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COMPLETE ASYMPTOTIC EXPANSIONS ASSOCIATED WITH EPSTEIN ZETA-FUNCTIONS II (SUMMARIZED VERSION)

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ABSTRACT. The present article announces the results in the forthcoming paper [Ka13].
Let $Q(u,v)=|u+vz|^2$ be a positive-definite quadratic form with a complex parameter $z=x+iy$ in the upper half-plane. The Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s;z)$ attached to Q is initially defined by (1.3) below. We have established in the preceding paper [Ka10] complete asymptotic expansions of $\zeta_{\mathbb{Z}^2}(s;x+iy)$ as $y\to+\infty$, and those of its weighted mean value (with respect to y) in the form of a Laplace-Mellin transform (1.4). The forthcoming paper [Ka13] proceeds further with our previous study to show that similar asymptotic series still exist for a more general Epstein zeta-function $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ defined by (1.2) below (see Theorem 1), and also for the Riemann-Liouville transform (1.5) of $\zeta_{\mathbb{Z}^2}(s;z)$ (see Theorem 2). Prior to the proofs of these asymptotic expansions, the meromorphic continuation of $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ over the whole s-plane is prepared by means of Mellin-Barnes integral transforms (see Proposition 1 in Section 3). This procedure, differs slightly from other previously known methods of analytic continuation, provides the meromorphic continuation of $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ in the form of a double infinite series (see (2.9) with (3.8) and (3.9)), which is most appropriate for deriving the asymptotic expansions in question. The use of Mellin-Barnes type integrals is crucial in all aspects of the proofs; several transformation and connection formulae for hypergeometric functions are especially applied with manipulation of these integrals.

1. INTRODUCTION

Throughout the present article, $s=\sigma+it$ is a complex variable, $z=x+iy$ a complex parameter in the upper half-plane, a, b, μ and ν real parameters, and the notation $e(s)=e^{2\pi is}$ is frequently used. Let $Q(u,v)=Au^2+2Buv+Cv^2$ be a real positive-definite quadratic form with $A>0$ and $AC-B^2=D>0$. The main object of the present article is the generalized Epstein zeta-function $\psi_Q(s;a,b;\mu,\nu)$ (attached to Q) initially defined by

$$\psi_Q(s;a,b;\mu,\nu)=\sum_{m,n=-\infty}^{\infty}{}' e((a+m)\mu+(b+n)\nu))Q(a+m,b+n)^{-s} \tag{1.1}$$

for $\operatorname{Re}s>1$, where (and in the sequel) the primed summation symbols indicate omission of the (possibly emerging) singular terms of the meaningless form 0^{-s}. This is normalized

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as

$$\psi_Q(s;a,b;\mu,\nu) = (y/\sqrt{D})^s \psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z),$$

where $x = B/A$, $y = \sqrt{D}/A$ and

$$\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) = \sum_{m,n=-\infty}^{\infty}{}' e((a+m)\mu + (b+n)\nu)|a+m+(b+n)z|^{-2s} \tag{1.2}$$

for $\operatorname{Re} s > 1$. Here (1.2) is the generalized Epstein zeta-function attached to the quadratic form $|u+vz|^2 = u^2 + 2(\operatorname{Re} z)uv + |z|^2 v^2$. We can therefore concentrate our study upon (1.2), instead of treating more general (1.1); the particular case $a, b, \mu, \nu \in \mathbb{Z}$ of (1.2) reduces to

$$\zeta_{\mathbb{Z}^2}(s;z) = \sum_{m,n=-\infty}^{\infty}{}' |m+nz|^{-2s} \qquad (\operatorname{Re} s > 1). \tag{1.3}$$

It is of importance from both theoretical and applicable point of views to study (1.3) and its generalizations. We cite here several of such instances. Chowla-Selberg [CS1][CS2] first established the Fourier series expansion of (1.3) for the purpose of applying it to study arithmetics of quadratic forms, while the twisted analogues (with respect to Dirichlet characters) of (1.3) was treated by Stark [St]. An extension of (1.3) to a form of matrix variables was investigated by Terras [Te]. Goldstein [Go] applied a generalization of (1.3) to give a new proof of a Kronecker limit formula for an arbitrary number field K, while another type of extensions of (1.3) was applied by Kohnen [Ko] to study certain Dirichlet series defined by a pair of Siegel cusp forms.

As for asymptotic aspects of various Eisenstein series, Matsumoto [Ma] obtained complete asymptotic expansions (with respect to z) of *holomorphic* Eisenstein series, while Noda [No] studied an asymptotic formula (as $t \to +\infty$) for the *non-holomorphic* Eisenstein series $E_0(s;z)$ (of weight 0). The author has established in the preceding paper [Ka10] complete asymptotic expansions for (1.3) in the descending order of $y = \operatorname{Im} z$ as $y \to +\infty$, and also those for the Laplace-Mellin transform of (1.3) (given in the form (1.4) below) as $Y \to +\infty$. Our preceding main formula in [Ka10, Theorem 1] is readily switched to that for $E_0(s;z)$ by the relation $E_0(s;z) = y^s \zeta_{\mathbb{Z}^2}(s;z)/2\zeta(s)$, where $\zeta(s)$ denotes the Riemann zeta-function; this can further be transferred to complete asymptotic expansions as $y \to +\infty$ for $E_k(s;z)$ (of any even weight k) by Katsurada-Noda [KN] upon using Maaß' weight change operators.

The forthcoming paper [Ka13] proceeds further with our previous study [Ka10] to explore asymptotic aspects of (1.2) when $y = \operatorname{Im} z \to +\infty$, and also of the Riemann-Liouville (or the Erdélyi-Köber) transform of (1.3) (given in the form (1.5) below) as $Y \to +\infty$. We have developed in [Ka10] a method of treating (1.3) by means of Mellin-Barnes type integrals (see for e.g. [Ka10, Formula (3.2) with (3.3)]). It is in fact possible to extend this method for investigating more general (1.2); this eventually leads us to establish complete asymptotic expansions of (1.2) in the descending order of y as $y \to +\infty$ (see Theorem 1) with explicit (vertical) t-estimates for the remainder terms (see (2.14)). The proof of this theorem in particular clarifies the key ingredients by which the functional equation of (1.2) is to be valid (see (2.16), (2.17) and Corollary 1.1). Moreover, the limiting case $s \to 1$ of our main formula (2.9) with (2.10)–(2.13) allows us to unify the (classical) first and second limit formulae of Kronecker (see (2.20)), while other specific cases naturally reduce to several of their variants for $\psi_{\mathbb{Z}^2}(m;0,0;\mu,\nu;z)$ $(m = 2, 3, \ldots)$ and $\psi'_{\mathbb{Z}^2}(-n;a,b;0,0;z)$ $(n = 0, 1, \ldots)$ (Corollaries 1.2 and 1.4), where the prime on $\psi_{\mathbb{Z}^2}$ indicates $\partial/\partial s$. It is further shown that our main formula (2.9) can be switched to a complete asymptotic expansion of (1.2) as $z \to 0$ through the sector $0 < \arg z < \pi$ (see Corollary 1.5) upon using the quasi-modular relation (2.26) below.

