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Linear Stochastic Evolutions[1]

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Abstract

We consider a discrete-time stochastic growth model on the d-dimensional lattice with non-negative real numbers as possible values per site. The growth model describes various interesting examples such as oriented site/bond percolation, directed polymers in random environment, time discretizations of the binary contact path process. We review some results on this model mainly from [11, 23, 29, 30].

Contents

1 Introduction

We write $\mathbb{N}=\{0,1,2,...\}$, $\mathbb{N}^*=\{1,2,...\}$ and $\mathbb{Z}=\{\pm x\ ;\ x\in\mathbb{N}\}$. For $x=(x_1,..,x_d)\in\mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x|=\sum_{i=1}^d|x_i|$. For $\xi=(\xi_x)_{x\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}^d}$, $|\xi|=\sum_{x\in\mathbb{Z}^d}|\xi_x|$. Let $(\Omega,\mathcal{F},P)$ be a probability space. We write $P[X]=\int X\,dP$ and $P[X:A]=\int_A X\,dP$ for a random variable X and an event A. For events $A,B\subset\Omega$, $A\subset B$ a.s. means that $P(A\backslash B)=0$. Similarly, $A=B$ a.s. means that $P(A\backslash B)=P(B\backslash A)=0$.

1.1 The oriented site percolation (OSP)

We start by discussing the *oriented site percolation* as a motivating example. Let $\eta_{t,y}$, $(t,y)\in\mathbb{N}^*\times\mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y}=1)=p\in(0,1)$. The site (t,y) with $\eta_{t,y}=1$ and $\eta_{t,y}=0$ are referred to respectively as *open* and *closed*. An *open oriented path* from $(0,0)$ to $(t,y)\in\mathbb{N}^*\times\mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N}\times\mathbb{Z}^d$ such that $x_0=0$, $x_t=y$, $|x_s-x_{s-1}|=1$, $\eta_{s,x_s}=1$ for all $s=1,..,t$. For oriented percolation, it is traditional to discuss the presence/absence of the open oriented paths to certain time-space location. On the other hand, the model exhibits another type of phase transition, if we look at not only the presence/absence of the open oriented paths, but also their number. Let $N_{t,y}$ be the number of open oriented paths from $(0,0)$ to (t,y) and let $|N_t|=\sum_{y\in\mathbb{Z}^d}N_{t,y}$ be the total number of open oriented paths from $(0,0)$ to the "level" t. If we regard each open oriented path $\{(s,x_s)\}_{s=0}^t$ as a trajectory of a particle, then $N_{t,y}$ is the number of the particles which occupy the site y at time t.

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We now note that $|\overline{N}_t| \stackrel{\text{def.}}{=} (2dp)^{-t}|N_t|$ is a martingale, since each open oriented path from $(0,0)$ to (t,y) branches and survives to the next level via $2d$ neighbors of y, each of which is open with probability p. Thus, by the martingale convergence theorem, the following limit exists a.s.:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t\to\infty} |\overline{N}_t|.$$

Moreover,

i) If $d \geq 3$ and p is large enough, then, $P(|\overline{N}_\infty| > 0) > 0$, which means that, at least with positive probability, the total number of paths $|N_t|$ is of the same order as its expectation $(2pd)^t$ as $t \to \infty$.

ii) If $d = 1, 2$, then for all $p \in (0,1)$, $P(|\overline{N}_\infty| = 0) = 1$, which means that the total number of paths $|N_t|$ is of smaller order than its expectation $(2pd)^t$ a.s. as $t \to \infty$. Moreover, there is a non-random constant $c > 0$ such that $|\overline{N}_t| = \mathcal{O}(\exp(-ct))$ a.s. as $t \to \infty$.

This phase transition was predicted by T. Shiga in late 1990's and the proof was given recently in [2, 29].

