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Higher moments of the Epstein zeta functions

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1 Introduction

Moments of the Riemann zeta function and other L-functions have been studied for about one hundred years, from the age of Hardy and Littlewood. Let $\zeta(s)$ be the Riemann zeta function. In 1918, Hardy and Littlewood [2] investigated the mean square (second moment) of $\zeta(s)$ on the critical line $\mathrm{Re}(s)=\frac{1}{2}$, and obtained the asymptotic formula

$$\int_{1}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2}dt=T\log T+O(T) \tag{1.1}$$

as $T\rightarrow\infty$. Further, in 1926, Ingham [6] considered the fourth moment of $\zeta(s)$ and proved that

$$\int_{1}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{4}dt=\frac{1}{2\pi^{2}}T(\log T)^{4}+O(T(\log T)^{3}) \tag{1.2}$$

holds as $T\rightarrow\infty$. The basic tools of them are the approximate functional equations for $\zeta(s)$ and $\zeta(s)^{2}$. Therefore, one might think that we can obtain the asymptotic formula for the higher moments (sixth moment, eighth moment, etc$\cdots$) of $\zeta(s)$ on the critical line $\mathrm{Re}(s)=\frac{1}{2}$ by using the approximate functional equations for $\zeta(s)^{k}$ $(k\geq 3)$. However, although these approximate functional equations are known, a straightforward application of them doesn't give the desirable results. In fact, it is generally conjectured that

$$\int_{1}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt\sim C_{k}T(\log T)^{k^{2}} \tag{1.3}$$

holds for all $k\geq 0$ with some constant C_{k}, but this has not been proved except for the cases $k=0,1,2$. Evaluating these moments is related to many topics in analytic number theory, for example, the zero-density estimate for $\zeta(s)$ or the order estimate for $\zeta(s)$ on the critical line. Also, the auther thinks this theme is sufficiently interesting in itself.

In this article [1], we consider the Epstein zeta function $\zeta(s;Q)$, where Q is a $n\times n$ positive definite symmetric matrix $(n\geq 4)$ which gives an integer-valued

[1] Almost all parts of this article are some generalizations or summaries of the contents of the auther's another paper [11], dealing with the fourth moment of the Epstein zeta functions.

quadratic form. We evaluate the moments of $\zeta(s;Q)$ on the line $\mathrm{Re}(s)=\frac{n-1}{2}$, and prove that the integral $\int_0^T|\zeta(\frac{n-1}{2}+it;Q)|^{2k}dt$ is evaluated by $O(T(\log T)^{k^2})$ as $T\to\infty$ under the assumption of a moment conjecture for the Dirichlet L-functions. Although the line $\mathrm{Re}(s)=\frac{n-1}{2}$ is not the center of the functional equation of $\zeta(s;Q)$, the auther thinks this formulation of problem is quite natural.

Let us introduce the basic idea of this article. For a $n\times n$ positive definite symmetric matrix Q, the quadratic form associated to Q is defined by $Q[\mathbf{x}]={}^t\mathbf{x}Q\mathbf{x}$ for $\mathbf{x}\in\mathbf{R}^n$. We assume that $Q[\mathbf{x}]\in\mathbf{N}$ for any $\mathbf{x}\in\mathbf{Z}^n\backslash\{\mathbf{0}\}$. For $l\in\mathbf{Z}_{\geq 0}$, we define $r_Q(l)$ by the number of $\mathbf{x}\in\mathbf{Z}^n$ which satisfies $Q[\mathbf{x}]=l$. Then the Epstein zeta function $\zeta(s;Q)$ is expressed by