Prior to the proof of Theorem 1, the analytic continuation of (1.2) over the whole s-plane is prepared (see Proposition 1) by means of the double sum in (3.9), which is most appropriate for establishing our main formula (2.9); crucial rôles here are played by the Mellin-Barnes type integral expression (3.2) with (3.3). We remark that the integrals of the type as in (3.3) have advantage over heuristic treatments in studying certain asymptotic aspects and transformation properties of zeta and theta functions (see for e.g. [Ka1]–[Ka9][Ka11][Ka12]).

Next let α and β be complex parameters specified later, and $\Gamma(s)$ the gamma function. We introduce here the Laplace-Mellin and the Riemann-Liouville (or the Erdélyi-Köber) transforms of (1.3), given respectively by

$$\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \zeta_{\mathbb{Z}^2}(s;x+iyY)y^{\alpha-1}e^{-y}dy \tag{1.4}$$

and

$$\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \zeta_{\mathbb{Z}^2}(s;x+iyY)y^{\alpha-1}(1-y)^{\beta-1}dy \tag{1.5}$$

for any $Y > 0$, with the normalization gamma multiples. Note that the integrals in (1.4) and (1.5) can be viewed as weighted mean values of $\zeta_{\mathbb{Z}^2}(s;x+iy)$ in terms of y; the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integrals as $y \to +0$, while e^{-y} and $(1-y)^{\beta-1}$ have effects to extract the portions of $\zeta_{\mathbb{Z}^2}(s;z)$ corresponding to $y = O(Y)$. Furthermore the *confluence* operation

$$\mathcal{RL}^{\alpha,\beta}_{y;\beta Y}\zeta_{\mathbb{Z}^2}(s;x+iy) \xrightarrow[(\beta\to+\infty)]{} \mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) \tag{1.6}$$

is valid by the definitions in (1.4) and (1.5), since $\zeta_{\mathbb{Z}^2}(s;x+iy) = O(y^{\max(0,1-2\sigma)})$ holds as $y \to +\infty$ (see [Ka10, Theorem 1]).

It is expected from [Ka10, Theorem 2] that an asymptotic expansion similar to those of (1.3) and (1.4) still exists for the Riemann-Liouville transform (1.5), together with an improved t-estimate of the remainder term. The use of a Mellin-Barnes transform technique in fact allows us to establish this asymptotic expansion (see Theorem 2) with an expected t-estimate (see (2.38)). It is worth-while noting (in comparison with (2.14)) that a sharp t-bound can be obtained here by virtue of the weighted averaging with respect to y in (1.5), instead at the cost of loosing sharp y-bounds for the remainder terms. We further remark that the remainder term of the asymptotic expansion for (1.5) is expressible by means of the generalized hypergeometric function ${}_2F_3$ (see (2.35) and (2.36)); it is contrasted to the case of (1.4), where Gauß' hypergeometric function ${}_2F_1$ appears in the explicit formula for the remainder term. One can moreover see that the asymptotic expansion for (1.5) reduces precisely to that of (1.4) through the *confluence* operation (1.6) (see Corollary 2.1).

The present article is organized as follows. The main results (Theorems 1 and 2) of the forthcoming paper [Ka13], together with their corollaries, are stated in the next section. As for the derivation we here content ourselves only to prepare the analytic continuation of (1.2), and to prove Theorem 1 except the estimate (2.14), because their further details will appear in [Ka13].

2. Statement of results

We prepare several notations and terminology for describing our main results.

The symbol $A \ll B$ hereafter means that $A = O(B)$ holds, and $A \asymp B$ that both $A \ll B$ and $A \gg B$. Let c, d, κ and λ be real parameters with $c \geq 0$ and $d \geq 0$ in what follows. We first introduce the Lerch zeta-function $\phi(s, c, \kappa)$, together with its companion $\psi(s, c, \kappa)$, defined by

$$\phi(s,c,\kappa) = \sum_{k=0}^{\infty}{}' e(k\kappa)(c+k)^{-s} \qquad (\operatorname{Re} s > 1), \tag{2.1}$$

$$\psi(s,c,\kappa) = \sum_{k=0}^{\infty}{}' e((c+k)\kappa)(c+k)^{-s} = e(c\kappa)\phi(s,c,\kappa), \tag{2.2}$$

which can be continued to entire functions if $\kappa \in \mathbb{R} \setminus \mathbb{Z}$, while for $\kappa \in \mathbb{Z}$ the former (or for $\kappa = 0$ the latter) reduces to the Hurwitz zeta-function $\zeta(s, c)$, also for $\kappa \in \mathbb{R}$ and $c = 1$ to the exponential zeta-function $\zeta_\kappa(s) = e(\kappa)\phi(s, 1, \kappa) = \psi(s, 1, \kappa)$, and hence to the Riemann zeta-function $\zeta(s) = \zeta(s, 1) = \zeta_\kappa(s)$ if $\kappa \in \mathbb{Z}$. Note that

$$\phi(s,0,\kappa) = \phi(s,1,\kappa) \qquad \text{and} \qquad \psi(s,0,\kappa) = \psi(s,1,\kappa) \tag{2.3}$$

hold by the convention of primed summation symbols; this also asserts that $\zeta_\kappa(s) = \phi(s, 0, \kappa) = \psi(s, 0, \kappa)$ for $\kappa \in \mathbb{R}$. We further use the bilateral extensions of (2.1) and (2.2) defined respectively by

$$\phi_{\mathbb{Z}}(s,a,\lambda) = \sum_{l=-\infty}^{\infty}{}' e(l\lambda)|a+l|^{-s} \qquad (\operatorname{Re} s > 1), \tag{2.4}$$

$$\psi_{\mathbb{Z}}(s,a,\lambda) = \sum_{l=-\infty}^{\infty}{}' e((a+l)\lambda)|a+l|^{-s} = e(a\lambda)\phi_{\mathbb{Z}}(s,a,\lambda). \tag{2.5}$$

We next introduce

$$\begin{aligned} \Phi_{r,s}(c,d;\kappa,\lambda;e(z)) = \sum_{k,l=0}^{\infty}{}' & e((c+k)\kappa + (d+l)\lambda) \\ & \times (c+k)^r (d+l)^s e((c+k)(d+l)z) \end{aligned} \tag{2.6}$$

(with the possible replacement of z by $-\overline{z}$), where the series converges absolutely for all $(r, s) \in \mathbb{C}^2$ if $\operatorname{Im} z > 0$, and for $\operatorname{Re} r < -1$ and $\operatorname{Re} s < -1$ if $\operatorname{Im} z = 0$, since $|e(z)| = |e(-\overline{z})| = e^{-2\pi y}$; in each case it defines a holomorphic function of (r, s) in the region of absolute convergence. The particular case $c = d = 0$ and $\kappa, \lambda \in \mathbb{Z}$ of (2.6) is to be written shortly as

$$\Phi_{r,s}(e(z)) = \sum_{k,l=1}^{\infty} k^r l^s e(klz), \tag{2.7}$$

which was first introduced and studied by Ramanujan [Ra] (see also [Be]) in connection with various closed form evaluations of the holomorphic Eisenstein series $E_k(z)$ ($k = 2, 4, 6$).

