数理解析研究所講究録
第 1806 巻 2012 年 79-93

Joint universality of periodic zeta-functions: continuous and discrete cases

Roma Kačinskaitė
Šiauliai University, Lithuania

Abstract

In this paper, we give a survey on universality theorems of the collection of various zeta-functions, when one of them has an Euler product and other has no. We present some results on both, continuous and discrete, cases.
Keywords and phrases: approximation, limit theorem, periodic sequence, probability measure, space of analytic functions, universality, weak convergence.
AMS classification: 11M41, 11M06, 11M35.

1 Introduction

As usual, by $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ we denote the set of all primes, positive integers, integers, real and complex numbers, respectively, and let $s = \sigma + it$ be a complex variable.

The most important zeta-function is the well-known Riemann zeta-function $\zeta(s)$, for $\sigma > 1$, defined by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

respectively by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1}$$

The investigation of statistical properties of the Riemann zeta-function was initiated by H. Bohr in 1910 and developed by many mathematicians. For example, B. Bagchi, V. Borchsenius, P.D.T.A. Elliott, R. Garunkštis, A. Ghosh, A. Good, J. Ignatavičiūtė, B. Jessen, A. Laurinčikas, K. Matsumoto, H. Mishou, H. Nagoshi, T. Nakamura, A. Selberg, E. Stankus, J. Steuding, R. Steuding (formerly Šleževičienė), W. Schwarz, A. Wintner, and others. For more details, see [18], [19], [26], [27].

Limit theorems we can formulate in the terminology of the weak convergence of probability measures. By $\mathcal{B}(S)$ we denote the family of Borel subsets of the space S. Let P_n and P be probability measures on the space $(S, \mathcal{B}(S))$. We say that P_n converges weakly to P as n tends to infinity if, for all bounded continuous functions $f : S \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_S f \mathrm{d}P_n = \int_S f \mathrm{d}P.$$

Denote by $\mathrm{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and, for $T > 0$, define

$$\nu_T(\ldots) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \ldots\},$$

where in place of dots a condition satisfied by τ is to be written.

We can construct limit theorems in various functional spaces. In this paper, the main attention we devote to the limit theorems in the space of analytic functions.

Let $H(G)$ be the set of all analytic on the region G functions with the topology of uniform convergence on compacta. Let $\{K_j\}$ be a sequence of compact subsets of G such that:

(1) $G = \bigcup_{j=1}^{\infty} K_j$;

(2) $K_j \subset K_{j+1}$ for every $j \in \mathbb{N}$;

(3) if K is compact and $K \subset G$, then $K \subset K_j$ for some $j \in \mathbb{N}$.

Now, for every functions $f, g \in H(G)$, let

$$\rho_j(f,g) = \max_{s\in K_j} |f(s) - g(s)|,$$

and define

$$\rho(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\rho_j(f,g)}{1+\rho_j(f,g)}.$$

Then ρ is a metric on $H(G)$ which induces its topology. It is well-known that the space $H(G)$ is a separable complete metric space [3].

In [1], B. Bagchi proved following statement for the Riemann zeta-function $\zeta(s)$. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$.

Theorem 1 ([1]) *There exists a probability measure Q_H on $(H(D), \mathcal{B}(H(D)))$ such that*

$$\frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

weakly converges to Q_H as $T \to \infty$.

Also, the explicit form of the probability measure is obtained, i.e. it is proved that the probability measure Q_H coincides with the distribution of the random element for the function $\zeta(s)$.

A natural generalization without Euler product of the function $\zeta(s)$ is the Hurwitz zeta-function. Let α be a fixed parameter, $0 < \alpha \leq 1$. The Hurwitz zeta-function $\zeta(s,\alpha)$ in the half-plane $\sigma > 1$ is defined by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and has an analytic continuation to the whole complex plane except a simple pole at $s = 1$ with residue 1. If $\alpha = 1$, then the Hurwitz zeta-function $\zeta(s,1)$ becomes the Riemann zeta-function $\zeta(s)$. On the other hand, when $\alpha \neq 1$ the situations of the study of the value distribution of $\zeta(s,\alpha)$ are completely different according to the arithmetical nature of α. When $\alpha = \frac{a}{q}$, $a,q \in \mathbb{N}$, is rational number $\neq \frac{1}{2}, 1$, the Hurwitz zeta-function can be represented as a sum of Dirichlet L-functions

$$\zeta\left(s,\frac{a}{q}\right) = q^s \sum_{\chi} \overline{\chi}(a) L(s,\chi),$$

where χ runs over the set of Dirichlet characters modulo q. We recall that the Dirichlet L-function $L(s,\chi)$ attached to a character $\chi \mod d$, $d \in \mathbb{N}$, on the half-plane $\sigma > 1$, is given by the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

If χ_0 is the principal character modulo d, then $L(s,\chi_0)$ is analytic for $\sigma > 1$, and, if χ is a non-principal character, then $L(s,\chi)$ is analytic in the half-plane $\sigma > 0$. For $\sigma > 1$, the function $L(s,\chi)$ has the Euler product representation

$$L(s,\chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

When α is a transcendental real number, then the function $\zeta(s,\alpha)$ has no such expression as (1). Instead, it follows from the transcendency of α that the set $\{\log(m+\alpha) : m \in \mathbb{N} \cup \{0\}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. In both cases, some statistical properties of the Hurwitz zeta-function have been obtained (see, for example, B. Bagchi [1], S.M. Gonek [5]).

