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Double zeta functions constructed by absolute tensor products

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The aim of the present article is to report recent progress on the theory of the absolute tensor product initiated by Kurokawa [K], in particular, to introduce a new result (Theorem 3). This article is based on the author's talk at the workshop "Analytic Number Theory –related Multiple aspects of Arithmetic Functions". This article can be regarded as a continuation of the author's previous RIMS report [A1].

This article is organized as follows. In §1 we recall the absolute tensor product. In §2 we introduce some results on Euler products for double zeta functions constructed by absolute tensor products. In §3 we give a proof of the new result.

I would like to express my sincere gratitude to the organizer, Takumi Noda, for giving me an opportunity of a presentation in the workshop.

1 Absolute tensor product

Definition 1.1 (zeta regularized product). Let $m : \mathbb{C} \to \mathbb{Z}$ be a support discrete function. Put

$$\zeta_m(w,s) := \sum_{\substack{\rho\in\mathbb{C} \\ m(\rho)\neq 0}} \frac{m(\rho)}{(s-\rho)^w}, \tag{1.1}$$

where $(s-\rho)^w := \exp[w\log(s-\rho)]$ and we choose the logarithmic branch by $\arg(s-\rho) \in (-\pi, \pi]$. Here we assume the following two conditions for each $\mathrm{Re}(s) \gg 1$:

1. the Dirichlet series (1.1) converges absolutely in $\mathrm{Re}(w) > C$ for some $C \in \mathbb{R}$.

2. $\zeta_m(w,s)$ has a meromorphic continuation with respect to w in a region including $w = 0$.

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Then the *zeta regularized product* is defined by

$$\coprod_{\rho\in\mathbb{C}}(s-\rho)^{m(\rho)} := \exp\left[-\mathrm{Res}_{w=0}\frac{\zeta_m(w,s)}{w^2}\right]. \tag{1.2}$$

The zeta regularized product has the following property.

Proposition 1.2. [I, Theorem 1] *We assume the two conditions* 1 *and* 2 *in Definition* 1.1. *Then the zeta regularized product* (1.2), *which is initially defined in* $\mathrm{Re}(s) \gg 1$, *has a meromorphic continuation in the whole complex plane* $s \in \mathbb{C}$. *Moreover, the analytic continuation of* (1.2) *has zeros at* $s=\rho$ *with the order* $m(\rho)$. (*If* $m(\rho)$ *is negative, then* $s=\rho$ *is a pole with the order* $|m(\rho)|$.)

Here we give two examples of zeta regularized product expressions.

Example 1.3. (1) *Let* $M>1$. *Then in* $\mathrm{Re}(s)>0$ *we have*

$$(1-M^{-s})^{-1} = \coprod_{n\in\mathbb{Z}}\left(s-\frac{2\pi i n}{\log M}\right)^{-1}.$$

(2) [Den, Theorem 1.1] *We denote* $\zeta(s)$ *by the Riemann-zeta function. Then in* $\mathrm{Re}(s)>1$ *we have*

$$\zeta(s) = \frac{\displaystyle\coprod_{\rho:\text{distinct}}(s-\rho)^{m(\rho)}\coprod_{n=1}^{\infty}(s+2n)}{s-1},$$

where ρ *runs over the distinct nontrivial zeros of* $\zeta(s)$ *and* $m(\rho)$ *is the order of the zero of* $\zeta(s)$ *at* $s=\rho$.

Definition 1.4 (absolute tensor product). We assume that $Z_j(s)$ have the following zeta regularized products for any $j\in\{1,\ldots,r\}$:

$$Z_j(s) \cong \coprod_{\rho\in\mathbb{C}}(s-\rho)^{m_j(\rho)},$$

where $F(s)\cong G(s)$ means that there exists $Q(s)\in\mathbb{C}[s]$ such that $F(s)=e^{Q(s)}G(s)$ holds. Then their *absolute tensor product* $(Z_1\otimes\cdots\otimes Z_r)(s)$ is defined by

$$(Z_1\otimes\cdots\otimes Z_r)(s) = \coprod_{\rho_1,\ldots,\rho_r\in\mathbb{C}}(s-\rho_1-\cdots-\rho_r)^{m(\rho_1,\ldots,\rho_r)},$$

where

$$m(\rho_1,\ldots,\rho_r) = m_1(\rho_1)\cdots m_r(\rho_r)\times\begin{cases}1 & \text{if } \mathrm{Im}(\rho_j)\geq 0 \text{ for any } j,\\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_j)<0 \text{ for any } j,\\ 0 & \text{otherwise.}\end{cases}$$

Keep the setting as in Definition 1.4. In addition we assume that $Z_j(s)$ are entire for any $j \in \{1, \ldots, r\}$. Then, by definition $(Z_1 \otimes \cdots \otimes Z_r)(s)$ has the following additive structure on zeros and poles:

$$\begin{aligned} &Z_j(\rho_j) = 0 \text{ for any } j \text{ and } \mathrm{Im}(\rho_j) \text{ are all nonnegative or all negative} \\ &\Longrightarrow (Z_1 \otimes \cdots \otimes Z_r)(\rho_1 + \cdots + \rho_r) = 0 \text{ or } \infty. \end{aligned}$$

In particular, taking $Z_j = Z$ and $\rho_j = \rho$, we have

$$Z(\rho) = 0 \Longrightarrow Z^{\otimes r}(r\rho) = 0 \text{ or } \infty.$$

Thus, if for some $A_r, B_r \in \mathbb{R}$ it holds that

$$Z^{\otimes r}(\rho') = 0 \text{ or } \infty \Longrightarrow A_r \leq \mathrm{Re}(\rho') \leq B_r, \tag{1.3}$$

then we obtain

$$Z(\rho) = 0 \Longrightarrow \frac{A_r}{r} \leq \mathrm{Re}(\rho) \leq \frac{B_r}{r}.$$

Therefore, if for *any* r there exist A_r and B_r such that (1.3) holds and there exists C, which does not depend on r, such that $B_r - A_r \leq C$, then we see that all the zeros of Z have a same real part. The above strategy, which is a trial to extend Deligne's proof [Del] for the Weil conjecture, was proposed by Kurokawa.