$$\zeta(s;Q)=\sum_{l=1}^{\infty}\frac{r_Q(l)}{l^s} \tag{1.4}$$

for $\mathrm{Re}(s)>\frac{n}{2}$. The corresponding theta series

$$\theta(z;Q)=\sum_{l=0}^{\infty}r_Q(l)e^{2\pi ilz}$$

becomes a modular form of weight $\frac{n}{2}$ and decomposes into the sum of an Eisenstein series and a cusp form. Therefore, $\zeta(s;Q)$ decomposes into the sum of the L-function associated to the Eisenstein series and the L-function associated to the cusp form. Hence to obtain the upper bound for the momens of $\zeta(s;Q)$, it suffices to evaluate the integrals of these two L-functions. By using a classical method in analytic number theory, we can prove that the moments of the L-function associated to the cusp form is evaluated by $O(T)$, and our main problem is to evaluate the moment of L-function associated to the Eisenstein series. For this purpose, we use the classical theories due to Hecke ([5]), Malyshev ([8]), and Siegel ([10]). By using their theorems, we prove that the L-function associated to the Eisenstein series is expressed by some finite or infinite series consisting of the Dirichlet L-functions and thus we can obtain some upper bounds for the moments of $\zeta(s;Q)$ by assuming a conjecture for the moments of the Dirichlet L-functions.

As the easiest example, we take $Q=I_4$, the 4×4 unit matrix. Then the Epstein zeta function $\zeta(s;I_4)$ is expressed by

$$\zeta(s;I_4)=8(1-2^{1-s})\zeta(s)\zeta(s-1). \tag{1.5}$$

Since the factor $(1-2^{1-s})\zeta(s)$ is bounded on the line $\mathrm{Re}(s)=\frac{3}{2}$, the $2k$-th moment $\int_0^T|\zeta(\frac{3}{2}+it;I_4)|^{2k}dt$ is evaluated by $O(T(\log T)^{k^2})$ as $T\to\infty$ if we assume that the conjecture (1.3) is valid. Of course, the general case is much more complicated, but the underlying idea is similar. Among others, desirable upper bounds for the fourth moment of $\zeta(s;Q)$ are obtained unconditionally, since we have the unconditional results for the fourth moment of the Riemann zeta function or the Dirichlet L-functions.

2 Moments of Epstein zeta functions

2.1 Notation and some basic results

Let n be a positive integer and Q be a $n \times n$ positive definite symmetric matrix. The Epstein zeta function associated to Q is defined by

$$\zeta(s;Q) = \sum_{\mathbf{x} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} Q[\mathbf{x}]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right)$$

where $Q[\mathbf{x}] := {}^t\mathbf{x}Q\mathbf{x}$. Like the Riemann zeta function, this function has the meromorphic continuation to the whole s-plane and satisfies the following functional equation:

$$\pi^{-s}\Gamma(s)\zeta(s;Q) = (\det Q)^{-\frac{1}{2}}\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\zeta\left(\frac{n}{2}-s;Q^{-1}\right). \tag{2.1}$$

$\zeta(s;Q)$ is holomorphic everywhere except for a simple pole at $s = \frac{n}{2}$ with residue $\pi^{\frac{n}{2}}/(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})$. Throughout this article, we assume that $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}$. Let $r_Q(l)$ be the number of $\mathbf{x} \in \mathbf{Z}^n$ which satisfies $Q[\mathbf{x}] = l$. Then $\zeta(s;Q)$ has the following Dirichlet series expansion in $\mathrm{Re}(s) > \frac{n}{2}$:

$$\zeta(s;Q) = \sum_{l=1}^{\infty} \frac{r_Q(l)}{l^s}.$$

Hereafter, we assume that $n \geq 4$. We consider the theta series corresponding to $\zeta(s;Q)$ defined by

$$\theta(z;Q) = \sum_{l=0}^{\infty} r_Q(l)e^{2\pi ilz}.$$

It is known that $\theta(z;Q)$ is decomposed into the sum of an Eisenstein series and a cusp form:

$$\theta(z;Q) = E(z) + S(z) \tag{2.2}$$

where

$$E(z) = \sum_{l=0}^{\infty} e(l)e^{2\pi ilz}$$

is the Eisenstein series and

$$S(z) = \sum_{l=1}^{\infty} s(l)e^{2\pi ilz}$$

is the cusp form. Moreover, it is known that the coefficient $s(l)$ of $S(z)$ is evaluated by