Also, interesting objects are so called periodic zeta-functions, i.e., the zeta-functions with periodic coefficients.

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic with the least period $k \in \mathbb{N}$ sequence of complex numbers. The periodic zeta-function $\zeta(s;\mathfrak{a})$, for $\sigma > 1$, is defined by the series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere. From the periodicity of sequence $\mathfrak{a}$ follows that, for $\sigma > 1$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta\left(s, \frac{m}{k}\right), \tag{2}$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function. Equality (2) gives an analytic continuation to the whole complex plane for the function $\zeta(s;\mathfrak{a})$, except, maybe for the point $s=1$ with residue

$$a = \frac{1}{k} \sum_{m=1}^{k} a_m.$$

If $a = 0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

Note, if the sequence $\mathfrak{a}$ is completely multiplicative, then the periodic zeta-function $\zeta(s;\mathfrak{a})$ coincides with the Dirichlet L-function (we say that the sequence $\mathfrak{a}$ is completely multiplicative if, for all $m,n \in \mathbb{N}$, the equality $a_{mn} = a_m \cdot a_n$ holds).

The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ with a fixed parameter α, $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s},$$

where $\mathfrak{b} = \{b_m : m \in \mathbb{N} \cup \{0\}\}$ is a periodic sequence of complex numbers b_m with a minimal period $l \in \mathbb{N}$. From the periodicity of $\mathfrak{b}$, for $\sigma > 1$, we have

$$\zeta(s,\alpha;\mathfrak{b}) = \frac{1}{l^s} \sum_{m=0}^{l-1} b_m \zeta\left(s, \frac{m+\alpha}{l}\right).$$

This gives an analytic continuation of the function $\zeta(s,\alpha;\mathfrak{b})$ to the whole complex plane, except, for a simple pole at $s=1$ with residue

$$b = \frac{1}{l} \sum_{m=0}^{l-1} b_m.$$

If $b = 0$, then periodic Hurwitz zeta-function is an entire function.

Many authors, among them A. Javtokas, A. Kačėnas, A. Laurinčikas, R. Macaitienė, J. Steuding, D. Šiaučiūnas, the author and other mathematicians investigated the value distribution of periodic zeta-functions (see [6], [7], [8], [15], [17], [25]).

Functional limit theorems characterize the asymptotic behaviour of the zeta-functions. In [1], B. Bagchi noted that they can be applied to the proof of universality.

In [29], S.M. Voronin proved that every analytic non-vanishing function on compact subsets can be approximated by the shifts of the Riemann zeta-function $\zeta(s)$. Now this property we call as universality.

Theorem 2 ([29]) *Let $0 < r < \frac{1}{4}$, and let $f(s)$ be any non-vanishing continuous function on the disc $|s| \leq r$ which is analytic in the interior of this disc. Then, for every $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right| < \varepsilon.$$

We can state it in modern terminology.

Theorem 3 ([1]) *Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement. Let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|\zeta(s+i\tau)-f(s)| < \varepsilon\right) > 0.$$

Theorem 3 shows that the set of translations of the Riemann zeta-function which approximate a given analytic function $f(s)$ has positive lower density. Because of the uniqueness of factorization in prime numbers, the set $\{\log p : p \text{ is prime}\}$ is linearly independent over $\mathbb{Q}$. This fact and the Euler product representation for $\zeta(s)$ play essential role in the proof of the universality theorem.

The universality property holds for several zeta-functions with Euler product. We mention some results. Concerning zeta-functions over algebraic number fields, A. Reich obtained the universality for Dedekind zeta-functions [24], H. Mishou obtained the universality for Hecke L-functions in the Grössencharacter aspect [20]. Let f be a Hecke eigen-cusp form. If f is holomorphic, the universality property for the automorphic L-function $L(s,f)$ was obtained by A. Laurinčikas and K. Matsumoto [16]. H. Nagoshi proved the universality for $L(s,f)$ in the case where f is a Maass cusp form [23]. Further, A. Laurinčikas [13] investigated the Matsumoto zeta-function, for which he found a condition for the universality.

There exists a conjecture of Linnik-Ibragimov that all functions in some half-plane defined by Dirichlet series, analytically continuable to the left of absolute convergence half-plane and satisfying some natural growth conditions are universal in Voronin sense.

2 Joint value-distribution of different zeta-functions

The first result on joint value-distribution of zeta-functions belongs to S.M. Voronin [28]. He investigated the collection of Dirichlet L-functions with pairwise non-equivalent characters.