In the above strategy the point is to give (1.3) for many r. Here a question arises: how do we obtain zero-free regions like (1.3)? We recall the basic fact that the Riemann zeta-function is zero-free in $\mathrm{Re}(s) > 1$. This follows that its Euler product converges absolutely in $\mathrm{Re}(s) > 1$. Thus Euler products for $Z^{\otimes r}(s)$ seem important in order to find B_r although existence and appearance of the Euler products are unclear. On the other hand, to give A_r, we would like a functional equation for $Z^{\otimes r}(s)$ in addition to the Euler product. If $Z(s)$ has a functional equation between $Z(s) \longleftrightarrow Z(d-s)$, then $Z^{\otimes r}(s)$ also has a functional equation between $Z^{\otimes r}(s) \longleftrightarrow Z^{\otimes r}(rd-s)^{(-1)^{r-1}}$, which follows from the definition of the absolute tensor product.[1]

For further details of absolute tensor products, see [M, S].

2 Euler products for double zeta functions

In this article we consider Euler products for

$$(\zeta_p \otimes \zeta_q)(s), \qquad (\zeta \otimes \zeta)(s), \qquad (\zeta \otimes \zeta_p)(s),$$

[1]However it seems difficult to write down the functional equation for $Z^{\otimes r}(s)$ as an explicit equation in general.

where $\zeta(s)$ is the Riemann zeta-function and $\zeta_p(s)$ is an Euler factor of the Riemann zeta-function, that is,

$$\zeta(s) := \prod_{p:\text{primes}} \zeta_p(s), \qquad \zeta_p(s) := (1-p^{-s})^{-1}.$$

First of all, we recall $(\zeta_p \otimes \zeta_q)(s)$. From Example 1.3 (1), $(\zeta_p \otimes \zeta_q)(s)$ is given by

$$(\zeta_p \otimes \zeta_q)(s) = \prod_{m,n=0}^{\infty} \left(s - \frac{2\pi i m}{\log p} - \frac{2\pi i n}{\log q} \right) \times \prod_{m,n=1}^{\infty} \left(s + \frac{2\pi i m}{\log p} + \frac{2\pi i n}{\log q} \right)^{-1}.$$

As was shown by Koyama and Kurokawa [KK1], $(\zeta_p \otimes \zeta_q)(s)$ has an expression similar to $\zeta_p(s) = \exp[\sum_{n=1}^{\infty} n^{-1} p^{-ns}]$ as follows:

Theorem 1. [KK1, Theorems 1 and 4] *Let p and q be prime numbers. Then,*
(1) *When $p \neq q$, in* $\mathrm{Re}(s) > 0$ *we have*

$$(\zeta_p \otimes \zeta_q)(s) \cong \exp\left(-\sum_{n=1}^{\infty} \frac{p^{-ns}}{n(1-e(n\frac{\log p}{\log q}))} - \sum_{n=1}^{\infty} \frac{q^{-ns}}{n(1-e(n\frac{\log q}{\log p}))} \right), \tag{2.1}$$

where $e(x) := e^{2\pi i x}$.
(2) *When $p = q$, in* $\mathrm{Re}(s) > 0$ *we have*

$$(\zeta_p \otimes \zeta_p)(s) \cong \exp\left(-\frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p^{-ns}}{n^2} - \left(1 + \frac{s \log p}{2\pi i}\right) \sum_{n=1}^{\infty} \frac{p^{-ns}}{n} \right). \tag{2.2}$$

Remark 2.1. All the sums in Theorem 1 converge absolutely in $\mathrm{Re}(s) > 0$. While it is easy to check the convergence of the sums in (2.2), the sums in (2.1) have a delicate problem because of the denominators of the summands. In fact, we need a result on linear forms in logarithms (see [B, Theorem 3.1]), which says that for any distinct prime numbers p and q there exists $C = C_{p,q} > 0$ such that

$$\left\| n \frac{\log p}{\log q} \right\| \geq n^{-C} \tag{2.3}$$

for any $n \in \mathbb{Z}_{\geq 2}$, where $\|x\| := \min_{m \in \mathbb{Z}} |x - m|$. The desired convergence in $\mathrm{Re}(s) > 0$ follows from (2.3) together with

$$|1 - e(\alpha)| = 2|\sin(\pi\alpha)| = 2\sin(\pi\|\alpha\|) \geq 4\|\alpha\| \tag{2.4}$$

for any $\alpha \in \mathbb{R}$. We note that we can prove the absolute convergence only in $\mathrm{Re}(s) > 1$ by using a more elementary inequality (3.13).

Similar formulas for $(\zeta_{p_1} \otimes \cdots \otimes \zeta_{p_r})(s)$ were obtained, for example, in [A2, KW].