$$s(l) \ll l^{\frac{n}{4}-\frac{1}{2}+\epsilon} \tag{2.3}$$

if n is even, and

$$s(l) \ll l^{\frac{n}{4}-\frac{1}{4}+\epsilon} \tag{2.4}$$

if n is odd, where ϵ is always an arbitrary positive number throughout this article. Firstly, since the coefficient $s(l)$ is relatively small, the integral $\int_0^T |\hat{S}(\frac{n-1}{2}+it)|^{2k}dt$ also becomes relatively small. That is, by using a classical method in analytic number theory, the following lemma is obtained:

Lemma 2.1. *When $T \to \infty$, we have*

$$\int_0^T \left|\hat{S}\left(\frac{n-1}{2}+it\right)\right|^{2k} dt = O(T). \tag{2.5}$$

Thus our main problem is to evaluate the integral $\int_0^T |\hat{E}(\frac{n-1}{2}+it)|^{2k}dt$. For this purpose, we use the relations between the L-function associated to the Eisenstein series and the Dirichlet L-functions. Let $L(s,\chi)$ be the Dirichlet L-function associated to a Dirichlet character χ. As an analogue of the moment conjecture (1.3) for the Riemann zeta function, the following conjecture seems to be natural:

Conjecture 2.2. As $q \to \infty$, $T \to \infty$, we have

$$\sum_{\chi(\mathrm{mod}q)} \int_1^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{2k} dt \ll qT(\log qT)^{k^2} \tag{2.6}$$

for any positive number k. Here, $\sum_{\chi(\mathrm{mod}q)}$ denotes the sum over all Dirichlet characters modulo q.

Remark 2.3. In the book [9], Montgomery mentioned that the estimate

$$\sum\nolimits^*_{\chi(\mathrm{mod}q)} \int_1^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{4} dt \ll \phi(q)T(\log qT)^4 \tag{2.7}$$

holds unconditionally. Here, $\sum^*_{\chi(\mathrm{mod}q)}$ denotes the sum over all primitive Dirichlet characters modulo q and ϕ denotes Euler's ϕ-function. As an easy consequence of (2.7) (in detail, see [4]), the estimate (2.6) holds unconditionally in the case of $k=2$.

The following lemma is famous hybrid bounds for Dirichlet L-functions, proved by Heath-Brown (see [3]):

Lemma 2.4. *Let $L(s,\chi)$ be a Dirichlet L-function associated to a Dirichlet character modulo q. Then. when $t \to \infty$, the following estimates hold:*

$$L\left(\frac{1}{2}+it,\chi\right) \ll q^{\frac{1}{2}}t^{\frac{1}{8}}\log(qt), \tag{2.8}$$

$$L\left(\frac{1}{2}+it,\chi\right) \ll (qt)^{\frac{3}{16}+\epsilon}. \tag{2.9}$$

Further, we prepare the following inequality:

Lemma 2.5. *For $k \geq \frac{1}{2}$ and $x_1, \cdots, x_m \geq 0$, we have*

$$x_1^{\frac{1}{2k}} + \cdots + x_m^{\frac{1}{2k}} \leq m^{1-\frac{1}{2k}} (x_1 + \cdots + x_m)^{\frac{1}{2k}}. \tag{2.10}$$

Proof. The inequality (2.10) is equivalent to

$$\frac{x_1^{\frac{1}{2k}} + \cdots + x_m^{\frac{1}{2k}}}{m} \leq \left(\frac{x_1 + \cdots + x_m}{m} \right)^{\frac{1}{2k}},$$

and we can easily prove this inequality by using the convexity of the function $f(x) = x^{\frac{1}{2k}}$. □

Now the main theorem is stated as follows:

Theorem 2.6. *Assume that n is even and $n \geq 4$, or n is odd and $n \geq 7$. Then, under the assumption of the Conjecture 2.2, for $k \geq \frac{1}{2}$, the following estimate holds:*

$$\int_0^T \left| \zeta\left(\frac{n-1}{2} + it; Q \right) \right|^{2k} dt = O(T(\log T)^{k^2}). \tag{2.11}$$