More complicated situation we have in the two-dimensional case when one of the zeta-functions has Euler product but the other has no.

2.1 Some joint limit theorems of continuous case

Joint limit theorems in the sense of the weakly convergent probability measures for different zeta-functions were obtained in particular by H. Mishou [21], [22]. He investigated the joint value distribution of the Riemann zeta-function $\zeta(s)$ and the Hurwitz zeta-function $\zeta(s,\alpha)$ with the transcendental parameter α.

In the proof of the limit theorem, the fact that if α is transcendental number, the set

$$\{\log(m+\alpha) : n \in \mathbb{N} \cup \{0\}\} \cup \{\log p : p \text{ is prime}\}$$

is also linearly independent over $\mathbb{Q}$, plays an important role.

Let D_0 be the half-plane $D_0 = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$. Denote by $H^2(D_0)$ the Cartesian product of the spaces of analytic on D_0 functions equipped with the topology of uniform convergence on compact subsets $H(D_0)$, i.e., $H^2(D_0) = H(D_0) \times H(D_0)$.

Let γ be the unit circle on the complex plane, i.e, $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N} \cup \{0\}$. By the Tikhonov theorem, the infinite-dimensional tori Ω_1 and Ω_2 with product topology and pointwise multiplication are compact topological Abelian groups. Then on the space $(\Omega_j, \mathscr{B}(\Omega_j))$ there exists a probability Haar measure m_{Hj}, $j = 1,2$. This leads to a probability space $(\Omega_j, \mathscr{B}(\Omega_j), m_{Hj})$, $j = 1,2$. Let $\Omega = \Omega_1 \times \Omega_2$. Then Ω also is a compact topological Abelian group, and $(\Omega, \mathscr{B}(\Omega), m_H)$ is a probability space, where m_H is the product of Haar measures m_{H1} and m_{H2} on the probability spaces $(\Omega_1, \mathscr{B}(\Omega_1))$ and $(\Omega_1, \mathscr{B}(\Omega_2))$, respectively, i.e., $m_H = m_{H1} \times m_{H2}$. Denote by $\omega_1(p)$ the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_p for any p, and, for any positive integer m, define

$$\omega_1(m) = \prod_{p^g \| m} \omega_1^g(p),$$

where $p^g \| m$ means that $p^g | m$ but $p^{g+1} \nmid m$. Also, denote by $\omega_2(m)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space γ_m for any $m \in \mathbb{N} \cup \{0\}$.

For $\sigma > \frac{1}{2}$ and $(\omega_1, \omega_2) \in \Omega$, we define

$$\underline{Z}(s,\omega) = \big(\zeta(s,\omega_1), \zeta(s,\alpha,\omega_2)\big), \tag{3}$$

where

$$\zeta(s,\omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m)}{m^s} \quad \text{and} \quad \zeta(s,\alpha,\omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

Since, for almost all $\omega \in \Omega$, these series converge uniformly on compact subsets of D_0, $\underline{Z}(s,\omega)$ is an $H^2(D_0)$-valued random element on the probability space $(\Omega, \mathscr{B}(\Omega), m_H)$. Denote by $P_{\underline{Z}}$ the distribution of the random element $\underline{Z}(s,\omega)$, i.e.,

$$P_{\underline{Z}}(A) = m_H\big(\omega \in \Omega : \underline{Z}(s,\omega) \in A\big), \quad A \in \mathscr{B}(H^2(D_0)),$$

and

$$\underline{Z}(s) = \big(\zeta(s), \zeta(s,\alpha)\big).$$

Theorem 4 ([21]) *Suppose that α is transcendental real number such that $0 < \alpha < 1$. Then the probability measure*

$$\nu_T\big(\underline{Z}(s+i\tau) \in A\big), \quad A \in \mathscr{B}(H^2(D_0)),$$

converges weakly to the probability measure $P_{\underline{Z}}$ as $T \to \infty$.

In [12], A. Laurinčikas and the author obtained the joint value distribution of periodic zeta-function and periodic Hurwitz zeta-function [12].

Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote $H^2(D) = H(D) \times H(D)$. Furthermore, define

$$\zeta(s,\omega_1;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}, \quad \omega_1 \in \Omega_1,$$

and

$$\zeta(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}, \quad \omega_2 \in \Omega_2.$$

Since the sequences $\mathfrak{a}$ and $\mathfrak{b}$ (the same as in Introduction) are bounded, by a standard way, using the Rademacher theorem on series of pairwise orthogonal random variables, it can be proved that the series for $\zeta(s,\omega_1;\mathfrak{a})$ and $\zeta(s,\alpha,\omega_2;\mathfrak{b})$ converge uniformly on compact subsets of D for almost all ω_1 and ω_2, respectively, and thus they define $H(D)$-valued random elements on the probability spaces $(\Omega_1, \mathscr{B}(\Omega_1), m_{H1})$ and $(\Omega_2, \mathscr{B}(\Omega_2), m_{H2})$, respectively. Moreover, since the sequence $\mathfrak{a}$ is multiplicative, we have that, for almost all $\omega_1 \in \Omega_1$,

$$\zeta(s,\omega_1;\mathfrak{a}) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{a_{p^k} \omega_1^k(p)}{p^{ks}}\right), \quad s \in D.$$