Next we treat $(\zeta \otimes \zeta)(s)$. By Example 1.3 (2), $(\zeta \otimes \zeta)(s)$ is given by

$$
\begin{aligned}
&(\zeta \otimes \zeta)(s) \\
&= \frac{\displaystyle\prod_{\mathrm{Im}(\rho_j)>0} (s-\rho_1-\rho_2)\Bigg(\prod_{\substack{\mathrm{Im}(\rho)>0,\\ n\geq 1}} (s-\rho+2n)\Bigg)^2 (s-2)\prod_{n_j\geq 1}(s+2n_1+2n_2)}{\displaystyle\prod_{\mathrm{Im}(\rho_j)<0} (s-\rho_1-\rho_2)\Bigg(\prod_{\mathrm{Im}(\rho)>0} (s-1-\rho)\Bigg)^2 \Bigg(\prod_{n\geq 1}(s+2n-1)\Bigg)^2},
\end{aligned}
$$

where ρ, ρ_1 and ρ_2 run over the nontrivial zeros of $\zeta(s)$ in the given range counted with multiplicity.[2] In [A3] the author gave an analogue of the Euler product expression for $(\zeta \otimes \zeta)(s)$ as follows:

Theorem 2. [A3, Theorem 1.3] *In* $\mathrm{Re}(s) > 2$ *we have*

$$
\begin{aligned}
(\zeta \otimes \zeta)(s) \;=\; \exp\Bigg(& \frac{1}{\pi i}\sum_{p}\sum_{m=1}^{\infty}\sum_{q}\sum_{\substack{n=1\\ q^n\neq p^m}}^{\infty} \frac{p^{-m(s-1)}q^{-n}\log p}{n(m\log p - n\log q)} \\
& - \frac{1}{\pi i}\sum_{p}\sum_{m=1}^{\infty}\sum_{q}\sum_{n=1}^{\infty} \frac{p^{-ms}q^{-n}\log p}{n(m\log p + n\log q)} \\
& + \frac{1}{\pi i}\int_0^1 \frac{\zeta'}{\zeta}(s-u)\log|\zeta(u)|du\Bigg)\zeta(s-1)^{-1}\times R(s),
\end{aligned}
$$

where p and q run over the prime numbers and $R(s)$ is a holomorphic function having no zeros in $\mathrm{Re}(s) > 1$.

Remark 2.2. We can express $R(s)$ in terms of sums over the prime numbers: see [A3].

The first sum and the third integral converge absolutely and locally uniformly in $\mathrm{Re}(s) > 2$ while the second sum converges absolutely and locally uniformly in $\mathrm{Re}(s) > 1$. Furthermore it is impossible to improve the convergent domain for the first sum and the third integral because we can show that

$$
\begin{aligned}
\frac{1}{\pi i}\sum_{p}\sum_{m=1}^{\infty}\sum_{q}\sum_{\substack{n=1\\ q^n\neq p^m}}^{\infty} \frac{p^{-m(s-1)}q^{-n}\log p}{n(m\log p - n\log q)} &\sim \frac{1}{2\pi i}(\log(s-2))^2, \\
\frac{1}{\pi i}\int_0^1 \frac{\zeta'}{\zeta}(s-u)\log|\zeta(u)|du &\sim -\frac{1}{2\pi i}(\log(s-2))^2
\end{aligned}
$$

[2]Instead of writing multiplicity functions like Example 1.3 (2), we count zeros considering the multiplicity.

as $s \to 2$. See [A3, Proposition 7.1].

Koyama and Kurokawa [KK2] also obtained an Euler product for $(\zeta \otimes \zeta)(s)$ by an entirely different method. However the Euler product in [KK2] includes an extra parameter and is more complicated than that in Theorem 2.

For $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$ let $F_j(s)$ and $G_k(s)$ be meromorphic functions having zeta regularized product expressions. Then by definition we have

$$\left[\left(\prod_{j=1}^{m} F_j\right) \otimes \left(\prod_{k=1}^{n} G_k\right)\right](s) = \prod_{j=1}^{m}\prod_{k=1}^{n}(F_j \otimes G_k)(s). \tag{2.5}$$

Therefore we may hope that

$$(\zeta \otimes \zeta)(s) \stackrel{?}{=} \prod_{p,q}(\zeta_p \otimes \zeta_q)(s), \tag{2.6}$$

where p and q are taken over the prime numbers. However we cannot interchange the order of the limit process and the zeta regularized product in general. To make matters worse, we expect that the right hand side of (2.6) does not converge absolutely for any $s \in \mathbb{C}$: see Theorem 1.

From the viewpoint of (2.5), it is also interesting to compare $(\zeta \otimes \zeta)(s)$ with $(\zeta \otimes \zeta_p)(s)$ and $(\zeta \otimes \zeta_p)(s)$ with $(\zeta_p \otimes \zeta_q)(s)$. As a first step to investigate relationships among these three functions, we give an Euler product expression for $(\zeta \otimes \zeta_p)(s)$.

First of all we recall that $(\zeta \otimes \zeta_p)(s)$ is given by

$$(\zeta \otimes \zeta_p)(s) = \frac{\displaystyle\prod_{\substack{\mathrm{Im}(\rho)<0,\\ n>0}} \left(s - \rho + \frac{2\pi i n}{\log p}\right) \prod_{n \geq 0} \left(s - 1 - \frac{2\pi i n}{\log p}\right)}{\displaystyle\prod_{\substack{\mathrm{Im}(\rho)>0,\\ n\geq 0}} \left(s - \rho - \frac{2\pi i n}{\log p}\right) \prod_{\substack{m\geq 1,\\ n\geq 0}} \left(s + 2m - \frac{2\pi i n}{\log p}\right)}, \tag{2.7}$$

where ρ runs over the nontrivial zeros of $\zeta(s)$ counted with multiplicity. This function has the following Euler product expression:

Theorem 3. *Let p be a prime number. Then in* $\mathrm{Re}(s) > 2$ *we have*

$$(\zeta \otimes \zeta_p)(s) = \exp\left(\sum_{j=1}^{10} E_j(s)\right),$$

where

$$E_1(s) = -\sum_{q \neq p} \sum_{m=1}^{\infty} \frac{q^{-ms}}{m(1 - e(m\frac{\log q}{\log p}))},$$

$$E_2(s) = \frac{1}{2\pi i}\sum_{m=1}^{\infty}\sum_{q}\sum_{\substack{n=1,\\ q^n\neq p^m}}^{\infty}\frac{p^{-m(s-1)}q^{-n}\log p}{n(m\log p - n\log q)},$$

$$E_3(s) = -\frac{s-1}{2\pi i}\sum_{m=1}^{\infty}\frac{p^{-ms}\log p}{m} - \frac{1}{2}\sum_{m=1}^{\infty}\frac{p^{-ms}}{m},$$

$$E_4(s) = -\frac{1}{2\pi i}\sum_{m=1}^{\infty}\sum_{q}\sum_{n=1}^{\infty}\frac{p^{-ms}q^{-n}\log p}{n(m\log p + n\log q)},$$

$$E_5(s) = \left(\frac{1}{4} + \frac{\gamma + \log(2\pi)}{2\pi i}\right)\left(\sum_{m=1}^{\infty}\frac{p^{-ms}}{m} + \sum_{m=1}^{\infty}\frac{p^{-ms}}{m^2\log p}\right),$$

$$E_6(s) = \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{p^{-ms}}{m}\times\frac{\Gamma'}{\Gamma}\left(\frac{m\log p}{\pi i}\right),$$

$$E_7(s) = -\frac{\log p}{2\pi i}\int_0^1\frac{\log|\zeta(u)|}{p^{s-u}-1}du,$$

$$E_8(s) = -\frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{p^{-ms}}{m^2\log p}\int_0^{\infty}\frac{u}{e^u-1}\frac{du}{u+m\log p},$$

$$E_9(s) = -\frac{1}{2}\sum_{m=1}^{\infty}\frac{p^{-m(s-1)}}{m}, \qquad E_{10}(s) = \sum_{m=1}^{\infty}\frac{p^{-m(s+2)}}{m(1-p^{-2m})},$$

where q appearing in $E_1(s)$ and $E_2(s)$ run over the prime numbers and γ appearing in $E_5(s)$ is the Euler constant.

Remark 2.3. The sum in $E_1(s)$ converges absolutely and locally uniformly in $\mathrm{Re}(s) > 2$. The sums and the integral in $E_2(s)$, $E_7(s)$ and $E_9(s)$ converge absolutely and locally uniformly in $\mathrm{Re}(s) > 1$. On the other hand the sums and the integral in $E_3(s)$, $E_4(s)$, $E_5(s)$ and $E_8(s)$ converge absolutely and locally uniformly in $\mathrm{Re}(s) > 0$ and the sum in $E_{10}(s)$ converges absolutely and locally uniformly in $\mathrm{Re}(s) > -2$.

3 Proof of Theorem 3

First of all we recall the theory of Cramér [C] and Guinand [G]. Put

$$\theta(t) := \sum_{\mathrm{Re}(\tau)>0} e^{-\tau t}, \qquad U(t) := \theta(t) + \frac{\log t}{4\pi\sin(t/2)},$$

where τ runs over the complex numbers with $\mathrm{Re}(\tau) > 0$ such that $\frac{1}{2}+i\tau$ are nontrivial zeros of $\zeta(s)$, counted with multiplicity. The sum converges absolutely and locally uniformly in $\mathrm{Re}(t) > 0$, so that $\theta(t)$ and $U(t)$ are originally defined in $|\arg(t)| < \pi/2$. Cramér and Guinand showed that

- an expression for $\theta(t)$ in terms of sums over prime numbers,
- a single-valued meromorphic continuation to the whole complex plane $t \in \mathbb{C}$ for $U(t)$,
- a functional equation for $U(t)$ between $t \longleftrightarrow -t$.

See [G, Theorem 3] and [C, p.114, (13)] for the precise statement. From the second property $\theta(t)$ has a single-valued meromorphic continuation to $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$. We rewrite it by the same notation $\theta(t)$ and put

$$\theta^*(t) := \theta(t) - e^{-it/2},$$

which is originally defined in $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$. For $\theta^*(t)$ we can read the results of Cramér and Guinand as follows:

Lemma 3.1. (cf. [A3, Lemmas 2.2 and 2.3]) (1) *In $t \in \mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ we have*

$$\begin{aligned}\theta^*(t) \quad = \quad & -\frac{t}{2\pi i} e^{-it/2} \sum_q \sum_{n=1}^{\infty} \frac{q^{-n}}{n(t - in\log q)} + \frac{t}{2\pi i} e^{it/2} \sum_q \sum_{n=1}^{\infty} \frac{q^{-n}}{n(t + in\log q)} \\ & -e^{it/2}\left(\frac{1}{4} + \frac{\gamma + \log(2\pi)}{2\pi i}\right)\left(1 - \frac{1}{it}\right) - \frac{e^{it/2}}{2\pi i}\frac{\Gamma'}{\Gamma}\left(-\frac{t}{\pi}\right) \\ & -\frac{t}{2\pi} e^{it/2} \int_0^1 e^{-itu} \log|\zeta(u)| du + \frac{e^{it/2}}{2\pi t} \int_0^{\infty} \frac{u}{e^u - 1} \frac{du}{u - it} \\ & -\frac{1}{2} e^{-it/2} + \frac{e^{3it/2}}{e^{it} - e^{-it}},\end{aligned}$$