We denote the density of the population by:

$$\rho_t(x) = \frac{N_{t,x}}{|N_t|}\mathbf{1}_{\{|N_t|>0\}}, \quad t \in \mathbb{N}, x \in \mathbb{Z}^d. \tag{1.1}$$

Here and in what follows, we adopt the following convention. For a random variable X defined on an event A, we define the random variable $X\mathbf{1}_A$ by $X\mathbf{1}_A = X$ on A and $X\mathbf{1}_A = 0$ outside A. Interesting objects related to the density would be

$$\rho_t^* = \max_{x\in\mathbb{Z}^d} \rho_t(x), \quad \text{and } \mathcal{R}_t = |\rho_t^2| = \sum_{x\in\mathbb{Z}^d} \rho_t(x)^2. \tag{1.2}$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that two particles picked up randomly from the total population at time t are at the same site. We call $\mathcal{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathcal{R}_t$ indicate that most of the particles are concentrated on small numbers of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over large number of sites (*delocalization*).

As applications of results in this paper, we get the following result. It says that, in the presence of an infinite open path, the slow growth $|\overline{N}_\infty| = 0$ is equivalent to a localization property $\overline{\lim}_{t\to\infty} \mathcal{R}_t \geq c > 0$. Here, and in what follows, a *constant* always means a *non-random constant.*

Theorem 1.1.1 a) *If $P(|\overline{N}_\infty| > 0) > 0$, then, $\sum_{t\geq 1} \mathcal{R}_t < \infty$ a.s.*

b) *If $P(|\overline{N}_\infty| = 0) = 1$, then, there exists a constant $c > 0$ such that:*

$$\{|N_t| > 0 \textit{ for all } t \in \mathbb{N}\} = \left\{\overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c\right\} \quad a.s. \tag{1.3}$$

Note that $P(|\overline{N}_\infty| = 0) = 1$ for all $p \in (0,1)$ if $d \leq 2$. Thus, (1.3) in particular means that, if $d \leq 2$, the path localization $\overline{\lim}_{t\to\infty} \mathcal{R}_t \geq c$ occurs a.s. on the event of percolation.

1.2 The linear stochastic evolution

We now introduce the framework of this article. Let $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t \in \mathbb{N}^*$ be a sequence of random matrices on a probability space $(\Omega, \mathcal{F}, P)$ such that:

$$A_1, A_2, \dots \text{ are i.i.d.} \tag{1.4}$$

Here are the set of assumptions we assume for A_1:

$$A_1 \text{ is not a constant matrix.} \tag{1.5}$$

$$A_{1,x,y} \geq 0 \text{ for all } x, y \in \mathbb{Z}^d. \tag{1.6}$$

$$\text{The columns } \{A_{1,\cdot,y}\}_{y\in\mathbb{Z}^d} \text{ are independent.} \tag{1.7}$$

$$P[A_{1,x,y}^3] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d, \tag{1.8}$$

$$A_{1,x,y} = 0 \text{ a.s. if } |x-y| > r_A \text{ for some non-random } r_A \in \mathbb{N}. \tag{1.9}$$

$$(A_{1,x+z,y+z})_{x,y\in\mathbb{Z}^d} \stackrel{\text{law}}{=} A_1 \text{ for all } z \in \mathbb{Z}^d. \tag{1.10}$$

$$\text{The set } \{x \in \mathbb{Z}^d \ ; \ \textstyle\sum_{y\in\mathbb{Z}^d} a_{x+y}a_y \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \text{ where } a_y = P[A_{1,0,y}]. \tag{1.11}$$

Depending on the results we prove in the sequel, some of these conditions can be relaxed. However, we choose not to bother ourselves with the pursuit of the minimum assumptions for each result.