Let $\omega = (\omega_1, \omega_2)$, and define

$$\underline{\zeta}(s) = \underline{\zeta}(s,\alpha;\mathfrak{a};\mathfrak{b}) = (\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$$

and

$$\underline{\zeta}(s,\omega) = \underline{\zeta}(s,\alpha,\omega;\mathfrak{a};\mathfrak{b}) = (\zeta(s,\omega_1;\mathfrak{a}), \zeta(s,\alpha,\omega_2;\mathfrak{b})).$$

Then $\underline{\zeta}(s,\omega)$ is an $H^2(D)$-valued random element defined on the probability space $(\Omega, \mathscr{B}(\Omega), m_H)$. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s,\omega)$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H(\omega \in \Omega : \underline{\zeta}(s,\alpha,\omega;\mathfrak{a};\mathfrak{b}) \in A), \quad A \in \mathscr{B}(H^2(D)).$$

Theorem 5 ([12]) *Let $\mathfrak{a}$ be a multiplicative periodic sequence and $\mathfrak{b}$ be another periodic sequence. Suppose that α is transcendental. Then the probability measure*

$$\frac{1}{T}\text{meas}(\tau \in [0,T] : \underline{\zeta}(s+i\tau) \in A), \quad A \in \mathscr{B}(H^2(D)),$$

converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.

In [14], A. Laurinčikas studied the joint value distribution of zeta-functions in the multidimensional space of analytic functions for the set of functions $\zeta(s;\mathfrak{a}_1)$, ..., $\zeta(s;\mathfrak{a}_{r_1}), \zeta(s,\alpha_1;\mathfrak{b}_1), ..., \zeta(s,\alpha_{r_2};\mathfrak{b}_{r_2})$.

Let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$, and let $\zeta(s;\mathfrak{a}_j)$ be the corresponding periodic zeta-function, $j = 1,...,r_1$, $r_1 > 1$. Define the matrix

$$B = \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \cdots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_1} & \cdots & a_{r_1\eta_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1\eta_{\phi(k)}} & a_{2\eta_{\phi(k)}} & \cdots & a_{r_1\eta_{\phi(k)}} \end{pmatrix},$$

where coefficients denote the reduced system of residues modulo k by $\eta_1,...,\eta_{\phi(k)}$, and k is the least common multiple of $k_1,...,k_{r_1}$ with Euler function $\phi(k)$. Let $\mathfrak{b}_j = \{b_{jm} : m \in \mathbb{N} \cup \{0\}\}$ be an another periodic sequence of complex numbers with minimal period $l_j \in \mathbb{N}$, and let $\zeta(s,\alpha_j;\mathfrak{b}_j)$ be the corresponding periodic Hurwitz zeta-function with fixed parameter α_j, $0 < \alpha_j \leq 1$.

By $H_{r_1,r_2}(D)$ denote the Cartesian product of $r_1 + r_2$ spaces of analytic functions in D. Let

$$\underline{\Omega} = \Omega_1 \times \hat{\Omega}_1 \times ... \times \hat{\Omega}_{r_2},$$

where $\hat{\Omega}_j = \Omega_2$ for all $j = 1,...,r_2$. Then $\underline{\Omega}$ is a compact topological group, and we obtain the probability space $(\underline{\Omega}, \mathscr{B}(\underline{\Omega}), \underline{m_H})$, where $\underline{m_H}$ is the product of Haar measures m_{H1} and $\hat{m}_{1H},...,\hat{m}_{r_2H}$ with the probability Haar measures $\hat{m}_{jH}$ on $(\hat{\Omega}_j, \mathscr{B}(\hat{\Omega}_j))$, $j = 1,...,r_2$. Denote by $\hat{\omega}_j(m)$ the projection of an element $\hat{\omega}_j \in \hat{\Omega}_j$ to the coordinate space γ_m, $m \in \mathbb{N} \cup \{0\}$, $j = 1,...,r_2$.

Let $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_r)$, $\underline{\omega} = (\omega_1, \hat{\omega}_1, ..., \hat{\omega}_{r_2})$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, ..., \mathfrak{a}_{r_1})$, $\underline{\mathfrak{b}} = (\mathfrak{b}_1, ..., \mathfrak{b}_{r_2})$, and define an $H_{r_1,r_2}(D)$-valued random element $\underline{\zeta}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},\underline{\mathfrak{b}})$ on the probability space $(\underline{\Omega}, \mathscr{B}(\underline{\Omega}), \underline{m_H})$ by the formula

$$\begin{aligned}&\underline{\zeta}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},\underline{\mathfrak{b}}) \\ &\quad = (\zeta(s,\omega_1;\mathfrak{a}_1), ..., \zeta(s,\omega_1;\mathfrak{a}_{r_1}), \zeta(s,\hat{\alpha}_1,\omega_1;\mathfrak{b}_1), ..., \zeta(s,\alpha_{r_2},\hat{\omega}_{r_2};\mathfrak{b}_{r_2})),\end{aligned}$$

where

$$\zeta(s,\omega_1;\mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_1(m)}{m^s}, \quad j = 1,...,r_1,$$

$$\zeta(s,\alpha_j,\hat{\omega}_j;\mathfrak{b}_j)=\sum_{m=0}^{\infty}\frac{b_{jm}\hat{\omega}_j(m)}{(m+\alpha_j)^s},\quad j=1,\ldots,r_2.$$

