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Transformations of L-values

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Abstract

In our recent work with M. Rogers on resolving some Boyd's conjectures on two-variate Mahler measures, a new analytical machinery was introduced to write the values $L(E,2)$ of L-series of elliptic curves as periods in the sense of Kontsevich and Zagier. Here we outline, in slightly more general settings, the novelty of our method with Rogers, and provide a simple illustrative example.

Throughout the note we keep the notation $q=e^{2\pi i\tau}$ for τ from the upper half-plane $\operatorname{Re}\tau>0$, so that $|q|<1$. Our basic constructor of modular forms and functions is Dedekind's eta-function

$$\eta(\tau):=q^{1/24}\prod_{m=1}^{\infty}(1-q^{m})=\sum_{n=-\infty}^{\infty}(-1)^{n}q^{(6n+1)^{2}/24}$$

with is modular involution

$$\eta(-1/\tau)=\sqrt{-i\tau}\eta(\tau). \tag{1}$$

We also set $\eta_{k}:=\eta(k\tau)$ for short.

We first describe a part of the general machinery from our joint works [6, 7] with M. Rogers on an example of computing the value $L(E_{32},2)$ of the L-series associated with a conductor 32 elliptic curve. It is known [3] that the corresponding cusp form in this case is $f_{32}(\tau):=\eta_{4}^{2}\eta_{8}^{2}$, so that $L(E_{32},s)=L(f_{32},s)$. We choose the conductor 32 case here because it is not discussed in [6, 7].

Note the (Lambert series) expansion

$$\frac{\eta_{8}^{4}}{\eta_{4}^{2}}=\sum_{m\geq 1}\left(\frac{-4}{m}\right)\frac{q^{m}}{1-q^{2m}}=\sum_{\substack{m,n\geq 1\\ n\text{ odd}}}\left(\frac{-4}{m}\right)q^{mn}, \tag{2}$$

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where $\left(\frac{-4}{m}\right)$ is the quadratic residue character modulo 4. In notation $\delta_{2|n} = 1$ if $2 \mid n$ and 0 if n is odd, we can write (2) as

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m,n\ge1} a(m)b(n)q^{mn}, \qquad \text{where} \quad a(m) := \left(\frac{-4}{m}\right), \quad b(n) := 1 - \delta_{2|n}.$$

Then

$$\begin{aligned} f_{32}(it) &= \frac{\eta_8^4}{\eta_4^2}\frac{\eta_4^4}{\eta_8^2}\bigg|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=it} \cdot \frac{1}{2t}\frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=i/(32t)} \\ &= \frac{1}{2t}\sum_{m_1,n_1\ge1} a(m_1)b(n_1)e^{-2\pi m_1 n_1 t} \sum_{m_2,n_2\ge1} b(m_2)a(n_2)e^{-2\pi m_2 n_2/(32t)}, \end{aligned}$$

where $t > 0$ and the modular involution (1) was used.

Now,

$$\begin{aligned} L(E_{32},2) = L(f_{32},2) &= \int_0^1 f_{32}\log q\,\frac{dq}{q} = -4\pi^2\int_0^\infty f_{32}(it)t\,dt \\ &= -2\pi^2\int_0^\infty \sum_{m_1,n_1,m_2,n_2\ge1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\qquad \times \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)dt \\ &= -2\pi^2 \sum_{m_1,n_1,m_2,n_2\ge1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\qquad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)dt. \end{aligned}$$

Here comes the crucial transformation of purely analytical origin: we make the change of variable $t = n_2u/n_1$. It does not change the form of the integrand but affects the differential, and we obtain

$$\begin{aligned} L(E_{32},2) &= -2\pi^2 \sum_{m_1,n_1,m_2,n_2\ge1} \frac{a(m_1)b(n_1)b(m_2)a(n_2)n_2}{n_1} \\ &\qquad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_2u + \frac{m_2n_1}{32u}\right)\right)du \\ &= -2\pi^2\int_0^\infty \sum_{m_1,n_2\ge1} a(m_1)a(n_2)n_2e^{-2\pi m_1n_2u} \\ &\qquad \times \sum_{m_2,n_1\ge1} \frac{b(m_2)b(n_1)}{n_1}e^{-2\pi m_2n_1/(32u)}du. \end{aligned}$$