We define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $[0,\infty)^{\mathbb{Z}^d}$ by:

$$\sum_{x\in\mathbb{Z}^d} N_{t-1,x}A_{t,x,y} = N_{t,y}, \quad t \in \mathbb{N}^*. \tag{1.12}$$

In this article, we suppose that the initial state N_0 is given by "a single particle at the origin":

$$N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d} \tag{1.13}$$

Here and in what follows, $\delta_{x,y} = \mathbf{1}_{\{x=y\}}$ for $x, y \in \mathbb{Z}^d$. If we regard $N_t \in [0,\infty)^{\mathbb{Z}^d}$ as a row vector, (1.12) can be interpreted as:

$$N_t = N_0 A_1 A_2 \cdots A_t, \quad t = 1, 2, \dots$$

The Markov chain defined above can be thought of as the time discretization of the linear particle system considered in the last Chapter in T. Liggett's book [17, Chapter IX]. Thanks to the time discretization, the definition is considerably simpler here. Though we *do not* assume in general that $(N_t)_{t\in\mathbb{N}}$ takes values in $\mathbb{N}^{\mathbb{Z}^d}$, we refer $N_{t,y}$ as the "number of particles" at time-space (t,y), and $|N_t|$ as the "total number of particles" at time t.

We now see that various interesting examples are included in this framework. We recall the notation a_y from (1.11).

• **Generalized oriented site percolation (GOSP):** We generalize OSP as follows. Let $\eta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in [0,1]$ and let $\zeta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. To exclude trivialities, we assume that either p or q is in $(0,1)$. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.12) with:

$$A_{t,x,y} = \mathbf{1}_{|x-y|=1}\eta_{t,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented site percolation* (GOSP). Thus, the OSP is the special case ($q = 0$) of GOSP. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} q & \text{if } x = \widetilde{x} = y, \\ p & \text{if } |x-y| = |\widetilde{x}-y| = 1, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.14}$$

In particular, we have $|a| = 2dp + q$ (Recall that $|a| = \sum_y a_y$).

• **Generalized oriented bond percolation (GOBP):** Let $\eta_{t,x,y}$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d.random variables with $P(\eta_{t,x,y} = 1) = p \in [0,1]$ and let $\zeta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.12) with:

$$A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}}\eta_{t,x,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented bond percolation* (GOBP). We call the special case $q = 0$ *oriented bond percolation* (OBP). To interpret the definition, let us call the pair of time-space points $\langle (t-1,x),(t,y) \rangle$ a *bond* if $|x-y| \le 1$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$. A bond $\langle (t-1,x),(t,y) \rangle$ with $|x-y| = 1$ is said to be *open* if $\eta_{t,x,y} = 1$, and a bond $\langle (t-1,y),(t,y) \rangle$ is said to be *open* if $\zeta_{t,y} = 1$. For GOBP, an *open oriented path* from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$ and bonds $\langle (s-1,x_{s-1}),(s,x_s) \rangle$ are open for all $s = 1,..,t$. If $N_0 = (\delta_{0,y})_{y\in\mathbb{Z}^d}$, then, the number of open oriented paths from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is given by $N_{t,y}$.

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.15}$$

In particular, we have $|a| = 2dp + q$.

• **Directed polymers in random environment (DPRE):** Let $\{\eta_{t,y}\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be i.i.d. with $\exp(\lambda(\beta)) \stackrel{\text{def.}}{=} P[\exp(\beta\eta_{t,y})] < \infty$ for any $\beta \in (0,\infty)$. The following expectation is called the partition function of the *directed polymers in random environment*:

$$N_{t,y} = P_S^0\left[\exp\left(\beta\sum_{u=1}^t \eta_{u,S_u}\right) : S_t = y\right], \quad (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d,$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the simple random walk on $\mathbb{Z}^d$. We refer the reader to a review paper [7] and the references therein for more information. Starting from $N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.12) with:

$$A_{t,x,y} = \frac{\mathbf{1}_{|x-y|=1}}{2d}\exp(\beta\eta_{t,y}).$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{e^{\lambda(\beta)}\mathbf{1}_{\{|y|=1\}}}{2d}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = e^{\lambda(2\beta)-2\lambda(\beta)}a_{y-x}a_{y-\widetilde{x}} \tag{1.16}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

