leq p \leq 1)$

$$\nu_p(\xi_i) = (1-p)^{\frac{m}{2}} \frac{\Gamma(\frac{m}{2} + \xi_i)}{\Gamma(\frac{m}{2})} \frac{p^{\xi_i}}{\xi_i!}, \tag{2.5}$$

are reversible, and therefore stationary, measures of the SIP(m) process. In particular, for $m = 2$ one has marginal Geometric distributions with parameter $1 - p$.

3 Duality

Definition 3.1 *Let $\{X(t)\}_{t\geq 0}$ and $\{Y(t)\}_{t\geq 0}$ be two stochastic processes. We say that they are dual with duality function $D(\cdot,\cdot)$ if the following relation hold for all (x,y) and all times t*

$$\mathbb{E}_x(D(X(t), y)) = \mathbb{E}_y(D(x, Y(t))) . \tag{3.6}$$

On the left-hand side we have expectation with respect to the $X(t)$ process initialized at x, while on the right-hand side we have expectation with respect to the $Y(t)$ process initialized at y.

Theorem 3.1 *The process $\{z(t)\}_{t\geq 0}$ with generator $L^{BEP(m)}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $L^{SIP(m)}$ are dual, with duality functions*

$$D(z,\xi) = \prod_{i\in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i)} \tag{3.7}$$

Proof: Duality is a consequence of the fact that the generator of the BEP(m) process and the generator of the SIP(m) process correspond to the same abstract operator $\mathcal{L}_{(m)}$ in two different representations. The abstract operator is given by the ferromagnetic quantum spin chain on the graph G, with spins satisfying the $SU(1,1)$ algebra. Namely,

$$\mathcal{L}_{(m)} = \sum_{(i,j)\in E} \left(\mathcal{K}^+_{(m),i}\mathcal{K}^-_{(m),j} + \mathcal{K}^-_{(m),i}\mathcal{K}^+_{(m),j} - 2\mathcal{K}^o_{(m),i}\mathcal{K}^o_{(m),j} + \frac{m^2}{8} \right) \tag{3.8}$$

where the spins $\left\{\mathcal{K}^{+}_{(m),i}, \mathcal{K}^{-}_{(m),i}, \mathcal{K}^{o}_{(m),i}\right\}_{i\in V}$ satisfy the $SU(1,1)$ commutation relations:

$$[\mathcal{K}^{o}_{(m),i}, \mathcal{K}^{\pm}_{(m),j}] = \pm\delta_{i,j}\mathcal{K}^{\pm}_{(m),i} \qquad [\mathcal{K}^{-}_{(m),i}, \mathcal{K}^{+}_{(m),j}] = 2\delta_{i,j}\mathcal{K}^{o}_{(m),i} \tag{3.9}$$

The $SU(1,1)$ algebra admit the following two families (labelled by m) of infinite dimensional representations:

$$\begin{cases} \mathcal{K}^{+}_{(m),i} = z_i \\ \\ \mathcal{K}^{-}_{(m),i} = z_i\,\partial^2_{z_i} + \frac{m}{2}\partial_{z_i} \\ \\ \mathcal{K}^{0}_{(m),i} = z_i\,\partial_{z_i} + \frac{m}{4} \end{cases} \qquad \begin{cases} \mathcal{K}^{+}_{(m),i}|\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right)|\xi_i + 1\rangle \\ \\ \mathcal{K}^{-}_{(m),i}|\xi_i\rangle = \xi_i|\xi_i - 1\rangle \\ \\ \mathcal{K}^{o}_{(m),i}|\xi_i\rangle = (\xi_i + m)\,|\xi_i\rangle \end{cases} \tag{3.10}$$

The BEP(m) generator is obtained when writing the abstract operator (3.8) in the representation with second order differential operators; the SIP(m) generator is obtained when writing the abstract operator (3.8) in the representation with infinite dimensional matrices. The duality functions (3.7) are found by imposing on each site that the action of the two representations on $D(z_i, \xi_i)$ is the same.