where q runs over the prime numbers and γ is the Euler constant. All the sums and the integrals converge absolutely and locally uniformly in $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$.
(2) *Poles of $\theta^*(t)$ in $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ are located at $t = in\log q$ (q: prime numbers, $n \in \mathbb{Z}_{\geq 1}$) and $t = -\pi m$ ($m \in \mathbb{Z}_{\geq 1}$), and nowhere else.*
(3) *$\theta^*(t)$ has the following functional equation:*

$$\theta^*(t) + \theta^*(-t) = \begin{cases} -\dfrac{e^{it/2}}{2i\sin t} & \text{if } \mathrm{Re}(t) > 0, \\ \dfrac{e^{-it/2}}{2i\sin t} & \text{if } \mathrm{Re}(t) < 0. \end{cases}$$

We go back to the proof of Theorem 3. By the definition (2.7) of $(\zeta \otimes \zeta_p)(s)$, for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 2$ we would like to express

$$\sum_{\substack{\mathrm{Im}(\rho)<0, \\ n\geq 1}} \left(s - \rho + \frac{2\pi i n}{\log p}\right)^{-w} \tag{3.1}$$

in terms of sums over the prime numbers, where $\arg(s-\rho+\frac{2\pi in}{\log p}) \in (-\pi/2, \pi/2)$. We put $z := (s-\frac{1}{2})/i$ and $a := 2\pi/\log p$. Then (3.1) equals to

$$e^{-\pi iw/2} \sum_{\substack{\mathrm{Re}(\tau)>0,\\ n\geq 1}} (z+\tau+an)^{-w},$$

where $\arg(z+\tau+an) \in (-\pi, 0)$. Under $\mathrm{Re}(z) > 0$ in addition to $\mathrm{Im}(z) < -1/2$ and $\mathrm{Re}(w) > 2$, a standard calculation gives

$$\sum_{\substack{\mathrm{Re}(\tau)>0,\\ n\geq 1}} (z+\tau+an)^{-w} = \frac{1}{\Gamma(w)} \int_0^\infty e^{-zt}\theta(t)\theta_a(t)t^w \frac{dt}{t}, \tag{3.2}$$

where

$$\theta_a(t) := \sum_{n=1}^\infty e^{-ant} = \frac{1}{e^{at}-1}.$$

To connect (3.2) with sums over the prime numbers, we would consider

$$\frac{1}{\Gamma(w)} \int_{C_\varepsilon} e^{-zt}\widetilde{\theta}(t)\theta_a(t)t^w \frac{dt}{t}, \tag{3.3}$$

where $0 < \varepsilon < \log 2$, C_ε is a path connecting $+\infty \to \varepsilon$, $\varepsilon e^{i\theta}$ $(\theta : 0 \to 2\pi)$ and $\varepsilon e^{2\pi i} \to \infty e^{2\pi i}$, and $\widetilde{\theta}(t)$ is an analytic continuation of $\theta(t)$ to $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ with the initial domain $\{t \in \mathbb{C} \mid \mathrm{Re}(t) > 0, \mathrm{Im}(t) > 0\}$. Ignoring the convergence temporarily, we would have

$$\frac{1}{\Gamma(w)} \int_{C_\varepsilon} e^{-zt}\widetilde{\theta}(t)\theta_a(t)t^w \frac{dt}{t} \overset{?}{=} \frac{2\pi i}{\Gamma(w)} \sum_{\substack{\alpha\in\mathbb{C}\setminus\{0\},\\ \text{poles of } \widetilde{\theta}(t)\theta_a(t)}} \mathrm{Res}_{t=\alpha}\, e^{-zt}\widetilde{\theta}(t)\theta_a(t)t^{w-1} \tag{3.4}$$

by the residue theorem. However the right hand side of (3.4) does not converge absolutely for any $(s, w) \in \mathbb{C}^2$.

To avoid the above problem, we start with

$$\frac{1}{\Gamma(w)} \int_{P_\varepsilon} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t} \tag{3.5}$$

instead of (3.3), where $0 < \varepsilon < \log 2$ and P_ε is a path connecting $\infty e^{3\pi i/4} \to \varepsilon e^{3\pi i/4}$, $\varepsilon e^{i\theta}$ $(\theta : \frac{3\pi}{4} \to -\frac{\pi}{4})$ and $\varepsilon e^{-\pi i/4} \to \infty e^{-\pi i/4}$. The integral converges absolutely in $D := \{(z, w) \in \mathbb{C}^2 \mid -\frac{1}{2} - a < \mathrm{Re}(z) + \mathrm{Im}(z) < 5/2\}$. We recall $a = 2\pi/\log p$. We restrict z to $\mathrm{Im}(z) < -3/2$ in addition to $(z, w) \in D$. For positive integers M we put $T := \log(M + \frac{1}{2})$ and we take real numbers $R \geq T$. Let $P_\varepsilon(R, T)$ be a closed path connecting $\sqrt{2}Te^{3\pi i/4}$, $\varepsilon e^{3\pi i/4}$, $\varepsilon e^{i\theta}$ ($\theta : \frac{3\pi}{4} \to -\frac{\pi}{4}$), $R-iR$, $R+iT$ and $-T+iT$. Applying the residue theorem to the path $P_\varepsilon(R,T)$ and taking the limits $R \to \infty$ and $M \to \infty$, we have[3]