• **The binary contact path process (BCPP):** The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [12]. In this article, we consider a discrete-time variant as follows. Let

$$\{\eta_{t,y} = 0, 1 \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0, 1 \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y} \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be families of i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0,1]$, $P(\zeta_{t,y} = 1) = q \in [0,1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other. Starting from an $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t \in \mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by:

$$N_{t+1,y} = \eta_{t+1,y} N_{t,y-e_{t+1,y}} + \zeta_{t+1,y} N_{t,y}, \quad t \in \mathbb{N}.$$

We interpret the process as the spread of an infection, with $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y} N_{t,y}$ term above means that these individuals remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y} N_{t,y-e_{t+1,y}}$ term means that, with probability p, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t+1$. This Markov chain is obtained by (1.12) with:

$$A_{t,x,y} = \eta_{t,y} \mathbf{1}_{e_{t,y} = y-x} + \zeta_{t,y} \delta_{x,y}.$$

The covariances of $(A_{t,x,y})_{x,y \in \mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{p \mathbf{1}_{\{|y|=1\}}}{2d} + q \delta_{0,y}, \quad P[A_{t,x,y} A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ \delta_{x,y} q a_{y-\widetilde{x}} + \delta_{\widetilde{x},y} q a_{y-x} & \text{if } x \neq \widetilde{x}. \end{cases} \tag{1.17}$$

In particular, we have $|a| = p + q$.

Remark: The branching random walk in random environment considered in [9, 14, 15, 16, 24, 25, 26, 28] can also be considered as a "close relative" to the models considered here, although it does not exactly fall into our framework.

2 Basic Results

2.1 The regular and slow growth phases

We now recall the following facts and notion from [29, Lemmas 1.3.1 and 1.3.2]. Let $\mathcal{F}_t$ be the σ-field generated by $A_1, .., A_t$.

Lemma 2.1.1 *Define* $\overline{N}_t = \left(\overline{N}_{t,x}\right)_{x \in \mathbb{Z}^d}$ *by:*

$$\overline{N}_{t,x} = |a|^{-t} N_{t,x}. \tag{2.1}$$

a) $(|\overline{N}_t|, \mathcal{F}_t)_{t \in \mathbb{N}}$ *is a martingale, and therefore, the following limit exists a.s.*

$$|\overline{N}_\infty| = \lim_{t \to \infty} |\overline{N}_t|. \tag{2.2}$$

b) *Either*

$$P[|\overline{N}_\infty|] = 1 \;\; or \;\; 0. \tag{2.3}$$

Moreover, $P[|\overline{N}_\infty|] = 1$ *if and only if the limit (2.2) is convergent in* $\mathbb{L}^1(P)$.

We will refer to the former case of (2.3) as *regular growth phase* and the latter as *slow growth phase.*

The regular growth means that, at least with positive probability, the growth of the "total number" $|N_t|$ of particles is of the same order as its expectation $|a|^t|N_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|N_t|$ is slower than its expectation.

We now recall from [2, 29] the following criteria for regular/slow growth phases.

Proposition 2.1.2 a) *$P[|\overline{N}_\infty|] = 1$ if $d \geq 3$ and*

$$\sup_{t\geq 0} P[|\overline{N}_t|^2] < \infty. \tag{2.4}$$

b) *Supppose that $d = 1, 2$, or*

$$\sum_{y\in\mathbb{Z}^d} P\left[A_{1,0,y}\ln A_{1,0,y}\right] > |a|\ln|a|. \tag{2.5}$$

Then, $P[|\overline{N}_\infty|] = 0$. More precisely, there exists $c > 0$ such that

$$|\overline{N}_t| = \mathcal{O}(e^{-ct}), \quad a.s. \text{ as } t \to \infty. \tag{2.6}$$

For $d \geq 3$, the following is known [23, 29], where π_0 is the return probability of the simple random walk on $\mathbb{Z}^d$.

$$(2.4)\left\{\begin{array}{lll} \Longleftrightarrow & p > \pi_0 & \text{for OSP,} \\ \Longleftarrow & p\wedge q > \pi_0 & \text{for GOSP with } q \neq 0, \\ \Longleftrightarrow & \frac{2dp(1-p)+q(1-q)}{(2dp+q)^2} < 1-\pi_0 & \text{for GOBP,} \\ \Longleftrightarrow & \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE,} \\ \Longleftarrow & p\wedge q \text{ is large enough} & \text{for BCPP.} \end{array}\right. \tag{2.7}$$