The distribution of the random element $\underline{\zeta}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},\underline{\mathfrak{b}})$ we denote by

$$P_{\underline{\zeta}}^{H_{r_1,r_2}}=m_H(\underline{\omega}\in\underline{\Omega}:\underline{\zeta}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},\underline{\mathfrak{b}})\in A),\quad A\in\mathscr{B}(H_{r_1,r_2}(D)).$$

Let

$$\underline{\zeta}(s,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}})=\big(\zeta(s;\mathfrak{a}_1),\ldots,\zeta(s;\mathfrak{a}_{r_1}),\zeta(s,\alpha_1;\mathfrak{b}_1),\ldots,\zeta(s,\alpha_{r_2};\mathfrak{b}_{r_2})\big).$$

Theorem 6 ([14]) *Suppose that the sequences $\mathfrak{a}_1,\ldots,\mathfrak{a}_{r_1}$ are multiplicative and the numbers $\alpha_1,\ldots,\alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the measure*

$$\nu_T\big(\underline{\zeta}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}})\in A\big),\quad A\in\mathscr{B}(H_{r_1,r_2}(D))$$

converges weakly to $P_{\underline{\zeta}}^{H_{r_1,r_2}}$ as $T\to\infty$.

2.2 Joint discrete value-distribution

In continuous limit theorems we deal with mathematical objects given by integrals, while in the case of discrete limit theorems, trigonometric and other sums appear. Therefore, discrete theorems are more complicated, they depend on a chosen discrete set used to define relevant probability measures.

For $N\in\mathbb{N}\cup\{0\}$, define

$$\mu_N(\ldots)=\frac{1}{N+1}\sum_{\substack{r=0\\ \ldots}}^{N}1,$$

where in place of dots a condition satisfied by r is to be written.

In [11], D. Korsakienė and the author investigated joint discrete value distribution for the Dirichlet L-function $L(s,\chi)$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ (in this Section and later we use the same notations as before). For $s\in D$, define

$$L(s,\chi,\omega_1)=\sum_{m=1}^{\infty}\frac{\chi(m)\omega_1(m)}{m^s},\quad \omega_1\in\Omega_1,$$

and

$$\zeta(s,\alpha,\omega_2;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m\omega_2(m)}{(m+\alpha)^s},\quad \omega_2\in\Omega_2.$$

For $\omega=(\omega_1,\omega_2)$, we define

$$\underline{\zeta}(s+irh)=\underline{\zeta}(s+irh,\alpha;\chi;\mathfrak{b})=(L(s+irh,\chi),\zeta(s+irh,\alpha;\mathfrak{b})),$$

and

$$\underline{\zeta}(s,\omega)=\underline{\zeta}(s,\alpha,\omega;\chi;\mathfrak{b})=(L(s,\chi,\omega_1);\zeta(s,\alpha,\omega_2;\mathfrak{b})).$$

Then $\underline{\zeta}(s,\omega)$ is an $H^2(D)$-valued random element defined on the probability space $(\Omega, \mathscr{B}(\Omega), m_H)$. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s,\omega)$, i.e.,

$$P_{\underline{\zeta}}^{H^2}(A) = m_H(\omega \in \Omega : \underline{\zeta}(s,\alpha,\omega;\chi;\mathfrak{b}) \in A), \quad A \in \mathscr{B}(H^2(D)).$$

Consider the probability measure

$$P_N(A) = \mu_N(\underline{\zeta}(s+irh) \in A), \quad A \in \mathscr{B}(H^2(D)).$$

Theorem 7 ([11]) *Suppose that α is transcendental. Let $h>0$ be a fixed number such that $\exp\{\frac{2\pi}{h}\}$ is a rational number. Then the probability measure P_N converges weakly to $P_{\underline{\zeta}}^{H^2}$ as $N \to \infty$.*

3 Joint universality theorems

As in the case of joint theorems for the zeta-functions, the joint universality is more complicated problem.

The first result on joint approximation of a given collection of analytic functions by a collection of shifts of zeta-functions belongs to S.M. Voronin [28]. He proved a joint universality for Dirichlet L-functions.