Proof. Firstly, we assume that n is even and $n \geq 4$. Then, the Eisenstein series $\hat{E}(z)$ is a modular form of weight $\frac{n}{2}$ and level N, where N is a positive integer such that NA^{-1} becomes the integral matrix for $A = 2Q$ (see [7]). According to Hecke's paper [5], the series $\hat{E}(s)$ is expressed by some linear combination of the form

$$(t_1 t_2)^{-s} L(s, \chi_1) L\left(s - \frac{n}{2} + 1, \chi_2\right)$$

where t_1, t_2 are positive divisors of level N and χ_1, χ_2 are Dirichlet characters modulo $\frac{N}{t_1}$, $\frac{N}{t_2}$, respectively. Since $(t_1 t_2)^{-s} L(s, \chi_1)$ is bounded on the line $\mathrm{Re}(s) = \frac{n-1}{2}$, and since the Conjecture 2.2 indicates that each integral $\int_0^T |L(\frac{1}{2} + it, \chi_2)|^{2k} dt$ is evaluated by $O(T(\log T)^{k^2})$, by applying Minkowski's inequaliy, the $2k$-th moment of $\hat{E}(s)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$ is also evaluated by $O(T(\log T)^{k^2})$. Therefore, the statement of theorem is proved in this case.

Next, we assume that n is odd and $n \geq 7$. The computations below is a simple arrangement of the Fomenko's technique introduced in [1]. In this case, Malyshev, about fifty years ago, showed that the Fourier coefficient $e(l)$ of the Eisenstein series $E(s)$ has the following expression (see [8]):

$$e(l) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}} \Gamma(\frac{n}{2})} l^{\frac{n}{2}-1} H(Q; l)$$

where

$$H(Q; l) = \sum_{q=1}^{\infty} \left\{ \sum_{h (\mathrm{mod} q)}{}' q^{-n} S(hQ; q) e^{-2\pi i \frac{lh}{q}} \right\}$$

is a singular series, $\sum'$ means the sum over a reduced residue system, and

$$S(Q;q) = \sum_{x_1,\cdots,x_n=0}^{q-1} e^{\frac{2\pi i Q(x_1,\cdots,x_n)}{q}}$$

is a Gaussian sum. Therefore, the associated Dirichlet series is given by

$$\hat{E}(s) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} \sum_{l=1}^{\infty} \frac{1}{l^{s-\frac{n}{2}+1}} \sum_{q=1}^{\infty} \sideset{}{'}\sum_{h(\mathrm{mod}q)} q^{-n} S(hQ;q) e^{-2\pi i \frac{lh}{q}}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Let $(l,q) = d$, $l = k_1 d$, $q = q_1 d$, $(k_1, q_1) = 1$ and $k_1 = k_2 q_1 + m$, $(q_1, m) = 1$. Then the right hand side becomes

$$\frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} \sum_{d=1}^{\infty} \frac{1}{d^{s-\frac{n}{2}+1}} \sum_{q_1=1}^{\infty} \sideset{}{'}\sum_{h(\mathrm{mod}q_1 d)} (q_1 d)^{-n} S(hQ; q_1 d)$$
$$\cdot \sideset{}{'}\sum_{m(\mathrm{mod}q_1)} e^{-\frac{2\pi i h m}{q_1 d}} \sum_{k_1 \equiv m(\mathrm{mod}q_1)} \frac{1}{k_1^{s-\frac{n}{2}+1}}.$$