Further, let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n be the shifted factorial of s, write

$$\Gamma\binom{\alpha_1, \ldots, \alpha_m}{\beta_1, \ldots, \beta_n} = \frac{\prod_{h=1}^m \Gamma(\alpha_h)}{\prod_{k=1}^n \Gamma(\beta_k)}$$

for complex numbers α_h and β_k ($h = 1, \dots, m$; $k = 1, \dots, n$), and introduce the confluent hypergeometric function $U(\alpha;\gamma;Z)$ defined by

$$U(\alpha;\gamma;Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha)-1\}} \int_{\mathcal{C}} e^{-wZ} w^{\alpha-1}(1+w)^{\gamma-\alpha-1} dw \tag{2.8}$$

for $|\arg Z| < \pi/2$, where the path $\mathcal{C}$ starts from ∞, proceeds along the real axis to a small positive ε, rounds the origin counter-clockwise, and returns to ∞ along the real axis; $\arg w$ varies from 0 to 2π along $\mathcal{C}$ (cf. [Er1, p.273, Chap.6, 6.11.2(9)]; the notation $U(\alpha;\gamma;Z)$, due to Slater [Sl], is rather preferable to our later use). We finally denote by $\langle X \rangle = X - [X]$ the fractional part of $X \in \mathbb{R}$, and define the symbols $\delta(X)$ and $\widehat{Y}$ for $Y \in \mathbb{R}$ by

$$\delta(X) = \begin{cases} 1 & \text{if } X \in \mathbb{Z}; \\ 0 & \text{if } X \notin \mathbb{Z}, \end{cases} \qquad \text{and} \qquad \widehat{Y} = \begin{cases} Y & \text{if } Y > 0; \\ 1 & \text{if } Y = 0; \\ 0 & \text{if } Y < 0, \end{cases}$$

respectively.

We now state our first main result.

Theorem 1. *Let $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ be defined by (1.2), where a, b, μ and ν are arbitrary real parameters. Then the formula*

$$\begin{aligned} &\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) \\ &= \delta(b)\psi_{\mathbb{Z}}(2s,a,\mu) + \delta(\mu)e(a\mu)\sqrt{\pi}\Gamma\binom{s-1/2}{s}\psi_{\mathbb{Z}}(2s-1,b,\nu)y^{1-2s} \\ &\quad + \psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) \end{aligned} \tag{2.9}$$

holds, where the last term splits into four idempotent parts as

$$\begin{aligned} &\psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) \\ &= e(a\mu)\frac{(2\pi)^{2s}}{\Gamma(s)}\{T(s;1-\langle\mu\rangle,\langle b\rangle;a,\nu;z) + T(s;\langle\mu\rangle,1-\langle b\rangle;-a,-\nu;z) \\ &\quad + T(s;\langle\mu\rangle,\langle b\rangle;-a,\nu;-\overline{z}) + T(s;1-\langle\mu\rangle,1-\langle b\rangle;a,-\nu;-\overline{z})\}. \end{aligned} \tag{2.10}$$

Here the T's are further represented for any integer $N \geq 0$ as

$$T(s;c,d;\kappa,\lambda;z) = S_N(s;c,d;\kappa,\lambda;z) + R_N(s;c,d;\kappa,\lambda;z), \tag{2.11}$$

in the region $-N < \sigma < N+1$, where

$$\begin{aligned} S_N(s;c,d;\kappa,\lambda;z) = \sum_{n=0}^{N-1} &\frac{(-1)^n(s)_n(1-s)_n}{n!} \\ &\times \Phi_{s-n-1,-s-n}(c,d;\kappa,\lambda;e(z))(4\pi y)^{-s-n} \end{aligned} \tag{2.12}$$

is the asymptotic series in the descending order of y as $y \to +\infty$; the remainder term R_N is expressed as

$$\begin{aligned} R_N(s;c,d;\kappa,\lambda;z) = &\frac{(-1)^N(s)_N(1-s)_N}{(N-1)!} \\ &\times \sum_{k,l=0}^{\infty}{}' e((c+k)(d+l)z)e((c+k)\kappa+(d+l)\lambda)(c+k)^{2s-1} \\ &\times \int_0^1 \xi^{-s-N}(1-\xi)^{N-1}U(s+N;2s;4\pi(c+k)(d+l)y/\xi)d\xi \end{aligned} \tag{2.13}$$

for any $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the ξ-integration), and it satisfies the estimate

$$R_N(s; c, d; \kappa, \lambda; z) = O\{(|t|+1)^{2N} e^{-2\pi\hat{c}\hat{d}y} y^{-\sigma-N}\} \tag{2.14}$$

for any $y \geq y_0 > 0$ in the same region of s above, where the implied O-constant depends at most on N, σ, c, d and y_0. Here z may be replaced by $-1/\bar{z}$ in (2.11)–(2.13).

Remark. The largest order term of the sum in (2.6) is isolated to give

$$\begin{aligned} \Phi_{r,s}(c, d; \kappa, \lambda; e(z)) &= e(\hat{c}\kappa + \hat{d}\lambda)\hat{c}^r \hat{d}^s e(\hat{c}\hat{d}z) + O\{e^{-2\pi(\hat{c}+1)\hat{d}y}\} + O\{e^{-2\pi\hat{c}(\hat{d}+1)y}\} \\ &\asymp e^{-2\pi\hat{c}\hat{d}y} \qquad (\text{as } y \to +\infty), \end{aligned} \tag{2.15}$$

and therefore the n-th indexed term of the asymptotic series in (2.12) is of order $\asymp (|t|+1)^{2n} e^{-2\pi\hat{c}\hat{d}y} y^{-\sigma-n}$; this shows that the presence of the bound in (2.14) is reasonable.

We introduce here the functions $\phi_{\mathbf{Z}^2}$ and $\phi^*_{\mathbf{Z}^2}$ (accompanied with $\psi_{\mathbf{Z}^2}$ and $\psi^*_{\mathbf{Z}^2}$) defined respectively by

$$\begin{aligned} \psi_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z) &= e(a\mu + b\nu)\phi_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z), \\ \psi^*_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z) &= e(a\mu + b\nu)\phi^*_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z). \end{aligned} \tag{2.16}$$

Then the Mellin-Barnes integral expression (4.6) below (see (2.10)) in fact shows that the following functional equation of $\psi_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z)$ reduces eventually to the primary symmetry

$$\Phi_{r,s}(c, d; \kappa, \lambda; e(z)) = \Phi_{s,r}(d, c; \lambda, \kappa; e(z)). \tag{2.17}$$