$$\frac{1}{\Gamma(w)} \int_{P_\varepsilon} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t} = \frac{2\pi i}{\Gamma(w)} \lim_{M\to\infty} \sum_{\substack{q,n \\ q^n < M+\frac{1}{2}}} \mathrm{Res}_{t=in\log q}\, e^{-zt}\theta^*(t)\theta_a(t)t^{w-1}, \tag{3.6}$$

where q runs over the prime numbers and n runs over the positive integers with $q^n < e^T$. We calculate the residues explicitly by using Lemma 3.1 (1). For simplicity we write Lemma 3.1 (1) as

$$\theta^*(t) = -\frac{t}{2\pi i} e^{-it/2} \sum_Q \sum_{N=1}^{\infty} \frac{Q^{-N}}{N(t - iN\log Q)} + R_1(t),$$

where Q runs over the prime numbers. When $q \neq p$, $e^{-zt}\theta^*(t)\theta_a(t)t^{w-1}$ has simple poles at $t = in\log q$ and we have

$$\begin{aligned}
&\mathrm{Res}_{t=in\log q}\, e^{-zt}\theta^*(t)\theta_a(t)t^{w-1} \\
= \; &\mathrm{Res}_{t=in\log q}\, e^{-zt}\left(-\frac{t}{2\pi i}e^{-it/2}\frac{q^{-n}}{n(t-in\log q)}\right)\theta_a(t)t^{w-1} \\
= \; &\frac{e^{\pi i w/2}}{2\pi i}\frac{q^{-(iz+\frac{1}{2})n}(n\log q)^{w-1}\log q}{1-e(n\frac{\log q}{\log p})}.
\end{aligned} \tag{3.7}$$

When $q = p$, $e^{-zt}\theta^*(t)\theta_a(t)t^{w-1}$ has double poles at $t = in\log q$ and we have

$$\begin{aligned}
&\mathrm{Res}_{t=in\log p}\, e^{-zt}\theta^*(t)\theta_a(t)t^{w-1} \\
= \; &\mathrm{Res}_{t=in\log p}\, e^{-zt}\left(-\frac{t}{2\pi i}e^{-it/2}\frac{p^{-n}}{n(t-in\log p)}\right)\theta_a(t)t^{w-1} \\
&+\mathrm{Res}_{t=in\log p}\, e^{-zt}\left(-\frac{t}{2\pi i}e^{-it/2}\sum_Q \sum_{\substack{N=1 \\ Q^N\neq p^n}}^{\infty}\frac{Q^{-N}}{N(t-iN\log Q)}+R_1(t)\right)\theta_a(t)t^{w-1} \\
= \; &-\frac{e^{\pi i w/2}p^{-(iz+\frac{1}{2})n}(n\log p)^w}{2\pi i n}\left[\frac{w}{2\pi i n}-\frac{\log p}{2\pi}\left(z+\frac{i}{2}\right)-\frac{1}{2}\right] \\
&-\frac{e^{\pi i w/2}p^{-(iz-\frac{1}{2})n}(n\log p)^w\log p}{(2\pi i)^2}\sum_Q \sum_{\substack{N=1 \\ Q^N\neq p^n}}^{\infty}\frac{Q^{-N}}{N(n\log p - N\log Q)} \\
&+\frac{e^{\pi i w/2}p^{-inz}R_1(in\log p)(n\log p)^{w-1}\log p}{2\pi i}.
\end{aligned} \tag{3.8}$$

[3]Here we need some estimates for the integrand. See [A3, §2] for needed estimates of $\theta^*(t)$.

Next we express (3.5) in terms of Dirichlet series like the left hand side of (3.2). We note that the integral (3.5) is determined independently of a choice of $\varepsilon \in (0, \log 2)$, which is a typical application of Cauchy's theorem. We assume $\mathrm{Re}(w) > 2$ in addition to $(z, w) \in D$. Then taking the limit $\varepsilon \downarrow 0$ gives

$$\frac{1}{\Gamma(w)} \int_{P_\varepsilon} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t} = \frac{1}{\Gamma(w)} \left(\int_{\infty e^{3\pi i/4}}^{0} + \int_0^{\infty e^{-\pi i/4}} \right) e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t}. \tag{3.9}$$

We calculate the second integral. By definition we have

$$\begin{aligned} &\frac{1}{\Gamma(w)} \int_0^{\infty e^{-\pi i/4}} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t} \\ =\ & \frac{1}{\Gamma(w)} \int_0^{\infty e^{-\pi i/4}} e^{-zt} \left(\sum_{\mathrm{Re}(\tau)>0} e^{-\tau t} - e^{-it/2} \right) \left(\sum_{n=1}^{\infty} e^{-ant} \right) t^w \frac{dt}{t} \\ =\ & \sum_{\mathrm{Re}(\tau)>0} \sum_{n=1}^{\infty} (z + \tau + an)^{-w} - \sum_{n=1}^{\infty} \left(z + \frac{i}{2} + an \right)^{-w}, \end{aligned} \tag{3.10}$$

where $\arg(z + \tau + an), \arg(z + \frac{i}{2} + an) \in (-\pi/4, 3\pi/4)$.[4] Here in the last equality we used

$$\int_0^{\infty e^{-\pi i/4}} e^{-\alpha t} t^w \frac{dt}{t} = \Gamma(w)\alpha^{-w},$$

which is valid for $\arg(\alpha) \in (-\pi/4, 3\pi/4)$ and $\mathrm{Re}(w) > 0$. Next we calculate the first integral in the right hand side of (3.9). By Lemma 3.1 (3) and $\theta_a(-t) = -e^{at}\theta_a(t)$, we have

$$\begin{aligned} &\frac{1}{\Gamma(w)} \int_{\infty e^{3\pi i/4}}^{0} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t} \\ =\ & \frac{1}{\Gamma(w)} \int_{\infty e^{-\pi i/4}}^{0} e^{zt}\theta^*(-t)\theta_a(-t)t^w e^{\pi i w} \frac{dt}{t} \\ =\ & -\frac{1}{\Gamma(w)} \int_0^{\infty e^{-\pi i/4}} e^{zt} \left(-\theta^*(t) - \frac{e^{it/2}}{2i \sin t} \right) (-e^{at}\theta_a(t)) t^w e^{\pi i w} \frac{dt}{t}. \end{aligned}$$