The last sum above is rewritten by using Dirichlet L-functions. By applying the well-known identity

$$\sum_{\chi(\mathrm{mod}q_1)} \overline{\chi}(m)\chi(l) = \begin{cases} \phi(q_1) & (l \equiv m \quad (\mathrm{mod}q_1)) \\ 0 & (\text{otherwise}), \end{cases}$$

we have

$$\begin{aligned} \sum_{k_1 \equiv m(\mathrm{mod}q_1)} \frac{1}{k_1^{s-\frac{n}{2}+1}} &= \frac{1}{\phi(q_1)} \sum_{\chi(\mathrm{mod}q_1)} \overline{\chi}(m) \sum_{k_1=1}^{\infty} \frac{\chi(k_1)}{k_1^{s-\frac{n}{2}+1}} \\ &= \frac{1}{\phi(q_1)} \sum_{\chi(\mathrm{mod}q_1)} \overline{\chi}(m) L\left(s - \frac{n}{2} + 1, \chi\right) \end{aligned}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Therefore,

$$\begin{aligned} \hat{E}(s) =& \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} \sum_{d=1}^{\infty} \frac{1}{d^{s-\frac{n}{2}+1}} \sum_{q_1=1}^{\infty} \sideset{}{'}\sum_{h(\mathrm{mod}q_1 d)} \frac{S(hQ;q_1 d)}{(q_1 d)^n} \\ & \cdot \sideset{}{'}\sum_{m(\mathrm{mod}q_1)} e^{-\frac{2\pi i h m}{q_1}} \frac{1}{\phi(q_1)} \sum_{\chi(\mathrm{mod}q_1)} \overline{\chi}(m) L\left(s - \frac{n}{2} + 1, \chi\right) \end{aligned} \tag{2.12}$$

holds for $\mathrm{Re}(s) > \frac{n}{2}$. It is known that the following estimate holds (see [8]):

$$S(hQ;q) \ll q^{\frac{n}{2}}.$$

The estimate above is not dependent on h. Therefore, the absolute value of the right hand side of (2.12) is estimated by

$$
\begin{aligned}
&\ll \sum_{d=1}^{\infty} \frac{1}{d^{\sigma-\frac{n}{2}+1}} \sum_{q_1=1}^{\infty} \phi(q_1 d) \frac{(q_1 d)^{\frac{n}{2}}}{(q_1 d)^n} \cdot \phi(q_1) \frac{1}{\phi(q_1)} \sum_{\chi(\mathrm{mod} q_1)} \left| L\left(s - \frac{n}{2} + 1, \chi\right)\right| \\
&\ll \sum_{d=1}^{\infty} \frac{1}{d^{\sigma}} \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi(\mathrm{mod} q_1)} \left| L\left(s - \frac{n}{2} + 1, \chi\right)\right| .
\end{aligned}
\tag{2.13}
$$

The estimate (2.9) yields the right hand side of (2.13) converges on the line $\mathrm{Re}(s) = \frac{n-1}{2}$, hence $\hat{E}(s)$ is continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{n-1}{2}$ by (2.12) and the estimate

$$
\left|\hat{E}\left(\frac{n-1}{2} + it\right)\right| \ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi(\mathrm{mod} q_1)} \left| L\left(\frac{1}{2} + it, \chi\right)\right| \tag{2.14}
$$

holds. By applying Minkowski's inequality to (2.14), we have

$$
\begin{aligned}
&\left(\int_0^T \left|\hat{E}\left(\frac{n-1}{2} + it\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \\
&\quad \ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi(\mathrm{mod} q_1)} \left(\int_0^T \left| L\left(\frac{1}{2} + it, \chi\right)\right|^{2k} dt\right)^{\frac{1}{2k}}
\end{aligned}
\tag{2.15}
$$

By applying the inequality (2.10) to the sum in $\chi(\mathrm{mod} q_1)$ and using the estimate (2.6), the right hand side of (2.15) is evaluated by

$$
\begin{aligned}
&\le \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \phi(q_1)^{1-\frac{1}{2k}} \left(\sum_{\chi(\mathrm{mod} q_1)} \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \\
&\ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} q_1^{1-\frac{1}{2k}} (q_1 T (\log q_1 T)^{k^2})^{\frac{1}{2k}} \\
&\ll \left(\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\epsilon}} \right) T^{\frac{1}{2k}} (\log T)^{\frac{k}{2}} .
\end{aligned}
$$