Corollary 1.1. *For any real a, b, μ and ν, and any complex z with $y = \operatorname{Im} z > 0$ the functional equation*

$$\Gamma(s)(y/\pi)^s \psi^*_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z) = \Gamma(1-s)(y/\pi)^{1-s}\phi^*_{\mathbf{Z}^2}(1-s; -\nu, \mu; b, -a; z) \tag{2.18}$$

holds, and this with the functional equation of $\psi_{\mathbf{Z}}(s, a, \lambda)$ (see (3.5) below) implies that

$$\Gamma(s)(y/\pi)^s \psi_{\mathbf{Z}^2}(s; a, b; \mu, \nu; z) = \Gamma(1-s)(y/\pi)^{1-s}\phi_{\mathbf{Z}^2}(1-s; -\nu, \mu; b, -a; z).$$

We next present (in a unified form) the first and the second limit formulae of Kronecker, together with their several variants. For this let $\gamma_0 = -\Gamma'(1)$ be the $0th$ Euler constant, set

$$w_1 = -\nu + \langle\mu\rangle z \qquad \text{and} \qquad c_1(\nu) = \begin{cases} \gamma_0 & \text{if } \nu \in \mathbf{Z}; \\ -\log|2\sin\pi\nu| & \text{if } \nu \notin \mathbf{Z} \end{cases}$$

and further let $B_k(x)$ $(k = 0, 1, \ldots)$ denote Bernoulli polynomials (cf. [Er1, p.36, 1.13(2)]), and $\eta(z)$ and $\vartheta_1(w, z)$ the Dedekind eta and the elliptic theta functions (cf. [Si, p.15, Sect.2; p.30, Sect.3]) defined respectively by

$$\begin{aligned} \eta(z) &= e(z/24)\prod_{l=1}^{\infty}\{1 - e(lz)\} \\ \vartheta_1(w, z) &= 2\sin(\pi w)e(z/8)\prod_{l=1}^{\infty}\{1 - e(w + lz)\}\{1 - e(-w + lz)\}\{1 - e(lz)\}. \end{aligned} \tag{2.19}$$

We will frequently write $e(z) = q$ and hence $e(-\bar{z}) = \overline{e(z)} = \bar{q}$ for brevity.

Corollary 1.2. *For any real μ and ν, and any complex $z = x + iy$ with $y > 0$, the following formulae hold:*

i) *when* $s \to 1$,

$$\lim_{s\to 1}\left\{\psi_{\mathbb{Z}^2}(s;0,0;\mu,\nu;z) - \frac{\delta(\mu)\delta(\nu)\pi/y}{s-1}\right\} \tag{2.20}$$
$$= 2\pi^2 B_2(\langle\mu\rangle) - \frac{2\pi}{y}\{\delta(\mu)\delta(\nu)\log(2y) - \delta(\mu)c_1(\nu)\} + \psi^*_{\mathbb{Z}^2}(1;0,0;\mu,\nu;z)$$
$$= 2\pi^2\langle\mu\rangle^2 - \frac{2\pi}{y}\left\{\delta(\mu)\delta(\nu) - \delta(\mu)c_1(\nu) - \log\left|\frac{\vartheta_1(w_1,z)}{(2\sin\pi\nu)^{\delta(\mu)}\eta(z)}\right|\right\};$$

ii) *when* $s = m$ $(m = 2, 3, \ldots)$,

$$\psi_{\mathbb{Z}^2}(m;0,0;\mu,\nu;z) \tag{2.21}$$
$$= \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}(\langle\mu\rangle)}{(2m)!} + \delta(\mu)\frac{(2m-2)!\pi}{\{2^{m-1}(m-1)!\}^2}$$
$$\times\{\zeta_\nu(2m-1) + \zeta_{-\nu}(2m-1)\}y^{1-2m} + \psi^*_{\mathbb{Z}^2}(m;0,0;\mu,\nu;z).$$

Here, in both the cases above,

$$\psi^*_{\mathbb{Z}^2}(m;0,0;\mu,\nu;z) = \frac{(2\pi)^{2m}}{\{(m-1)!\}^2}\sum_{n=0}^{m-1}\binom{m-1}{n}(m-n-1)! \tag{2.22}$$
$$\times\{\Phi_{m-n-1,-m-n}(1-\langle\mu\rangle,0;0,\nu;q) + \Phi_{m-n-1,-m-n}(\langle\mu\rangle,1;0,-\nu;q)$$
$$+ \Phi_{m-n-1,-m-n}(\langle\mu\rangle,0;0,\nu;\overline{q}) + \Phi_{m-n-1,-m-n}(1-\langle\mu\rangle,1;0-\nu;\overline{q})\}.$$

Remark. The last logarithmic term in (2.20) when $(\mu,\nu)\in\mathbb{Z}^2$ is interpreted to be the 'vertical' limit

$$\lim_{(\mu,\nu')\to(\mu,\nu)\in\mathbb{Z}^2}\log\left|\frac{\vartheta_1(\nu',z)}{(2\sin\pi\nu')^{\delta(\mu)}\eta(z)}\right| = 2\log|\eta(z)|$$

(see (2.19)); a similar interpretation is also applied for that of (2.23) below when $(a,b)\in\mathbb{Z}^2$ upon the 'horizontal' limiting $(a',b)\to(a,b)\in\mathbb{Z}^2$.

Corollary 1.3. *For any integer* $m \geq 0$, *any real* a *and* b, *and any complex* $z = x + iy$ *with* $y > 0$, *we have*

$$\psi_{\mathbb{Z}^2}(-m;a,b;0,0;z) = \begin{cases} -\delta(a)\delta(b) & \text{if } m = 0;\\ 0 & \text{if } m \geq 1.\end{cases}$$

To state further variants of (2.20) we set

$$w_0 = a + \langle b\rangle z \qquad\text{and}\qquad c_0(a) = \begin{cases} -\log(2\pi) & \text{if } a\in\mathbb{Z};\\ -\log|2\sin\pi a| & \text{if } a\notin\mathbb{Z}.\end{cases}$$

Corollary 1.4. *For any real* a *and* b, *and any complex* $z = x+iy$ *with* $y > 0$, *the following formulae hold:*

i) *when* $s = 0$,

$$\psi'_{\mathbb{Z}^2}(0;a,b;0,0;z) = 2\delta(b)c_0(a) + 2\pi B_2(\langle b\rangle)y + (\psi^*_{\mathbb{Z}^2})'(0;a,b;0,0;z) \tag{2.23}$$
$$= 2\delta(b)c_0(a) + 2\pi\langle b\rangle^2 y - 2\log\left|\frac{\vartheta_1(w_0,z)}{(2\sin\pi a)^{\delta(b)}\eta(z)}\right|;$$

ii) *when* $s=-m$ $(m=1,2,\ldots)$,

$$\psi'_{\mathbf{Z}^2}(-m;a,b;0,0;z)=\delta(b)\frac{(-1)^m(2m)!}{(2\pi)^{2m}}\{\zeta_a(2m+1)+\zeta_{-a}(2m+1)\} + \frac{2\pi(2^m m!)^2B_{2m+2}(\langle b\rangle)}{(2m+1)!(m+1)}y^{2m+1}+(\psi^*_{\mathbf{Z}^2})'(-m;a,b;0,0;z). \tag{2.24}$$