The condition (2.5) roughly says that the matrix A_1 is "random enough". It is easy to see that

$$(2.5) \Longleftrightarrow \left\{\begin{array}{ll} 2dp+q<1 & \text{for GOSP and GOBP,} \\ \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) & \text{for DPRE,} \\ p+q<1 & \text{for BCPP.} \end{array}\right. \tag{2.8}$$

2.2 The localization and delocalization

We introduce the following additional condition, which says that the entries of the matrix A_1 are positively correlated in the following weak sense: there is a constant $\gamma \in (1,\infty)$ such that:

$$\sum_{x,\widetilde{x},y\in\mathbb{Z}^d} \left(P[A_{1,x,y}A_{1,\widetilde{x},y}] - \gamma a_{y-x}a_{y-\widetilde{x}}\right)\xi_x\xi_{\widetilde{x}} \geq 0 \tag{2.9}$$

for all $\xi \in [0,\infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$.

Remark: Clearly, (2.9) is satisfied if there is a constant $\gamma \in (1,\infty)$ such that:

$$P[A_{1,x,y}, A_{1,\widetilde{x},y}] \geq \gamma a_{y-x}a_{y-\widetilde{x}} \quad \text{for all } x,\widetilde{x},y \in \mathbb{Z}^d. \tag{2.10}$$

For OSP and DPRE, we see from (1.14) and (1.16) that (2.10) holds with:

$$\gamma = 1/p \text{ and } \exp(\lambda(2\beta) - 2\lambda(\beta))$$

respectively for OSP and DPRE. For GOSP, GOBP and BCPP, (2.10) is no longer true. However, one can check (2.9) for them with:

$$\gamma = 1 + \begin{cases} \frac{2dp(1-p)+q(1-q)}{(2dp+q)^2} & \text{for GOSP and GOBP,} \\ \frac{p(1-p)+q(1-q)}{(p+q)^2} & \text{for BCPP} \end{cases}$$

[29, Remarks after Theorem 3.2.1].

We define the density $\rho_t(x)$ and the replica overlap $\mathcal{R}_t$ in the same way as (1.1) and (1.2).

We first show that, on the event of survival, the slow growth is equivalent to the localization:

Theorem 2.2.1 *Suppose (2.9).*

a) *If* $P(|\overline{N}_\infty| > 0) > 0$, *then* $\sum_{t \geq 0} \mathcal{R}_t < \infty$ *a.s.*

b) *If* $P(|\overline{N}_\infty| = 0) = 1$, *then*

$$\{\text{survival}\} = \left\{ \sum_{t \geq 0} \mathcal{R}_t = \infty \right\} \quad a.s. \tag{2.11}$$

where $\{\text{survival}\} \stackrel{\text{def}}{=} \{|N_t| > 0 \text{ for all } t \in \mathbb{N}\}$. *Moreover, there exists a constant* $c > 0$ *such that almost surely,*

$$|\overline{N}_t| \leq \exp\left(-c \sum_{1 \leq s \leq t-1} \mathcal{R}_s \right) \quad \text{for all large enough } t\text{'s} \tag{2.12}$$

Remark: As can be seen from the proof, (2.11) is true even without assuming (2.9) and with (1.8) replaced by a weaker assumption:

$$P[A_{1,x,y}^2] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{2.13}$$

Theorem 2.2.1 says that, conditionally on survival, the slow growth $|\overline{N}_\infty| = 0$ is equivalent to the localization $\sum_{t \geq 0} \mathcal{R}_t = \infty$. We emphasize that this is the first case in which a result of this type is obtained for models with positive probability of extinction at finite time (i.e.,$P(|N_t| = 0) > 0$ for finite t). Similar results have been known before only in the case where no extinction at finite time is allowed, i.e., $|N_t| > 0$ for all $t \geq 0$, e.g., [5, Theorem 1.1], [6, Theorem 1.1], [8, Theorem 2.3.2], [16, Theorem 1.3.1]. The argument in the previous literature is roughly to show that