The series $\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\epsilon}}$ converge when $n > 6$. Therefore, the estimate

$$
\left(\int_0^T \left|\hat{E}\left(\frac{n-1}{2} + it\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \ll T^{\frac{1}{2k}} (\log T)^{\frac{k}{2}}
$$

holds when $n \ge 7$, hence the statement of theorem is proved. □

Next, we consider the case of $n = 5$. In this case, we cannot use the method we used in the proof of Theorem 2.6, since the right hand side of (2.13) may not converge on the line $\mathrm{Re}(s) = 2$ in the case of $n = 5$. To obtain the upper bound for the moments of $\hat{E}(s)$, we use another formula proved by Siegel ([10]) under some additional conditions.

Theorem 2.7. *Let Q be a 5×5 positive definite symmetric integer matrix which satisfies* $\det Q = 1$. *Then, for $k > \frac{1}{2}$, under the assumption of the Conjecture 2.2, we have*

$$\int_0^T |\zeta(2+it;Q)|^{2k} dt = O(T(\log T)^{k^2}) \tag{2.16}$$

as $T \to \infty$.

Proof. Assume that Q satisfies the conditions of theorem. In this case, Siegel showed that $\hat{E}(s)$ has the following expression (see [10], Theorem 12):

$$\hat{E}(s) = 2\pi^s \frac{\Gamma(\frac{5}{2}-s)}{\Gamma(\frac{5}{2})} \left\{ \psi(s) + \psi\left(\frac{5}{2}-s\right) \right\} \tag{2.17}$$

for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where the function $\psi(s)$ is defined by

$$\begin{aligned} \psi(s) =& 2^{s-\frac{5}{2}} \left\{ \cos\frac{\pi}{4}(2s-5) \sum_{a,b\ b\equiv 1 (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} \right. \\ & \left. + \cos\frac{\pi}{4}(2s+5) \sum_{a,b\ b\equiv 3 (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} \right\} \end{aligned} \tag{2.18}$$

and $\chi_b(a) = (\frac{a}{b})$ denoting the Legendre-Jacobi symbol. For fixed b, we have

$$\sum_a \chi_b(a) a^{s-\frac{5}{2}} = L\left(\frac{5}{2}-s, \chi_b\right)$$

for $\mathrm{Re}(s) < \frac{3}{2}$. Therefore,

$$\sum_{a,b\ b\equiv j (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} = \sum_{b\equiv j (\mathrm{mod} 4)} b^{-s} L\left(\frac{5}{2}-s, \chi_b\right) \tag{2.19}$$

(j=1,3) holds for $\mathrm{Re}(s) < \frac{3}{2}$. By using the estimate (2.8), the series of the right hand side of (2.19) converge absolutely on $\mathrm{Re}(s) = 2$, so the left hand side of (2.19) can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by (2.19). Therefore, $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by

$$\begin{aligned} \psi(s) =& 2^{s-\frac{5}{2}} \left\{ \cos\frac{\pi}{4}(2s-5) \sum_{b\equiv 1 (\mathrm{mod} 4)} b^{-s} L\left(\frac{5}{2}-s, \chi_b\right) \right. \\ & \left. + \cos\frac{\pi}{4}(2s+5) \sum_{b\equiv 3 (\mathrm{mod} 4)} b^{-s} L\left(\frac{5}{2}-s, \chi_b\right) \right\}. \end{aligned} \tag{2.20}$$

On the other hand, for fixed a,

$$\sum_{b,\ b\equiv j(\mathrm{mod}4)} \chi_b(a)b^{-s}$$

$$= \frac{1}{\phi(4)} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(j) \sum_{b=1}^{\infty} \chi(b)\chi_b(a)b^{-s}$$

$$= \frac{1}{\phi(4)} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(j) L(s, \tilde{\chi}_{a,\chi})$$

(j=1,3) holds for $\mathrm{Re}(s) > 1$, where

$$\tilde{\chi}_{a,\chi}(b) = \chi(b)\chi_b(a) = \chi(b)\left(\frac{a}{b}\right). \tag{2.21}$$