Here, in both the cases above,

$$(\psi^*_{\mathbf{Z}^2})'(-m;a,b;0,0;z)=\frac{(-1)^m}{(2\pi)^{2m}}\sum_{n=0}^{m}\binom{m}{n}(m+n)! \times\{\Phi_{-m-n-1,m-n}(1,\langle b\rangle;a,0;q)+\Phi_{-m-n-1,m-n}(0,1-\langle b\rangle;-a,0;q) + \Phi_{-m-n-1,m-n}(0,\langle b\rangle;-a,0;\overline{q})+\Phi_{-m-n-1,m-n}(1,1-\langle b\rangle;a,0;\overline{q})\}. \tag{2.25}$$

We next present an asymptotic expansion of (1.2) as $z\to 0$ through $0<\arg z<\pi$. For this it is convenient to set $z=i\tau$ with $|\arg\tau|<\pi/2$. Then one can see from (1.2) the quasi-modular relation

$$\psi_{\mathbf{Z}^2}(s;a,b;\mu,\nu;i\tau)=|\tau|^{-2s}\psi_{\mathbf{Z}^2}(s;-b,a;-\nu,\mu;i/\tau), \tag{2.26}$$

from which our main formula (2.9) with (2.10)–(2.13) is readily switched to the following.

Corollary 1.5. *For any real* a, b, μ *and* ν, *and any complex* τ *with* $|\arg\tau|<\pi/2$, *the formula*

$$\psi_{\mathbf{Z}^2}(s;a,b;\mu,\nu;i\tau)=\delta(a)\psi_{\mathbf{Z}}(2s,-b,-\nu)|\tau|^{-2s}+\delta(\nu)e(b\nu)\sqrt{\pi}\Gamma\binom{s-1/2}{s} \times\psi_{\mathbf{Z}}(2s-1,a,\mu)\cos^{1-2s}(\arg\tau)|\tau|^{-1}+\psi^*_{\mathbf{Z}^2}(s;-b,a;-\nu,\mu;i/\tau)|\tau|^{-2s} \tag{2.27}$$

holds, where

$$\psi^*_{\mathbf{Z}^2}(s;-b,a;-\nu,\mu;i/\tau)=e(b\nu)\frac{(2\pi)^{2s}}{\Gamma(s)} \times\{T(s;1-\langle-\nu\rangle,\langle a\rangle;-b,\mu;i/\tau)+T(s;\langle-\nu\rangle,1-\langle a\rangle;b,-\mu;i/\tau) + T(s;\langle-\nu\rangle,\langle a\rangle;b,\mu;i/\overline{\tau})+T(s;1-\langle-\nu\rangle,1-\langle a\rangle;-b,-\mu;i/\overline{\tau})\}. \tag{2.28}$$

Here the T*'s are represented for any integer* $N\geq 0$ *as*

$$T(s;c,d;\kappa,\lambda;i/\tau)=S_N(s;c,d;\kappa,\lambda;i/\tau)+R_N(s;c,d;\kappa,\lambda;i/\tau) \tag{2.29}$$

in the region $-N<\sigma<N+1$, *where*

$$S_N(s;c,d;\kappa,\lambda;i/\tau)=\sum_{n=0}^{N-1}\frac{(-1)^n(s)_n(1-s)_n}{n!} \times\Phi_{s-n-1,-s-n}(c,d;\kappa,\lambda;e^{-2\pi/\tau})\{|\tau|/4\pi\cos(\arg\tau)\}^{s+n} \tag{2.30}$$

is the asymptotic series in the ascending order of τ *as* $\tau\to 0$ *through* $|\arg\tau|<\pi/2$; *the remainder term* R_N *is expressed as in* (2.13) *with* (y,z) *replaced by* $(\cos(\arg\tau)/|\tau|,i/\tau)$, *and it satisfies the estimate*

$$R_N(s;c,d;\kappa,\lambda;e^{-2\pi/\tau})=O(|\tau|^{\sigma+N}e^{-2\pi\sin\delta/|\tau|}) \tag{2.31}$$

as $\tau\to 0$ *through the sector* $|\arg\tau|\leq\pi/2-\delta$ *with any small* $\delta>0$, *where the constant implied in the O-symbol depends at most on* N, s, c, d *and* δ. *Here* τ *may be replaced by* $\overline{\tau}$ *in* (2.29)–(2.31).

We proceed to state our second main result. For this we set $e^*(z) = e(z) + \overline{e(z)} = e(z) + e(-\overline{z})$ and

$$\Phi^*_{r,s}(e(z)) = \Phi_{r,s}(e(z)) + \Phi_{r,s}(e(-\overline{z})) = \sum_{k,l=1}^{\infty} k^r l^s e^*(klz), \tag{2.32}$$

and let ${}_mF_n$ denote the generalized hypergeometric function defined by

$${}_mF_n\left({\alpha_1, \ldots, \alpha_m \atop \beta_1, \ldots, \beta_n}; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_n)_k k!} z^k$$

for $|z| < +\infty$ if $m < n+1$, and for $|z| < 1$ if $m = n+1$.

Theorem 2. *Let $\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s; x+iy)$ be defined by (1.5), where α and β are complex numbers with $\operatorname{Re}\alpha > 1$ and $\operatorname{Re}\beta > 0$. Then for any integer $N \geq 0$, and any real x and Y with $Y > 0$ the formula*

$$\begin{aligned}\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s; x+iy) &= 2\zeta(2s) + 2\sqrt{\pi}\Gamma\left({s-1/2, \alpha+\beta, \alpha+1-2s \atop s, \alpha, \alpha+\beta+1-2s}\right)\zeta(2s-1)Y^{1-2s} \\ &\quad + 2\pi^{2s}\Gamma\left({\alpha+\beta \atop s}\right)\{S_{\alpha,\beta,N}(s,x;Y) + R_{\alpha,\beta,N}(s,x;Y)\}\end{aligned} \tag{2.33}$$

holds in the region $\sigma < \operatorname{Re}\alpha/2$. Here

$$\begin{aligned}S_{\alpha,\beta,N}(s,x;Y) = \sum_{n=0}^{N-1} &\frac{(-1)^n(\alpha)_n}{n!}\Gamma\left({(\alpha+n+1)/2-s \atop (\alpha+n+1)/2, \beta-n}\right) \\ &\times \Phi^*_{2s-1-\alpha-n,-\alpha-n}(e(x))(2\pi Y)^{-\alpha-n}\end{aligned} \tag{2.34}$$

is the asymptotic series in the descending order of Y; the remainder term $R_{\alpha,\beta,N}$ is expressed as

$$\begin{aligned}R_{\alpha,\beta,N}(s,x;Y) = &\frac{2^{2s}(-1)^N(\alpha)_N}{(N-1)!}\sum_{k,l=1}^{\infty} e^*(klx)k^{2s-1} \\ &\times \int_0^1 \xi^{-\alpha-N}(1-\xi)^{N-1}F_{\alpha+N,\beta-N}(s; 2\pi klY/\xi)d\xi\end{aligned} \tag{2.35}$$