What are the resulting series in the product? The first one corresponds to

$$\sum_{m,n\ge1} a(m)a(n)n\,q^{mn} = \sum_{m,n\ge1}\left(\frac{-4}{mn}\right)n\,q^{mn} = \sum_{n\ge1} n\left(\frac{-4}{n}\right)\frac{nq^n}{1+q^{2n}} = \frac{\eta_2^4\eta_8^4}{\eta_4^4},$$

while the second one is

$$\sum_{m,n\ge1}\frac{b(m)b(n)}{n}q^{mn}=\sum_{m,n\ge1}\frac{q^{mn}}{n}-\frac{q^{(2m)n}}{n}-\frac{q^{m(2n)}}{2n}+\frac{q^{(2m)(2n)}}{2n}$$
$$=\frac{1}{2}\sum_{m,n\ge1}\frac{2q^{mn}-3q^{2mn}+q^{4mn}}{n}$$
$$=-\frac{1}{2}\log\prod_{m\ge1}\frac{(1-q^m)^2(1-q^{4m})}{(1-q^{2m})^3}=-\frac{1}{2}\log\frac{\eta_1^2\eta_4}{\eta_2^3},$$

hence

$$L(E_{32},2)=\pi^2\int_0^\infty\frac{\eta_2^4\eta_8^4}{\eta_4^4}\bigg|_{\tau=iu}\cdot\log\frac{\eta_1^2\eta_4}{\eta_2^3}\bigg|_{\tau=i/(32u)}\,du.$$

Applying the involution (1) to the eta quotient under the logarithm sign we obtain

$$L(E_{32},2)=\pi^2\int_0^\infty\frac{\eta_2^4\eta_8^4}{\eta_4^4}\log\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\bigg|_{\tau=iu}\,du.$$

Now comes the modular magic: assisted with Ramanujan's knowledge [1] we choose a particular modular function $x(\tau):=\eta_2^4\eta_8^2/\eta_4^6$, which ranges from 1 to 0 when $\tau\in(0,i\infty)$, and verify that

$$\frac{1}{2\pi i}\frac{x\,dx}{2\sqrt{1-x^4}}=-\frac{\eta_2^4\eta_8^4}{\eta_4^4}\,d\tau\quad\text{and}\quad\left(\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\right)^2=\frac{1-x}{1+x}.$$

Thus,

$$L(E_{32},2)=\frac{\pi}{8}\int_0^1\frac{x}{\sqrt{1-x^4}}\log\frac{1+x}{1-x}\,dx.$$

The result is a *period* in the sense of [2], and as such it can be compared with several other objects like values of generalized hypergeometric functions or even Mahler measures [4, 5]. This however involves a different set of routines which we do not touch here.

To summarize, in our evaluation of $L(E,2)=L(f,2)$ we first split $f(\tau)$ into a product of two Eisenstein series of weight 1 and at the end we arrive at a product of two Eisenstein(-like) series $g_2(\tau)$ and $g_0(\tau)$ of weights 2 and 0, respectively, so that $L(f,2)=cL(g_2g_0,1)$ for some algebraic constant c. The latter object is doomed to be a period as $g_0(\tau)$ is a logarithm of a modular function, while $2\pi i\,g_2(\tau)\,d\tau$ is, up to a modular function multiple, the differential of a modular function, and finally any two modular functions are tied up by an algebraic relation over $\overline{\mathbb{Q}}$.

The method however can be formalized to even more general settings, and it is this extension which we attempt to outline below.

For two *bounded* sequences $a(m)$, $b(n)$, we refer to an expression of the form

$$g_k(\tau)=a+\sum_{m,n\ge1}a(m)b(n)n^{k-1}q^{mn},\qquad q:=e^{2\pi i\tau},\tag{3}$$

as to an Eisenstein-like series of weight k, especially in the case when $g_k(\tau)$ is a modular form of certain level, that is, when it transforms sufficiently 'nice' under $\tau \mapsto -1/(N\tau)$ for some positive integer N. This automatically happens when $g_k(\tau)$ is indeed an Eisenstein series (for example, when $a(m) = 1$ and $b(n)$ is a Dirichlet character modulo N of designated parity, $b(-1) = (-1)^k$), in which case $\widehat{g}_k(\tau) := g_k(-1/(N\tau))(\sqrt{-N}\tau)^{-k}$ is again an Eisenstein series. It is worth mentioning that the above notion has perfect sense in case $k \le 0$ as well. Indeed, modular units, or week modular forms of weight 0, that are the logarithms of modular functions are examples of Eisenstein-like series $g_0(\tau)$. Also, for $k \le 0$ examples are given by Eichler integrals, the $(1-k)$th τ-derivatives of holomorphic Eisenstein series of weight $2-k$, a consequence of the famous lemma of Hecke [8, Section 5].