In the same manner as (3.10) we obtain

$$\frac{1}{\Gamma(w)} \int_{\infty e^{3\pi i/4}}^{0} e^{-zt}\theta^*(t)\theta_a(t)t^w \frac{dt}{t}$$

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[4]From the assumption $-\frac{1}{2} - a < \mathrm{Re}(z) + \mathrm{Im}(z) < 5/2$ we see $\mathrm{Re}(z+\tau+an) + \mathrm{Im}(z+\tau+an) > 0$ and $\mathrm{Re}(z+\frac{i}{2}+an) + \mathrm{Im}(z+\frac{i}{2}+an) > 0$. These imply $\arg(z+\tau+an), \arg(z+\frac{i}{2}+an) \in (-\pi/4, 3\pi/4)$.

$$= -e^{\pi i w}\sum_{\mathrm{Re}(\tau)>0}\sum_{n=0}^{\infty}(\tau+an-z)^{-w} - e^{\pi i w}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\left(2m+\frac{1}{2}\right)i+an-z\right)^{-w}, \tag{3.11}$$

where $\arg(\tau+an-z), \arg((2m+\frac{1}{2})i+an-z) \in (-\pi/4, 3\pi/4)$.

Combining (3.6)–(3.11), we obtain

$$\begin{aligned}
&\sum_{\mathrm{Re}(\tau)>0}\sum_{n=1}^{\infty}(z+\tau+an)^{-w} - \sum_{n=1}^{\infty}\left(z+\frac{i}{2}+an\right)^{-w} \\
&-e^{\pi i w}\sum_{\mathrm{Re}(\tau)>0}\sum_{n=0}^{\infty}(\tau+an-z)^{-w} - e^{\pi i w}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\left(2m+\frac{1}{2}\right)i+an-z\right)^{-w} \\
= &\frac{e^{\pi i w/2}}{\Gamma(w)}\sum_{q\neq p}\sum_{n=1}^{\infty}\frac{q^{-(iz+\frac{1}{2})n}(n\log q)^{w-1}\log q}{1-e(n\frac{\log q}{\log p})} \\
&-\frac{e^{\pi i w/2}}{\Gamma(w)}\sum_{n=1}^{\infty}\frac{p^{-(iz+\frac{1}{2})n}(n\log p)^{w}}{n}\left[\frac{w}{2\pi i n}-\frac{\log p}{2\pi}\left(z+\frac{i}{2}\right)-\frac{1}{2}\right] \\
&-\frac{e^{\pi i w/2}}{2\pi i\Gamma(w)}\sum_{n=1}^{\infty}\sum_{Q}\sum_{\substack{N\geq 1,\\ Q^N\neq p^n}}\frac{p^{-(iz-\frac{1}{2})n}Q^{-N}(n\log p)^{w}\log p}{N(n\log p-N\log Q)} \\
&+\frac{e^{\pi i w/2}}{\Gamma(w)}\sum_{n=1}^{\infty}p^{-inz}R_1(in\log p)(n\log p)^{w-1}\log p,
\end{aligned}$$

provided $-\frac{1}{2}-a < \mathrm{Re}(z)+\mathrm{Im}(z) < \frac{5}{2}$, $\mathrm{Im}(z) < -3/2$ and $\mathrm{Re}(w) > 2$. Putting $z=(s-\frac{1}{2})/i$ and dividing both sides by $-e^{\pi i w/2}$, we reach

$$\begin{aligned}
&-\sum_{\mathrm{Im}(\rho)<0}\sum_{n=1}^{\infty}\left(s-\rho+\frac{2\pi i n}{\log p}\right)^{-w} + \sum_{n=1}^{\infty}\left(s-1+\frac{2\pi i n}{\log p}\right)^{-w} \\
&+\sum_{\mathrm{Im}(\rho)>0}\sum_{n=0}^{\infty}\left(s-\rho-\frac{2\pi i n}{\log p}\right)^{-w} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(s+2m-\frac{2\pi i n}{\log p}\right)^{-w} \\
= &\frac{1}{\Gamma(w)}\left(-\sum_{q\neq p}\sum_{n=1}^{\infty}\frac{q^{-ns}(n\log q)^{w-1}\log q}{1-e(n\frac{\log q}{\log p})}\right. \\
&+\frac{w}{2\pi i}\sum_{n=1}^{\infty}\frac{p^{-ns}(n\log p)^{w}}{n^2} - \frac{s-1}{2\pi i}\sum_{n=1}^{\infty}\frac{p^{-ns}(n\log p)^{w}\log p}{n} \\
&-\frac{1}{2}\sum_{n=1}^{\infty}\frac{p^{-ns}(n\log p)^{w}}{n} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{Q}\sum_{\substack{N\geq 1,\\ Q^N\neq p^n}}\frac{p^{-n(s-1)}Q^{-N}(n\log p)^{w}\log p}{N(n\log p-N\log Q)}
\end{aligned}$$

$$-\sum_{n=1}^{\infty} p^{-n(s-\frac{1}{2})} R_1(in\log p)(n\log p)^{w-1}\log p\Bigg), \tag{3.12}$$

which is valid for $-2 < \mathrm{Re}(s) - \mathrm{Im}(s) < 1 + \frac{2\pi}{\log p}$, $\mathrm{Re}(s) > 2$ and $\mathrm{Re}(w) > 2$. Here the arguments in the left hand side are taken in $(-\pi/2, \pi/2)$.