for any $N \geq 0$, where

$$\begin{aligned}F_{\alpha,\beta}(s;Z) = &\Gamma\left({1-2s \atop 1-s, \alpha+\beta}\right){}_2F_3\left({\alpha/2, (\alpha+1)/2 \atop (\alpha+\beta)/2, (\alpha+\beta+1)/2, s+1/2}; Z^2/4\right) \\ &+ \Gamma\left({2s-1, \alpha+1-2s \atop s, \alpha, \alpha+\beta+1-2s}\right)(2Z)^{1-2s} \\ &\times {}_2F_3\left({(\alpha+1)/2-s, \alpha/2+1-s \atop (\alpha+\beta+1)/2-s, (\alpha+\beta)/2+1-s, 3/2-s}; Z^2/4\right)\end{aligned} \tag{2.36}$$

with α, β replaced by $\alpha+N$, $\beta-N$, and it satisfies the estimate

$$R_{\alpha,\beta,N}(s,x;Y) = O(Y^{-\operatorname{Re}\alpha-N}) \tag{2.37}$$

for any $Y \geq Y_0 > 0$ in the region $\sigma < \operatorname{Re}\alpha/2$ with the implied O-constant depends at most on α, β, N, s, t and Y_0. In particular, if α and β are real, a more explicit estimate

$$R_{\alpha,\beta,N}(s,x;Y) = O\{e^{-\pi|t|/2}(|t|+1)^{(\alpha+N)/2-\min(\sigma,\beta+1/2)}Y^{-\alpha-N}\} \tag{2.38}$$

follows for any $Y \geq Y_0 > 0$ in the region $\sigma < \alpha/2$, where the O-constant depends at most on α, β, N, σ and Y_0.

Remark. Since

$$\Phi_{r,s}(e(x)) \asymp 1 \tag{2.39}$$

for $\operatorname{Re} r < -1$ and $\operatorname{Re} s < -1$ by the definition (2.7), the n-th indexed term of the asymptotic series in (2.34) is of order $\asymp e^{-\pi|t|/2}(|t|+1)^{(\alpha+n)/2}Y^{-\operatorname{Re}\alpha-n}$; this shows that the presence of the bound in (2.38) is, though slightly weaker, reasonable.

Corollary 2.1. *Formula* (2.33) *with* (2.34)–(2.36) *precisely reduces to the complete asymptotic expansion of* (1.4) *as $Y \to +\infty$ (see* [Ka10, Theorem 2, (2.17)–(2.19)]) *through the confluence operation* (1.6).

3. Analytic continuation of the generalized Epstein zeta-function

The aim of this section is to prepare the analytic continuation of $\psi_{\mathbf{Z}^2}(s;a,b;\mu,\nu;z)$, through which we prove Propositions 1 and 2 below; the Mellin-Barnes type integrals in (3.3) play key rôles in the derivation. The basic frame of the proofs are almost the same as those in [Ka10, Sect.3], however the details fairly differ from the original.

We hereafter write $w = u + iv$ with real coordinates. Suppose temporarily that $\sigma > 1$, and classify each term of the double sum in (1.2) as

$$\begin{aligned}
&\psi_{\mathbf{Z}^2}(s;a,b;\mu,\nu;z) \\
&= \left\{\delta(b)\sum_{\substack{m=-\infty\\ n=-b}}^{\infty}{}' + \sum_{\substack{m,n=-\infty\\ n\neq -b}}^{\infty}{}'\right\} e((a+m)\mu+(b+n)\nu)|a+m+(b+n)z|^{-2s} \\
&= \delta(b)\psi_{\mathbf{Z}}(2s,a,\mu) + \sum_{n=-\infty}^{\infty}{}' e((b+n)(\nu-\mu x)) \sum_{m=-\infty}^{\infty} e((a+m+(b+n)x)\mu) \\
&\qquad \times \{|a+m+(b+n)x|^2 + (|b+n|y)^2\}^{-s}.
\end{aligned} \tag{3.1}$$

Here the last m-sum can be transformed by substituting the expression

$$\left[1+\left\{\frac{|a+m+(b+n)x|}{|b+n|y}\right\}^2\right]^{-s} = \frac{1}{2\pi i}\int_{(u_0)} \Gamma\binom{s+w,-w}{s}\left\{\frac{|a+m+(b+n)x|}{|b+n|y}\right\}^{2w} dw$$

into each term, where u_0 is a constant satisfying $1-\sigma < u_0 < 0$, and (u_0) denotes the vertical straight line from $u_0 - \infty$ to $u_0 + \infty$; this is obtained by taking $-W = \{|a+m+(b+n)x|/|b+n|y\}^2$ in the Mellin-Barnes formula [Ka10, (7.7)]. The second equality in (3.1) then becomes

$$\begin{aligned}
\psi_{\mathbf{Z}^2}(s;a,b;\mu,\nu;z) &= \delta(b)\psi_{\mathbf{Z}}(2s,a,\mu) + \sum_{n=-\infty}^{\infty}{}' e((b+n)(\nu-\mu x))(|b+n|y)^{-2s} \\
&\qquad \times I_\mu(s;a+(b+n)x,|b+n|y),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
I_\mu(s;X,Y) &= \frac{1}{2\pi i}\int_{(u_0)} \Gamma\binom{s+w,-w}{s}\psi_{\mathbf{Z}}(-2w,X,\mu)Y^{-2w}dw \\
&= \frac{1}{2\pi i\sqrt{\pi}}\int_{(u_0)} \Gamma\binom{s+w,w+1/2}{s}\phi_{\mathbf{Z}}(1+2w,-\mu,X)(\pi Y)^{-2w}dw.
\end{aligned} \tag{3.3}$$

Here the interchange of the order of summation and integration is justified in passing through the first equality, since the integrand is bounded by the inequalities in [Ka10, Lemma 4] and $\psi_{\mathbb{Z}}(-2w, X, \mu) \ll |v|^{A(u)}$ as $v \to \pm\infty$ with some constant $A(u) > 0$, while the second equality follows by substituting the functional equation (3.5) below, and then by using the duplication formula

$$\Gamma(2w) = 2^{2w-1}\pi^{-1/2}\Gamma(w)\Gamma(w+1/2). \tag{3.4}$$

Lemma 1. *For any real X and μ we have the functional equation*

$$\psi_{\mathbb{Z}}(r, X, \mu) = \frac{2\Gamma(1-r)\sin(\pi r/2)}{(2\pi)^{1-r}}\phi_{\mathbb{Z}}(1-r, -\mu, X), \tag{3.5}$$

where the identity $\phi_{\mathbb{Z}}(1-r, -\mu, X) = \phi_{\mathbb{Z}}(1-r, \mu, -X)$ holds on the right side by the definition (2.4).