□

4 Heat conduction

As an application to the heat conduction problem, we consider the case with $m = 1$ on a chain with N sites, and couple the BMP(1) process to two heat reservoirs (modeled by Ornstein-Uhlenbeck process) at different temperatures. Namely we consider the generator

$$L^{res} = L_1 + L_N + \sum_{i=1}^{N-1} L^{BMP(1)}_{i,i+1} \tag{4.11}$$

with

$$L_1 f = T_L \frac{\partial^2 f}{\partial x_1^2} - x_1 \frac{\partial f}{\partial x_1}$$
$$L_N f = T_R \frac{\partial^2 f}{\partial x_N^2} - x_N \frac{\partial f}{\partial x_N}$$

$$L_{i,i+1}^{BMP(1)} f = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 (f) \, .$$

The dual process will be a SIP(1) process with absorbing boundaries. Considering the two additional sites 0 and $N+1$ and configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1})$ the dual evolution will be given by the generator

$$L^{abs} = L_1^{abs} + L_N^{abs} + \sum_{i=1}^{N-1} L_{i,i+1}^{SIP(1)} \tag{4.12}$$

with

$$(L_1^{abs} f)(\bar{\xi}) = 2\xi_1 (f(\bar{\xi}^{1,0}) - f(\bar{\xi}))$$

$$(L_N^{abs} f)(\bar{\xi}) = 2\xi_N (f(\bar{\xi}^{N,N+1}) - f(\bar{\xi}))$$

$$(L_{i,i+1}^{SIP(1)} f)(\bar{\xi}) = \xi_i \left(\xi_{i+1} + \frac{1}{2} \right) [f(\bar{\xi}^{i,i+1}) - f(\bar{\xi})] + \xi_{i+1} \left(\xi_i + \frac{1}{2} \right) [f(\bar{\xi}^{i+1,i}) - f(\bar{\xi})]$$

Then one has the following

Proposition 4.1 *The process $\{x(t)\}_{t\geq 0}$ with generator L^{res} is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator L^{abs} on*

$$D(x, \bar{\xi}) = T_1^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_N^{\xi_{N+1}} \tag{4.13}$$

Proof: It follows from Theorem 3.1 and the identity

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \, .$$

The boundaries terms are checked with an explicit computation. □

Proposition 4.2 *Let $\mathbb{E}(\cdot)$ be the expectation in the stationary states of the non-equilibrium statistical mechanics process with generator L^{res} and let $\tilde{\xi} = (0, \xi_1, \ldots, \xi_N, 0)$. Then the following holds*

$$\mathbb{E}(D(x, \tilde{\xi})) = \sum_{a,b:\, a+b=|\xi|} T_1^a T_N^b \;\; \mathbb{P}_{\tilde{\xi}}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b) \tag{4.14}$$

where $|\xi| = \sum_{i=1}^N \xi_i$ denotes the total number of SIP dual walkers and $\mathbb{P}_{\tilde{\xi}}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b)$ denotes the probability that, starting from the configuration $\tilde{\xi}$, the number of dual walkers eventually absorbed in 0 is a and the number of those absorbed in $N+1$ is b.

Proof: It is a consequence of the fact that, asymptotically, the dual process voids the chain:

$$\begin{aligned}
\mathbb{E}(D(x,\tilde{\xi})) &= \lim_{t\to\infty}\int \mathbb{E}_x(D(x_t,\tilde{\xi}))\, d\mu(x) \\
&= \int \lim_{t\to\infty} \mathbb{E}_{\tilde{\xi}}(D(x,\tilde{\xi}_t))\, d\mu(x) \\
&= \int \sum_{a,b:a+b=|\xi|} T_1^a T_N^b \ \mathbb{P}_{\tilde{\xi}}(\xi_0(\infty)=a, \xi_{N+1}(\infty)=b)\, d\mu(x)\, .
\end{aligned}$$

□

Remark 4.1 *If one start from a chain with $m = 2$ and consider an instantaneous thermalization limit of the $BEP(2)$ model, then the KMP model [15] is recovered (see [11] for details).*

5 Open problems

A theory of transport in non-equilibrium models is under constrcution [1, 7, 21]. The problem of heat conduction for Hamiltonian systems is still under hot debate [2, 3, 5, 9, 16, 22]. Stochastic models are useful because they allow sometimes exact solutions [6, 7, 15]. A class of interacting diffusions - the BEP(m) model - have been introduced in [10] and further studied in [4, 11, 14, 19]. The dual stochastic process - the SIP(m) model [12] - resembles in the form of its rates the well-know exclusion process [8, 17, 20] but it has attractive interactions [13], rather than repulsive. In [11] the origin of duality has been traced back to the underlying geometrical structure of a group (see also [18]). Among the open problem we mention: i) the construction of duality between an asymmetric version of inclusion process $ASIP_q(m)$ and brownian energy process $ABEP_q(m)$. Following the scheme developed in [11] this should involve the deformed quantum group $SU_q(1,1)$; ii) a full ergodic theory of SIP/BEP model; iii) the explicit solution for the stationary measure for models with reservoirs, akin to the exclusion process.

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