Theorem 8 ([28]) *Let $\chi_1,\dots,\chi_n$ be pairwise non-equivalent Dirichlet characters, and $L(s,\chi_1),\dots,L(s,\chi_n)$ are the corresponding Dirichlet L-functions. For $j=1,\dots,n$, let K_j denote a compact subset of the strip D with connected complement, and $f_j(s)$ be a continuous non-vanishing function on K_j and analytic in the interior of K_j. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\le j\le n}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-f_j(s)|<\varepsilon\right)>0.$$

3.1 Continuous joint universality

As an application of Theorem 4, H. Mishou proved the joint universality theorem for the Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s,\alpha)$ attached to a transcendental parameter α [21].

Theorem 9 ([21]) *Suppose that α is a transcendental number such that $0<\alpha<1$. Let K_1 and K_2 be compact subsets of the strip $\frac{1}{2}<\sigma<1$ with connected complements. Assume that functions $f_j(s)$ are continuous on K_j and analytic in the interior of K_j for each $j=1,2$. In addition, we suppose that $f_1(s)$ does not vanish on K_1. Then, for all positive ε,*

$$\liminf_{T\to\infty} \nu_T\big(\max_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon, \max_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\big)>0.$$

The joint approximation of a given collection of analytic functions by a collection of shifts of periodic zeta-function and periodic Hurwitz zeta-function is obtained by A. Laurinčikas and the author in [12].

Theorem 10 ([12]) *Suppose that α is a transcendental number. Let K_1 and K_2 be a compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, $f_1(s)$ be a continuous non-vanishing function on K_1 which is analytic in the interior of K_1, and let $f_2(s)$ be a continuous function on K_2 which is analytic in the interior of K_2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \right) > 0.$$

The most general result on continuous joint universality of different zeta functions is obtained by A. Laurinčikas in [14].

Theorem 11 ([14]) *Suppose that the sequences $\mathfrak{a}_1, ..., \mathfrak{a}_{r_1}$ are multiplicative,* $\operatorname{rank}(B) = r_1$, *and, for all $p \in \mathbb{P}$, holds the inequality*

$$\sum_{j=1}^{\infty} \frac{|a_{jp^g}|}{p^{g/2}} < 1, \quad j = 1, ..., r_1.$$

Let $\alpha_1, ..., \alpha_{r_2}$ be algebraically independent over $\mathbb{Q}$. Suppose that $K_1, ..., K_{r_1}$ and $\hat{K}_1, ..., \hat{K}_{r_2}$ are compact subsets of the strip D, their complements are connected. Suppose that $f_1(s), ..., f(s)_{r_1}$ are continuous non-vanishing functions in $K_1, ..., K_{r_1}$ and analytic in interior $K_1, ..., K_{r_1}$, and $\hat{f}_1(s), ..., \hat{f}_{r_2}(s)$ are continuous in $\hat{K}_1, ...,$ $\hat{K}_{r_2}$ and analytic in interior $\hat{K}_1, ..., \hat{K}_{r_2}$, respectively. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \Bigg(\sup_{1\le j\le r_1} \sup_{s\in K_j} |\zeta(s+i\tau;\mathfrak{a}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1\le j\le r_2} \sup_{s\in \hat{K}_j} |\zeta(s+i\tau,\alpha_j;\mathfrak{b}_j) - \hat{f}_j(s)| < \varepsilon \Bigg) > 0.$$

The approximation of analytic functions by a collection containing the Riemann zeta-function and periodic Hurwitz zeta-functions is obtained by J. Genys, R. Macaitienė, S. Račkauskainė, D. Šiaučiūnas in [4]. They considered the joint universality of the Riemann zeta-function $\zeta(s)$ and the periodic Hurwitz zeta-functions $\zeta(s, \alpha_j; \mathfrak{b}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$.

Theorem 12 ([4]) *Let $\alpha_1, ..., \alpha_r$ be the same as in* Theorem 11. *Suppose that K_{jl} and f_{jl}, $j = 1, ..., r$, $l = 1, ..., l_j$, satisfies the same hypotheses as $\hat{f}_j(s)$ and $\hat{K}_j$, $j = 1, ..., r_2$, in* Theorem 11, *and let K and f be as K_1 and f_1 in* Theorem 7, *respectively. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \Bigg(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$

$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{b}_{jl})-f_{jl}(s)|<\varepsilon\Big)>0.$$

3.2 Some remarks on discrete universality

In [9], the author obtains joint discrete universality of Dirichlet L-function $L(s,\chi)$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$.

Theorem 13 ([9]) *Suppose that α, K_1, K_2, $f_1(s)$ and $f_2(s)$ are the same as in* Theorem 10. *Let $h>0$ be a fixed number such that* $\exp\{\frac{2\pi}{h}\}$ *is rational. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\mu_N\Big(\sup_{s\in K_1}|L(s+irh,\chi)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+irh,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon\Big)>0.$$

It is possible to generalize Theorem 7 and obtain joint discrete limit theorem in the sense of weakly convergent probability measures in the multidimensional space of analytic functions for the collection of functions $L(s,\chi_1),\dots,L(s,\chi_r)$, $\zeta(s,\alpha;\mathfrak{b})$.