Proof. We split the series representation for $\psi_{\mathbb{Z}}(r, X, \mu)$ (see (2.5)) with the summation conditions $l \leq -[X]-1$ and $l \geq -[X]$, to find by noting (2.2) and the property $\phi(r, X, \mu) = \phi(r, X, \mu+m)$ for $m \in \mathbb{Z}$ that

$$\begin{aligned}\psi_{\mathbb{Z}}(r, X, \mu) &= e(-(1-\langle X\rangle)\mu)\phi(r, 1-\langle X\rangle, 1-\langle\mu\rangle) + e(\langle X\rangle\mu)\phi(r, \langle X\rangle, \langle\mu\rangle)\\ &= \frac{\Gamma(1-r)}{(2\pi)^{1-r}}2\cos\left\{\frac{\pi}{2}(1-r)\right\}\{e(\langle X\rangle([\mu]+1))\phi(1-r, 1-\langle\mu\rangle, \langle X\rangle)\\ &\quad + e(\langle X\rangle[\mu])\phi(1-r, \langle\mu\rangle, -\langle X\rangle)\},\end{aligned} \tag{3.6}$$

where the last equality is obtained by applying the functional equation (cf. [Le] or [Ap])

$$\begin{aligned}\phi(s, a, \lambda) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{&e^{\pi i(1-s)/2}e(-a\lambda)\phi(1-s, \lambda, -a)\\ &+ e^{-\pi i(1-s)/2}e(a(1-\lambda))\phi(1-s, 1-\lambda, a)\}\end{aligned}$$

for any $a, \lambda \in [0, 1]$ with the former convention in (2.3). The last factor in the curly brackets on the rightmost side of (3.6) is further modified, upon using (2.2) and the properties $\psi_{\mathbb{Z}}(r, \mu+m, X) = \psi_{\mathbb{Z}}(r, \mu, X)$ and $\phi_{\mathbb{Z}}(r, \mu, X) = \phi_{\mathbb{Z}}(r, \mu, X+n)$ for $m, n \in \mathbb{Z}$, into

$$\begin{aligned}&e(\langle X\rangle\mu)\psi(1-r, 1-\langle\mu\rangle, \langle X\rangle) + e(\langle X\rangle\mu)\psi(1-r, \langle\mu\rangle, -\langle X\rangle)\\ &\quad = e(\langle X\rangle\mu)\psi_{\mathbb{Z}}(1-r, \langle\mu\rangle, -\langle X\rangle) = e(\langle X\rangle\mu)\psi_{\mathbb{Z}}(1-r, \mu, -\langle X\rangle)\\ &\quad = \phi_{\mathbb{Z}}(1-r, \mu, -\langle X\rangle) = \phi_{\mathbb{Z}}(1-r, \mu, -X),\end{aligned}$$

which implies the assertion (3.5) by replacing l with $-l$ in (2.4). □

Suppose temporarily that $x \notin \{-(a+m)/(b+n) \mid m, n \in \mathbb{Z}\}$, which is equivalent to that $X = a + (b+n)x \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$. Let $u_1 > 0$ be any constant. We can now move the path of integration in (3.3) from (u_0) to (u_1). The residues of the relevant (possible) poles at $w = -1/2$ and $w = 0$ are computed by using [Ka13, Lemma 6], and this gives

$$\begin{aligned}I_\mu(s; X, Y) = \delta(\mu)e(\mu X)\sqrt{\pi}\Gamma\binom{s-1/2}{s}Y + \frac{1}{2\pi i\sqrt{\pi}}\int_{(u_1)}\Gamma\binom{s+w, w+1/2}{s}\\ \times \phi_{\mathbb{Z}}(1+2w, -\mu, X)(\pi Y)^{-2w}dw.\end{aligned}$$

Substituting the series representation of $\phi_{\mathbb{Z}}(1+2w,-\mu,X)$ (see (2.4)), which converges absolutely on the path $\mathrm{Re}\,w=u_1$, into the integrand, we obtain

$$(3.7)\qquad I_\mu(s;X,Y)=\delta(\mu)e(\mu X)\sqrt{\pi}\Gamma\binom{s-1/2}{s}Y+\frac{1}{\sqrt{\pi}}\sum_{m=-\infty}^{\infty}{}' e(mX)|-\mu+m|^{-1} \times J(s;\pi|-\mu+m|Y),$$

where

$$(3.8)\qquad J(s;Z)=\frac{1}{2\pi i}\int_{(u_1)}\Gamma\binom{s+w,w+1/2}{s}Z^{-2w}dw.$$

The right side of (3.7) with $X=a+(b+n)x$ and $Y=|b+n|y$ is further substituted into each term of the sum in (3.2), to yield Formula (2.9) with

$$(3.9)\qquad \psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) =\frac{e(a\mu)}{\sqrt{\pi}}\sum_{m,n=-\infty}^{\infty}{}' e((-\mu+m)a+(b+n)\nu)e((-\mu+m)(b+n)x) \times|-\mu+m|^{-1}(|b+n|y)^{-2s}J(s;\pi|(-\mu+m)(b+n)|y).$$

This shows that $\psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ can be continued to an entire function over the s-plane, since from the estimate, by (3.8), $J(s;Z)=O(Z^{-2u_1})$ as $Z\to+\infty$ for any $u_1>0$, the last double sum is bounded as

$$\ll\sum_{m=-\infty}^{\infty}{}'|-\mu+m|^{-1-2u_1}\sum_{n=-\infty}^{\infty}{}'|b+n|^{-2\sigma-2u_1}<+\infty,$$

provided that u_1 is taken sufficiently large according to the location of s. Moreover, at this stage the restriction on x above can be removed by continuity. We therefore find the following.

Proposition 1. *Formula* (2.9) *with* (3.9) *provides the meromorphic continuation of* (1.2) *over the whole s-plane; its only singularity is a (possible) simple pole at $s=1$ with the residue $\delta(\mu)\delta(\nu)\pi/y$ which exactly comes from the second term on the right side of* (2.9), *while $\psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ in the last term is an entire function.*

Let $K_\nu(Z)$ denotes the modified Bessel function of the third kind (cf. [Er2, p.5, 7.22(13)]). Then we have shown in [Ka10, Lemmas 2 and 3] that $J(s;Z)$ can be expressed in terms of $K_\nu(Z)$ or the confluent hypergeometric function in (2.8):

Lemma 2. *For any complex s and any real $Z>0$ we have the equalities*

$$(3.10)\qquad J(s;Z)=\frac{2Z^{s+1/2}}{\Gamma(s)}K_{s-1/2}(2Z)=\frac{\sqrt{\pi}e^{-2Z}(2Z)^{2s}}{\Gamma(s)}U(s;2s;4Z).$$

Substituting the former expression of $J(s;Z)$ in (3.10) into each term of the sum in (3.7) we obtain the following.

Proposition 2. *The formula*

$$(3.11)\qquad \psi^*_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) =\frac{e(a\mu)\pi^s y^{1/2-s}}{\Gamma(s)}\sum_{m,n=-\infty}^{\infty}{}' e((-\mu+m)a+(b+n)\nu)e((-\mu+m)(b+n)x) \times|-\mu+m|^{s-1/2}|b+n|^{1/2-s}K_{s-1/2}(2\pi|(-\mu+m)(b+n)|y)$$

holds for any complex s and any $z = x + iy$ with $y = \operatorname{Im} z > 0$.