Suppose we are interested in the L-value $L(f, k_0)$ of a cusp form $f(\tau)$ of weight $k = k_1 + k_2$ which can be represented as a product (in general, as a linear combination of several products) of two Eisenstein(-like) series $g_{k_1}(\tau)$ and $\widehat{g}_{k_2}(\tau)$, where the first one vanishes at infinity ($a = g_{k_1}(i\infty) = 0$ in (3)) and the second one vanishes at zero ($\widehat{g}_{k_2}(i0) = 0$). (The vanishing happens because the product is a cusp form!) In reality, we need the series $g_{k_2}(\tau) := \widehat{g}_{k_2}(-1/(N\tau))(\sqrt{-N}\tau)^{-k_2}$ to be Eisenstein-like:

$$g_{k_1}(\tau) = \sum_{m,n\ge1} a_1(m)b_1(n)n^{k_1-1}q^{mn} \quad \text{and} \quad g_{k_2}(\tau) = \sum_{m,n\ge1} a_2(m)b_2(n)n^{k_2-1}q^{mn}.$$

We have

$$\begin{aligned}
L(f,k_0) = L(g_{k_1}\widehat{g}_{k_2},k_0) &= \frac{1}{(k_0-1)!}\int_0^1 g_{k_1}\widehat{g}_{k_2}\log^{k_0-1}q\,\frac{\mathrm{d}q}{q} \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!}\int_0^\infty g_{k_1}(it)\widehat{g}_{k_2}(it)t^{k_0-1}\,\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty g_{k_1}(it)g_{k_2}(i/(Nt))t^{k_0-k_2-1}\,\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty \sum_{m_1,n_1\ge1} a_1(m_1)b_1(n_1)n_1^{k_1-1}e^{-2\pi m_1n_1t} \\
&\qquad\times \sum_{m_2,n_2\ge1} a_2(m_2)b_2(n_2)n_2^{k_2-1}e^{-2\pi m_2n_2/(Nt)}t^{k_0-k_2-1}\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\sum_{m_1,n_1,m_2,n_2\ge1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1-1}n_2^{k_2-1} \\
&\qquad\times\int_0^\infty \exp\biggl(-2\pi\Bigl(m_1n_1t+\frac{m_2n_2}{Nt}\Bigr)\biggr)t^{k_0-k_2-1}\mathrm{d}t;
\end{aligned}$$

the interchange of integration and summation is legitimate because of the exponential decrease of the integrand at the endpoints. After performing the change of variable

$t = n_2u/n_1$ and interchanging back summation and integration we obtain

$$
\begin{aligned}
L(f,k_0) &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \sum_{m_1,n_1,m_2,n_2\ge 1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1+k_2-k_0-1}n_2^{k_0-1} \\
&\quad\times \int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_2u+\frac{m_2n_1}{Nu}\biggr)\biggr)u^{k_0-k_2-1}\mathrm{d}u \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \int_0^\infty \sum_{m_1,n_2\ge 1} a_1(m_1)b_2(n_2)n_2^{k_0-1}e^{-2\pi m_1n_2u} \\
&\quad\times \sum_{m_2,n_1\ge 1} a_2(m_2)b_1(n_1)n_1^{k_1+k_2-k_0-1}e^{-2\pi m_2n_1/(Nu)}u^{k_0-k_2-1}\mathrm{d}u \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}} \int_0^\infty g_{k_0}(iu)g_{k_1+k_2-k_0}(i/(Nu))u^{k_0-k_2-1}\,\mathrm{d}u.
\end{aligned}
$$

Assuming a modular transformation of the Eisenstein-like series $g_{k_1+k_2-k_0}(\tau)$ under $\tau\mapsto -1/(N\tau)$, we can realize the resulting integral as $c\pi^{k_0-k_1}L(g_{k_0}\widehat{g}_{k_1+k_2-k_0},k_1)$, where c is algebraic (plus extra terms when $g_{k_1+k_2-k_0}(\tau)$ is an Eichler integral). Alternatively, if $g_{k_0}(\tau)$ transforms under the involution, we perform the transformation and switch to the variable $v=1/(Nu)$ to arrive at $c\pi^{k_0-k_1}L(\widehat{g}_{k_0}g_{k_1+k_2-k_0},k_1)$. In both cases we obtain an identity which relates the starting L-value $L(f,k_0)$ to a different 'L-value' of a modular-like object of the same weight.

The case $k_1=k_2=1$ and $k_0=2$, discussed in [6, 7] and in our example above, allows one to reduce the L-values to periods. In our future work [9] we plan to address some examples with $k_0>2$.

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