By $\hat{H}(D)$ we denote the Cartesian product of $r+1$ spaces $H(D)$, i.e., $\hat{H}(D)=\underbrace{H(D)\times\dots\times H(D)}_{r+1}$. Let $\hat{\chi}=(\chi_1,\dots,\chi_r)$. On the probability space $(\Omega,\mathscr{B}(\Omega),m_H)$, define an $\hat{H}(D)$-valued random element $\hat{\zeta}(s,\hat{\chi},\alpha,\hat{\omega};\mathfrak{b})$ by

$$\hat{\zeta}(s,\hat{\chi},\alpha,\hat{\omega};\mathfrak{b})=(L(s,\chi_1,\omega_1),\dots,L(s,\chi_r,\omega_1),\zeta(s,\alpha,\omega_2;\mathfrak{b})),$$

where

$$L(s,\chi_j,\omega_1)=\sum_{m=1}^{\infty}\frac{\chi_j(m)\omega_1(m)}{m^s},\quad j=1,\dots,r,$$

is $H(D)$-valued random element defined on the probability space $(\Omega_1,\mathscr{B}(\Omega_1)$, $m_{H1})$. Denote by $P_{\hat{\zeta}}$ the distribution of the random element $\hat{\zeta}(s,\hat{\chi},\alpha,\hat{\omega};\mathfrak{b})$, i.e.,

$$\hat{P}_{\zeta}(A)=m_H(\hat{\omega}\in\Omega:\hat{\zeta}(s,\hat{\chi},\alpha,\hat{\omega};\mathfrak{b})\in A),\quad A\in\mathscr{B}(\hat{H}(D)).$$

We put

$$\hat{\zeta}(s+ilh,\hat{\chi},\alpha;\mathfrak{b})=(L(s+ilh,\chi_1),\dots,L(s+ilh,\chi_r),\zeta(s+ilh,\alpha;\mathfrak{b})).$$

Theorem 14 ([10]) *Suppose that α is a transcendental number such that* $0<\alpha<1$. *Let $h>0$ be a fixed number such that* $\exp\{\frac{2\pi}{h}\}$ *is rational. Suppose that $\chi_1,\dots$, χ_r are pairwise non-equivalent Dirichlet characters, and $L(s,\chi_1),\dots$, $L(s,\chi_r)$ are the corresponding Dirichlet L-functions. Then the probability measure*

$$\mu_N\Big(\hat{\zeta}(s+ikh,\hat{\chi},\alpha;\mathfrak{b})\in A\Big),\quad A\in\mathscr{B}(\hat{H}(D)),$$

weakly converges to the measure $\hat{P}_{\zeta}$ as $N\to\infty$.

The above mentioned theorem can be applied to the proof of the following statement on the universality of collection of Dirichlet L-functions and periodic Hurwitz zeta-function with transcendental parameter α.

Theorem 15 *Suppose that* α, h, $\chi_1,..,\chi_r$, $L(s,\chi_1),\dots,L(s,\chi_r)$ *satisfy the hypotheses of* Theorem 14, *and* $K_1,..,K_r$, $f_1(s),\dots,f_r(s)$ *satisfy the hypothesis of* Theorem 11. *Let* K_{r+1} *be a compact subset of the strip* D *with connected complement, and* $f_{r+1}(s)$ *be a continuous function on* K_{r+1} *which is analytic in the inside of* K_{r+1}. *Let* $h >$ *be a fixed number such that* $\exp\{\frac{2\pi}{h}\}$ *is rational. Then, for every* $\varepsilon > 0$,

$$\liminf_{n\to\infty} \mu_N \Bigg(\sup_{1\le j\le r}\sup_{s\in K_j} |L(s+ikh,\chi_j)-f_j(s)| < \varepsilon,$$
$$\sup_{s\in K_{r+1}} |\zeta(s+ikh,\alpha;\mathfrak{b})-f_{r+1}(s)| < \varepsilon \Bigg) > 0$$

Remark. The discrete universality theorem similar to Theorem 11 can be obtained if we extend the collection of functions noted at begining of this Section, namely to $L(s,\chi_1),\dots,L(s,\chi_{r_1})$, $\zeta(s,\alpha_1;\mathfrak{b}_1),\dots,\zeta(s,\alpha_{r_2};\mathfrak{b}_{r_2})$.

Acknowledgements. The author would like to thank to Professor Kohji Matsumoto for the invitation to Nagoya University and for the warm hospitality during this stay. Also, I would like to thank to Professor Takumi Noda from Nihon University for the possibility to give a lecture at the International Conference "Analytic Number Theory – related Multiple aspects of Arithmetic Functions" at RIMS of Kyoto University, 31 October – 2 November, 2011.

References

[1] B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and other allied Dirichlet Series*, PhD Thesis, Calcutta, Indian Statistical Institute, 1981.

[2] B. Bagchi, A joint universality theorem for Dirichlet L-functions, *Math. Z.*, **181**, 319–334 (1982).