Remark. Formula (2.9) with (3.11) reduces when $a, b \in \mathbb{Z}$ and $\mu = \nu = 0$ to the classical Fourier series expansion of $\zeta_{\mathbb{Z}^2}(s; z)$.

4. Derivation of the asymptotic expansion

The aim of this section is to deduce the asymptotic series (2.9) with (2.10)–(2.12), and the explicit formula (2.13).

Suppose temporarily that $0 < \sigma < 1$. We then classify each term of the sum in (3.9) according to the conditions: i) $m \geq [\mu] + 1$ and $n \geq -[b]$; ii) $m \leq [\mu]$ and $n \leq -[b] - 1$; iii) $m \leq [\mu]$ and $n \geq -[b]$; iv) $m \geq [\mu] + 1$ and $n \leq -[b] - 1$, to find that

$$\begin{aligned}(4.1)\qquad &\psi^*_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z)\\ &= \frac{e(a\mu)}{\sqrt{\pi}}\{\widetilde{T}(s; 1 - \langle\mu\rangle, \langle b\rangle; a, \nu; z) + \widetilde{T}(s; \langle\mu\rangle, 1 - \langle b\rangle; -a, -\nu; z)\\ &\quad + \widetilde{T}(s; \langle\mu\rangle, \langle b\rangle; -a, \nu; -\overline{z}) + \widetilde{T}(s; 1 - \langle\mu\rangle, 1 - \langle b\rangle; a, -\nu; -\overline{z})\}\end{aligned}$$

with

$$\begin{aligned}(4.2)\qquad \widetilde{T}(s; c, d; \kappa, \lambda; z) &= {\sum_{k,l=0}^{\infty}}' e((c+k)\kappa + (d+l)\lambda)\{(c+k)y\}^{-2s}(d+l)^{-1}\\ &\quad \times e((c+k)(d+l)x)J(s; \pi(c+k)(d+l)y)\\ &= \frac{(2\pi)^{2s}\sqrt{\pi}}{\Gamma(s)}T(s; c, d; \kappa, \lambda; z),\end{aligned}$$

say, where

$$\begin{aligned}(4.3)\qquad T(s; c, d; \kappa, \lambda; z) &= {\sum_{k,l=0}^{\infty}}' e((c+k)\kappa + (d+l)\lambda)(c+k)^{2s-1}\\ &\quad \times e((c+k)(d+l)z)U(s; 2s; 4\pi(c+k)(d+l)y),\end{aligned}$$

by substituting the latter expression in (3.10), and by using the property

$$(4.4)\qquad e(x)e^{-2\pi y} = e(z).$$

Note that (4.3) holds with $-\overline{z}$ instead of z. We therefore obtain the assertion (2.10) upon combining (4.1) with (4.2).

The expression in (4.3) can further be transformed upon incorporating the Mellin-Barnes formula

$$(4.5)\qquad U(\alpha; \gamma; Z) = \frac{1}{2\pi i}\int_{(u)} \Gamma\binom{\alpha + w, -w, 1 - \gamma - w}{\alpha, \alpha - \gamma + 1} Z^w dw$$

for $|\arg Z| < 3\pi/2$ and u satisfying $-\operatorname{Re}\alpha < u < \min(0, 1 - \operatorname{Re}\gamma)$ (cf. [Er1, p.256, 6.5(5)]); this shows that

$$\begin{aligned}(4.6)\qquad T(s; c, d; \kappa, \lambda; z) &= \frac{1}{2\pi i}\int_{(u_0)} \Gamma\binom{s + w, -w, 1 - 2s - w}{s, 1 - s}\\ &\quad \times \Phi_{2s-1+w,w}(c, d; \kappa, \lambda; e(z))(4\pi y)^w dw\end{aligned}$$

for $0 < \sigma < 1$, where u_0 is a constant satisfying $-\sigma < u_0 < \min(0, 1 - 2\sigma)$. Here the interchange of the order of summation and integration is justified, since the series expression of $\Phi_{2s-1+w,w}(c, d; \kappa, \lambda; e(z))$ (see (2.6)) converges absolutely for all complex s

and w. Note here that the path (u_0) in (4.6) separates the poles of the integrand at $w=-s-n$ $(n=0,1,\ldots)$, from those at $w=n$, $1-2s+n$ $(n=0,1,\ldots)$.

We are now ready to deduce the asymptotic series in (2.12). For this let $N\geq 0$ be any integer, and u_N a constant satisfying $-\sigma-N<u_N<-\sigma-N+1$. Then the path of integration in (4.6) can be moved from (u_0) to (u_N), upon passing over the poles at $w=-s-n$ $(n=0,1,\ldots,N-1)$. Collecting the residues of these poles, we obtain the assertion (2.11) with (2.12) and

$$R_N(s;c,d;\kappa,\lambda;z)=\frac{1}{2\pi i}\int_{(u_N)}\Gamma\binom{s+w,-w,1-2s-w}{s,1-s}\times\Phi_{2s-1+w,w}(c,d;\kappa,\lambda;e(z))(4\pi y)^w dw. \tag{4.7}$$

Here the temporarily restriction on σ can be relaxed into $-N<\sigma<N+1$; this allows to choose u_N such that

$$-\sigma-N<u_N<\min(1-2\sigma,0,-\sigma-N+1), \tag{4.8}$$

by which the path (u_N) separates the poles of the integrand at $w=-s-n$ $(n=N,N+1,\ldots)$, from those at $w=-s-n$ $(n=0,\ldots,N-1)$ and $w=n$, $1-2s+n$ $(n=0,1,\ldots)$. Furthermore, the same argument as in [Ka10, Sect.4] is also applicable to treat the integral in (4.7) for $N\geq 1$; this can be transformed, upon substituting

$$\Gamma(s+w)=\frac{(-1)^N\Gamma(s+N+w)}{(N-1)!}\int_0^1\xi^{-s-N-w}(1-\xi)^{N-1}d\xi,$$

being valid for $-N<\sigma<N+1$ with $N\geq 1$ and on the path $\operatorname{Re}w=u_N$, into the integrand, and then by integrating term-by-term. We therefore obtain

$$\begin{aligned}R_N(s;c,d;\kappa,\lambda;z)&=\frac{(-1)^N}{(N-1)!}\sideset{}{'}\sum_{k,l=0}^{\infty}e((c+k)\kappa+(d+l)\lambda)\\&\times(c+k)^{2s-1}e((c+k)(d+l)z)\int_0^1\xi^{-s-N}(1-\xi)^{N-1}\\&\times\frac{1}{2\pi i}\int_{(u_N)}\Gamma\binom{s+N+w,-w,1-2s-w}{s,1-s}\{4\pi(c+k)(d+l)y/\xi\}^w dw d\xi,\end{aligned}$$

which implies the assertion (2.13) for $N\geq 1$, since the resulting inner w-integral is equal to $(s)_N(1-s)_N U(s+N;2s;4\pi(c+k)(d+l)y/\xi)$ by (4.5). The remaining case $N=0$ can be treated in the same manner except inserting the ξ-integration. We finally remark that the whole derivation above can be carried out with $-\bar{z}$ instead of z.

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