We check the absolute convergence of the sums in (3.12). It is easy to see that the sums in the left hand side of (3.12) converge absolutely and locally uniformly in $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 2$. We can also prove that the sums in the right hand side except for the first and fifth sums converge absolutely and locally uniformly in $\mathrm{Re}(s) > 1$ and $w \in \mathbb{C}$ with no difficulty. We treat the first sum. To check the convergence, we give a *uniform* bound for $\|n\frac{\log q}{\log p}\|$ with respect to $q(\neq p)$ and n. For $m \in \mathbb{Z}$ we have

$$\begin{aligned}\left|n\frac{\log q}{\log p} - m\right| &= \frac{1}{\log p}\left|\log\frac{q^n}{p^m}\right| \geq \frac{1}{\log p}\min\left\{\log\frac{q^n}{q^n-1}, \log\frac{q^n+1}{q^n}\right\}\\ &= \frac{1}{\log p}\log\left(1+\frac{1}{q^n}\right) \geq \frac{1}{2q^n\log p}.\end{aligned}$$

Here in the last inequality we used $\log(1+x) \geq x/2$ for $0 \leq x \leq 1/2$. Therefore we get

$$\left\|n\frac{\log q}{\log p}\right\| \geq \frac{1}{2q^n\log p}. \tag{3.13}$$

This together with (2.4) guarantees that the first sum in the right hand side of (3.12) converges absolutely and locally uniformly in $\mathrm{Re}(s) > 2$ and $w \in \mathbb{C}$.

Next we deal with the convergence of the fifth sum in the right hand side of (3.12). To do this, we estimate

$$\sum_{Q}\sum_{\substack{N\geq 1,\\ Q^N\neq p^n}} \frac{Q^{-N}}{N|n\log p - N\log Q|} = \sum_{\substack{m\geq 2,\\ m\neq L}} \frac{\Lambda(m)}{m\log m|\log L - \log m|}, \tag{3.14}$$

where $\Lambda(m)$ is the von Mangoldt function and $L := p^n$. We divide the sum into $2 \leq m \leq \sqrt{L}$, $\sqrt{L} < m < L$, $L < m < L^2$ and $m \geq L^2$. Firstly we consider $2 \leq m \leq \sqrt{L}$. In this case we have $|\log L - \log m| = \log L - \log m \geq (\log L)/2$. Thus we have

$$\sum_{2\leq m\leq\sqrt{L}} \frac{\Lambda(m)}{m\log m|\log L - \log m|} \leq \frac{2}{\log L}\sum_{2\leq m\leq\sqrt{L}}\frac{\Lambda(m)}{m\log m} \ll \frac{\log\log L}{\log L}. \tag{3.15}$$

Here in the last inequality we used the prime number theorem. In the same manner as (3.14), the sum over $m \geq L^2$ is estimated as follows:

$$\sum_{m\geq L^2}\frac{\Lambda(m)}{m\log m|\log L - \log m|} \leq 2\sum_{m\geq L^2}\frac{\Lambda(m)}{m(\log m)^2} \ll \frac{1}{\log L}. \tag{3.16}$$

Next we deal with the sum over $\sqrt{L} < m < L$. We note that the mean value theorem gives

$$\log Y - \log X \geq \frac{Y - X}{Y}$$

for any $0 < X < Y$. Thus, together with $\Lambda(m) \leq \log m$ we have

$$\begin{aligned}
&\sum_{\sqrt{L}<m<L} \frac{\Lambda(m)}{m \log m |\log L - \log m|} \\
\leq\ & L \sum_{\sqrt{L}<m<L} \frac{\Lambda(m)}{m \log m (L-m)} \leq L \sum_{\sqrt{L}<m<L} \frac{1}{m(L-m)} \\
=\ & \sum_{\sqrt{L}<m<L} \frac{1}{m} + \sum_{\sqrt{L}<m<L} \frac{1}{L-m} \ll \log L
\end{aligned} \tag{3.17}$$

In the same manner we have

$$\sum_{L<m<L^2} \frac{\Lambda(m)}{m \log m |\log L - \log m|} \ll \log L. \tag{3.18}$$

Applying (3.15)–(3.18) to (3.14), we obtain

$$\sum_{Q} \sum_{\substack{N \geq 1, \\ Q^N \neq p^n}} \frac{Q^{-N}}{N |n \log p - N \log Q|} \ll n \log p. \tag{3.19}$$

By (3.19) we see that the fifth sum in the right hand side of (3.12) converges absolutely and locally uniformly in $\mathrm{Re}(s) > 1$ and $w \in \mathbb{C}$.

From the above observation (3.12) holds in $\mathrm{Re}(s) > 2$ and $\mathrm{Re}(w) > 2$ thanks to the coincidence principle. Taking the linear term of the Laurent expansion at $w = 0$ in (3.12), we obtain Theorem 3.[5]

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[5]Strictly speaking, we also need Example 1.3 (1) with $M = p$ and $s \mapsto s-1$ because the Dirichlet series corresponding to (2.7) is slightly different from (3.12).

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