[3] J.B. Conway, *Functions of one Complex Variable I*, 2nd edition, Springer, Berlin, Heidelberg, New York, 1978.

[4] J. Genys, R. Macaitienė, S. Račkauskienė, D. Šiaučiūnas, A mixed joint universality theorem for zeta-functions, *Math. Model. Anal.*, **15**(4), 431–446 (2010).

[5] S.M. Gonek, *Analytic Properties of Zeta and L-Functions*, PhD Thesis, University of Michigan, 1979.

[6] A. Javtokas, A. Laurinčikas, On the periodic Hurwitz zeta-function, *Hardy-Ramanujan J.*, **29**, 18–36 (2006).

[7] A. Kačėnas, A. Laurinčikas, On the periodic zeta-function, *Lith. Math. J.*, **42**(2), 168–177 (2001).

[8] A. Kačėnas, D. Šiaučiūnas, On the periodic zeta-function. III, in: *Analytic and Probabilistic Methods in Number Theory*, Proc. 3rd Intern. Conf., Palanga 24–28 September, 2001, A. Dubickas et al. (eds.), TEV, Vilnius, 99–108, 2002.

[9] R. Kačinskaitė, Joint discrete universality of periodic zeta-functions, *Int. Transf. Spec. Funct.*, **22** (8), 593–601 (2011).

[10] R. Kačinskaitė, On joint discrete value distribution of Dirichlet L-functions and periodic Hurwitz zeta-functions, *Analysis*, submitted.

[11] R. Kačinskaitė, D. Korsakienė, A joint discrete value distribution of periodic zeta-functions, in: Proceed. of the Würzburg Conf. *"New Directions in the Theory of Universality Zeta- and L-functions"*, 6–10 October, 2008, J. Steuding and R. Steuding (eds.), Shaker-Verlag, 99–111, 2009.

[12] R. Kačinskaitė, A. Laurinčikas, The joint distribution of periodic zeta-functions, *Acta Scient. Hungar.*, **48**(2), 257–279 (2011).

[13] A. Laurinčikas, On the Matsumoto zeta-function, *Acta Arith.*, **84**, 1–16 (1998).

[14] A. Laurinčikas, Joint universality of zeta-functions with periodic coefficients, *Izv. Math.*, **74**(3), 515–539 (2010) = *Izv. Ross. Akad. Nauk, Ser. Mat.*, **74**(3), 79–102 (2010) (in Russian).

[15] A. Laurinčikas, R. Macaitienė, On joint universality of periodic zeta functions, *Math. Notes*, **85**(1), 51–60 (2009).

[16] A. Laurinčikas, K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, *Acta Arith.*, **98**(4), 345–359 (2001).

[17] A. Laurinčikas, D. Šiaučiūnas, On the periodic zeta-function. II, *Liet. Matem. Rink.*, **41**(4), 461–476 (2001) (in Russian) = *Lith. Math. J*, **41**(4), 361–372 (2001).

[18] K. Matsumoto, Probabilistic value-distrubtion theory of zeta-functions, *Sugaku Expositions*, **17**(1), 51–71 (2004).

[19] K. Matsumoto, An introduction to the value-distribution theory of zeta-functions, *Šiauliai Math. Semin.*, **1** (**9**), 61–83 (2006).

[20] H. Mishou, The universality theorem for Hecke L-functions, *Acta Arith.*, **110**(1), 45–71 (2003).

[21] H. Mishou, The joint value-distribution of the Riemann zeta-function and Hurwitz zeta-functions, *Liet. Matem. Rink.*, **47**(1), 62–807 (2007) = *Lith. Math. J.*, **47**(1), 32–47 (2007).

[22] H. Mishou, *The joint value distribution of the Riemann zeta-function and Hurwitz zeta-functions. II*, Arch. Math., **90**(3), 230–238 (2008).

[23] H. Nagoshi, On the universality for L-functions attached to Maass forms, *Analysis (Munich)*, **25**(1), 1–22 (2005).

[24] A. Reich, Zur Universalitt und Hypertranszendenz der Dedekindschen Zetafunktion, *Abh. Braunschweig. Wiss. Ges.*, **33**, 197–203 (1982).

[25] J. Steuding, On Dirichlet series with periodic coefficients, *Ramanujan J.*, **6**, 295–306 (2002).

[26] J. Steuding, *Value-Distribution of L-functions*, Springer-Verlag, Berlin, Heidelberg, New York, 2007.

[27] J. Steuding, The mathematical work of Antanas Laurinčikas – an internim report, *Šiauliai Math. Semin.*, **3** (**11**), 7–51 (2008).

[28] S.M. Voronin, On the functional independence of Dirichlet L-functions, *Acta Arith.*, **27**, 493–503 (1975) (in Russian).

[29] S.M. Voronin, Theorem on the "universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **39**, 475–486 (1975) (in Russian) = *Math. USSR Izv.*, **9**, 443–453 (1975).