By a straightforward exercise, we can prove that $\tilde{\chi}_{a,\chi}$ becomes a Dirichlet character modulo $4a$. Therefore, we have proved that the identity

$$\begin{aligned}\psi(s) = \frac{2^{s-\frac{5}{2}}}{\phi(4)} \Bigg\{ & \cos\frac{\pi}{4}(2s-5) \sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(1) L(s, \tilde{\chi}_{a,\chi}) \\ & + \cos\frac{\pi}{4}(2s+5) \sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(3) L(s, \tilde{\chi}_{a,\chi}) \Bigg\}\end{aligned} \tag{2.22}$$

holds for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where $\tilde{\chi}_{a,\chi}$ is a Dirichlet character modulo $4a$. By using Heath-Brown's estimate (2.8) again, the right hand side of (2.22) converges absolutely at $s = \frac{1}{2} + it$, so $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{1}{2}$ by the identity (2.22). Therefore, by combining these results, the L-function $\hat{E}(s)$ has the following Dirichlet series expansion on the line $\mathrm{Re}(s) = 2$:

$$\begin{aligned}&\hat{E}(2+it) \\ &= 2^{\frac{1}{2}-it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})} \Bigg\{ \cos\frac{\pi}{4}(-1+2it) \sum_{b\equiv 1(\mathrm{mod}4)} b^{-2-it} L\left(\frac{1}{2}-it, \chi_b\right) \\ &+ \cos\frac{\pi}{4}(9+2it) \sum_{b\equiv 3(\mathrm{mod}4)} b^{-2-it} L\left(\frac{1}{2}-it, \chi_b\right) \Bigg\} \\ &+ 2^{-2-it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})} \Bigg\{ \cos\frac{\pi}{4}(-4-2it) \sum_{a=1}^{\infty} a^{-2-it} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(1) L\left(\frac{1}{2}-it, \tilde{\chi}_{a,\chi}\right) \\ &+ \cos\frac{\pi}{4}(6-2it) \sum_{a=1}^{\infty} a^{-2-it} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(3) L\left(\frac{1}{2}-it, \tilde{\chi}_{a,\chi}\right) \Bigg\}.\end{aligned} \tag{2.23}$$

Note that $\Gamma(\frac{1}{2}-it)\cos\frac{\pi}{4}(\cdot \pm 2it)$ (4 terms) are bounded when $t \to \infty$ (use Stirling's formula). Now, for $k > \frac{1}{2}$, by applying Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_0^T |\hat{E}(2+it)|^{2k}dt\right)^{\frac{1}{2k}} \\
&\ll \sum_{b\equiv 1(\mathrm{mod}4)} b^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\chi_b\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \\
&+ \sum_{b\equiv 3(\mathrm{mod}4)} b^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\chi_b\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \\
&+\sum_{a=1}^{\infty} a^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \\
&\ll \sum_{b\equiv 1,3(\mathrm{mod}4)} b^{-2}(bT(\log bT)^{k^2})^{\frac{1}{2k}} + \sum_{a=1}^{\infty} a^{-2}(aT(\log aT)^{k^2})^{\frac{1}{2k}} \\
&\ll T^{\frac{1}{2k}}(\log T)^{\frac{k}{2}}.
\end{aligned}$$

Therefore, the estimate

$$\int_0^T |\hat{E}(2+it)|^{2k}dt \ll T(\log T)^{k^2}$$

holds. Thus we obtain the estimate (2.16). □

Since the estimate (2.6) in Conjecture 2.2 holds unconditionally in the case of $k=2$, as a corollary of Theorem 2.6 and Theorem 2.7, we obtain the following result for the fourth moment of $\zeta(s;Q)$:

Corollary 2.8. *Unconditionally, for any $n \times n$ positive definite matrix Q in Theorem 2.6 or Theorem 2.7, we have*

$$\int_0^T \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^4 dt \ll T(\log T)^4 \tag{2.24}$$

as $T \to \infty$.

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