$$-\ln |\overline{N}_t| \asymp \sum_{u=0}^{t-1} \mathcal{R}_u \quad \text{a.s. as } t \to \infty \tag{2.14}$$

by using Doob's decomposition of the supermartingale $\ln |\overline{N}_t|$ ("$\asymp$" above means the asymptotic upper and lower bounds with positive multiplicative constants). This argument does not seem to be directly transportable to the case where the total population may get extinct at finite time, since $\ln |\overline{N}_t|$ is not even defined. To cope with this problem, we first characterize, in a general setting, the event on which an exponential martingale vanishes in the limit [30, Proposition 2.1.2]. We then apply this characterization to the martingale $|\overline{N}_t|$. See also [21] for the application of this idea to the continuous-time setting.

Next, we present a result which says that, under a mild assumption, we can replace

$$\sum_{t\geq 0} \mathcal{R}_t = \infty$$

in (2.11) by a stronger localization property:

$$\overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c,$$

where $c > 0$ is a constant. To state the theorem, we introduce some notation related to the random walk associated to our model. Let $((S_t)_{t\in\mathbb{N}}, P_S^x)$ be the random walk on $\mathbb{Z}^d$ such that:

$$P_S^x(S_0 = x) = 1 \text{ and } P_S^x(S_1 = y) = a_{y-x}/|a| \tag{2.15}$$

and let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be its independent copy. We then define:

$$\pi_d = P_S^0 \otimes P_{\widetilde{S}}^0(S_t = \widetilde{S}_t \text{ for some } t \geq 1). \tag{2.16}$$

Then, by (1.11),

$$\pi_d = 1 \text{ for } d = 1, 2 \text{ and } \pi_d < 1 \text{ for } d \geq 3 \tag{2.17}$$

Theorem 2.2.2 *Suppose (2.9) and either of*

a) $d = 1, 2,$

b) $P(|\overline{N}_\infty| = 0) = 1$ *and*

$$\gamma > \frac{1}{\pi_d}, \tag{2.18}$$

where γ and π_d are from (2.9) and (2.16).

Then, there exists a constant $c > 0$ such that:

$$\{\text{survival}\} = \left\{ \overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c \right\} \quad a.s. \tag{2.19}$$

This result generalizes [5, Theorem 1.2] and [6, Proposition 1.4 b)], which are obtained in the context of DPRE. The result can be carried over to the continuous-time model [21] and for branching random walks in random environment [14, 16]. To prove Theorem 2.2.2, we will use the argument which was initially applied to DPRE by P. Carmona and Y. Hu in [5] (See also [16]).

Remarks 1) We prove (2.19) by way of the following stronger estimate:

$$\underline{\lim_{t\nearrow\infty}} \frac{\sum_{s=0}^t \mathcal{R}_s^{3/2}}{\sum_{s=0}^t \mathcal{R}_s} \geq c_1, \quad a.s.$$

for some constant $c_1 > 0$. This in particular implies the following quantitative lower bound on the number of times at which the replica overlap is larger than a certain positive number:

$$\underline{\lim_{t\nearrow\infty}} \frac{\sum_{s=0}^t 1_{\{\mathcal{R}_s \geq c_2\}}}{\sum_{s=0}^t \mathcal{R}_s} \geq c_3, \quad a.s.$$

where c_2 and c_3 are positive constants (The inequality $r^{3/2} \leq \mathbf{1}\{r \geq c\} + \sqrt{c}r$ for $r, c \in [0, 1]$ can be used here).

2) (2.19) is in contrast with the following delocalization result by M. Nakashima [23]: if $d \geq 3$ and $\sup_{t\geq 0} P[|\overline{N}_t|^2] < \infty$, then,

$$\mathcal{R}_t = \mathcal{O}(t^{-d/2}) \text{ in probability }.$$

See also [20, 22] for the continuous-time case and [25, 28] for the case of branching random walk in random environment.

Finally, we state the following variant of Theorem 2.2.2, which says that even for $d \geq 3$, (2.18) can be dropped at the cost of some alternative assumptions. Following M. Birkner [3, page 81, (5.1)], we introduce the following condition:

$$\sup_{t\in\mathbb{N}, x\in\mathbb{Z}^d} \frac{P_S^0(S_t = x)}{P_S^0 \otimes P_{\widetilde{S}}^0(S_t = \widetilde{S}_t)} < \infty, \tag{2.20}$$

which is obviously true for the symmetric simple random walk on $\mathbb{Z}^d$.

Theorem 2.2.3 *Suppose $d \geq 3$, (2.9), (2.20) and that there exist mean-one i.i.d. random variables $\overline{\eta}_{t,y}$, $(t, y) \in \mathbb{N} \times \mathbb{Z}^d$ such that:*

$$A_{t,x,y} = \overline{\eta}_{t,y} a_{y-x}. \tag{2.21}$$

Then, the slow growth ($P(|N_\infty| = 0) = 1$) implies that there exists a constant $c > 0$ such that (2.19) holds.

Note that OSP and DPRE for $d \geq 3$ satisfy all the assumptions for Theorem 2.2.3. The proof of Theorem 2.2.3 is based on Theorem 2.2.2 and a criterion for the regular growth phase, which is essentially due to M. Birkner [4].

Proof of Theorem 1.1.1: The theorem follows from Theorem 2.2.1 and Theorem 2.2.3. □.

2.3 The central limit theorem

Suppose $d \geq 3$ and (2.4). Then, by Proposition 2.1.2, we are in the regular growth phase, which implies the delocalization via Theorem 2.2.1a. Further information on the large time behavior of the density $\rho_{t,x}$ in this regime is provided by the following central limit theorem.

Theorem 2.3.1 *[23] Suppose $d \geq 3$ and (2.4). Then, for all $f \in C_b(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left(\frac{x - mt}{\sqrt{t}}\right) \rho_{t,x} = \int_{\mathbb{R}^d} f d\nu, \quad a.s. \text{ on } \{\text{survival}\}. \tag{2.22}$$

where $m = (m_j)_{j=1}^d = \frac{1}{|a|}\sum_{x\in\mathbb{Z}^d} x a_x$, and ν is the centered Gaussian measure with

$$\int_{\mathbb{R}^d} x_i x_j d\nu(x) = \frac{1}{|a|}\sum_{x\in\mathbb{Z}^d}(x_i - m_i)(x_j - m_j)a_x, \quad i, j = 1, ..., d.$$

2.4 Dichotomy: exponential growth or extinction

So far, we have discussed the regular and slow growth phases of the linear stochastic evolutions and their correspondence to delocalization and localization. However, the following fundamental question remains: does the total population grow exponentially whenever it survives? As is well-known, the answer is affirmative for the classical Galton-Watson process, e.g. , [1, p.30, Theorem 20]. The following result confirms the dichotomy is true for the linear stochastic evolution.

Theorem 2.4.1 *[11] Suppose that $A_{x,y} \in \{0\} \cup [1, \infty)$ and that A is not deterministic. Then, there exists $c > 0$ such that*

$$\{\text{survival}\} \stackrel{\text{a.s.}}{=} \left\{ |N_t| \geq e^{ct} \text{ for large enough } t \right\}.$$

It is in fact shown in [11] that, on the event of survival, there exists an "open path" (suitably defined) oriented in time, along which the mass growth exponentially fast.

2.5 Continuous-time model

We go directly into the formal definition of the model, referring the reader to [20, 21] for relevant backgrounds. The class of growth models considered here is a reasonably ample subclass of the one considered in [17, Chapter IX] as "linear systems". We introduce a random vector $K = (K_x)_{x \in \mathbb{Z}^d}$ such that

$$0 \leq K_x \leq b_K \mathbf{1}_{\{|x| \leq r_K\}} \text{ a.s. for some constants } b_K, r_K \in [0, \infty), \tag{2.23}$$

$$\text{the set } \{x \in \mathbb{Z}^d \,;\, P[K_x] \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d. \tag{2.24}$$

The first condition (2.23) amounts to the standard boundedness and the finite range assumptions for the transition rate of interacting particle systems. The second condition (2.24) makes the model "truly d-dimensional".

Let $\tau^{z,i}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + \ldots + \tau^{z,i}$. Let also $K^{z,i} = (K_x^{z,i})_{x \in \mathbb{Z}^d}$ ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. random vectors with the same distributions as K, independent of $\{\tau^{z,i}\}_{z \in \mathbb{Z}^d, i \in \mathbb{N}^*}$. We suppose that the process (η_t) starts from a deterministic configuration $\eta_0 = (\eta_{0,x})_{x \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ with $|\eta_0| < \infty$. At time $t = T^{z,i}$, η_{t-} is replaced by η_t, where

$$\eta_{t,x} = \begin{cases} K_0^{z,i} \eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K_{x-z}^{z,i} \eta_{t-,z} & \text{if } x \neq z. \end{cases} \tag{2.25}$$

We also consider the *dual process* $\zeta_t \in [0, \infty)^{\mathbb{Z}^d}$, $t \geq 0$ which evolves in the same way as $(\eta_t)_{t \geq 0}$ except that (2.25) is replaced by its transpose:

$$\zeta_{t,x} = \begin{cases} \sum_{y \in \mathbb{Z}^d} K_{y-x}^{z,i} \zeta_{t-,y} & \text{if } x = z, \\ \zeta_{t-,x} & \text{if } x \neq z. \end{cases} \tag{2.26}$$

Here are some typical examples which fall into the above set-up:

• **The binary contact path process (BCPP):** The binary contact path process (BCPP), originally introduced by D. Griffeath [12] is a special case the model, where

$$K = \begin{cases} (\delta_{x,0} + \delta_{x,e})_{x \in \mathbb{Z}^d} & \text{with probability } \frac{\lambda}{2d\lambda+1}, \text{ for each } 2d \text{ neighbor } e \text{ of } 0 \\ 0 & \text{with probability } \frac{1}{2d\lambda+1}. \end{cases} \tag{2.27}$$

The process is interpreted as the spread of an infection, with $\eta_{t,x}$ infected individuals at time t at the site x. The first line of (2.27) says that, with probability $\frac{\lambda}{2d\lambda+1}$ for each $|e| = 1$, all the infected individuals at site $x - e$ are duplicated and added to those on the site x. On the other hand, the second line of (2.27) says that, all the infected individuals at a site become healthy with probability $\frac{1}{2d\lambda+1}$. A motivation to study the BCPP comes from the fact that the projected process $(\eta_{t,x} \wedge 1)_{x \in \mathbb{Z}^d}$, $t \geq 0$ is the basic contact process [12].

• **The potlatch/smoothing processes:** The potlatch process discussed in e.g. [13] and [17, Chapter IX] is also a special case of the above set-up, in which

$$K_x = Wk_x, \quad x \in \mathbb{Z}^d. \tag{2.28}$$

Here, $k = (k_x)_{x \in \mathbb{Z}^d} \in [0, \infty)^{\mathbb{Z}^d}$ is a non-random vector and W is a non-negative, bounded, mean-one random variable such that $P(W = 1) < 1$. The smoothing process is the dual process of the potlatch process. The potlatch/smoothing processes were first introduced in [27] for the case $W \equiv 1$ and discussed further in [18]. It was in [13] where case with $W \not\equiv 1$ was introduced and discussed. Note that we *do not* assume that k_x is a transition probability of an irreducible random walk, unlike in the literatures mentioned above.

Results for the discrete-time case (Theorem 2.2.1,Theorem 2.2.2,Theorem 2.3.1,Theorem 2.4.1) can be carried over to the continuous-time setting explained above. For the detail, we refer the readers to [19, 20, 21, 22].

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