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Dynamics and weights of polynomial skew products on $\mathbb{C}^2$

Kohei Ueno（上野康平）
Toba National College of Maritime Technology

Abstract

We study the dynamics of polynomial skew products on $\mathbb{C}^2$. By using suitable weights, we prove the existence of several types of Green functions. Largely, continuity and plurisubharmonicity follow. Moreover, it relates to the dynamics of the rational extensions to weighted projective spaces.

1 Introduction

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$ such that $p(z) = az^\delta + O(z^{\delta-1})$ and $q(z,w) = b(z)w^d + O_z(w^{d-1})$. Let $\gamma = \deg b$. We assume that $\delta \geq 2$ and $d \geq 2$. Then we may assume that polynomials p and b are monic by taking an affine conjugate; $p(z) = z^\delta + O(z^{\delta-1})$ and $b(z) = z^\gamma + O(z^{\gamma-1})$. Let $\lambda = \max\{\delta, d\}$, which coincides with the dynamical degree of f.

The dynamics of f consists of the dynamics on the base space and the dynamics on the fibers. The first component p defines the dynamics on the base space $\mathbb{C}$. Note that f preserves the set of vertical lines in $\mathbb{C}^2$. For this reason, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to the vertical line $\{z\} \times \mathbb{C}$ can be viewed as the composition of n polynomials on $\mathbb{C}$, $q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$.

A useful tool in the study of the dynamics of p on the base space is the Green function G_p of p,

$$G_p(z) = \lim_{n\to\infty} \frac{1}{\delta^n} \log^+ |p^n(z)|.$$

It is well known that G_p is defined, continuous and subharmonic on $\mathbb{C}$. More precisely, G_p is harmonic and positive on A_p and zero on K_p, and $G_p(z) = \log|z| + o(1)$ as $z \to \infty$. Here $A_p = \{z : p^n(z) \to \infty\}$ and $K_p = \{z : \{p^n(z)\}_{n\geq 1}$ bounded$\}$. By definition, $G_p(p(z)) = \delta G_p(z)$. In a similar fashion, we consider the fiberwise Green function of f,

$$G_z(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |Q_z^n(w)| \text{ or } G_z^\lambda(w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |Q_z^n(w)|,$$

where $Q_z^n = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$. By definition, $G_{p(z)}(q_z(w)) = dG_z(w)$ if it exists. Since G_p exists on $\mathbb{C}$, the existence of G_z implies that of the Green function G_f of f,

$$G_f(z,w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |f^n(z,w)|,$$

where $|(z,w)| = \max\{|z|,|w|\}$. Favre and Guedj proved the existence of G_z on $K_p \times \mathbb{C}$ in [1, Theorem 6.1], which is continuous and plurisubharmonic if $b^{-1}(0) \cap K_p = \emptyset$. Hence the remaining problem lies in investigating the existence of G_z on $A_p \times \mathbb{C}$. In [2], with the assumption $\gamma = 0$, we studied the existence of G_z and concluded that the weighted Green function G_f^α of f,

$$G_f^\alpha(z,w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |f^n(z,w)|_\alpha,$$

where $|(z,w)|_\alpha = \max\{|z|^\alpha,|w|\}$, is defined, continuous and plurisubharmonic on $\mathbb{C}^2$ for a suitable rational number $\alpha \geq 0$. Moreover, f extends to an algebraically stable map on a weighted projective space, whose dynamics relates to G_f^α.

In this report, assuming $\gamma \neq 0$, we investigate the existence of G_z on $A_p \times \mathbb{C}$, which implies the existence of G_f and G_f^α. Although the dynamics becomes much more difficult without the condition $\gamma = 0$, the idea of imposing suitable weights is still effective. We also show the existence of other Green functions such as

$$G_z^\alpha(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left| \frac{Q_z^n(w)}{p^n(z)^\alpha} \right| \text{ and } G(z,w) = \lim_{n\to\infty} \frac{1}{d^n} \log \left| \frac{Q_z^n(w)}{p^n(z)^{n\frac{\gamma}{d}}} \right|$$

in the cases $\delta \neq d$ and $\delta = d$, respectively.

2 Weights

Let $q(z,w) = z^\gamma w^d + \sum_j c_j z^{n_j} w^{m_j}$, where $c_j \neq 0$, and define α as

- $\min\{l \in \mathbb{Q} : l\delta \geq \gamma + ld$ and $l\delta \geq n_j + lm_j$ for any $j\}$ if $\delta > d$,
- $\min\{l \in \mathbb{Q} : \gamma + ld \geq l\delta$ and $\gamma + ld \geq n_j + lm_j$ for any $j\}$ if $\delta < d$,
- $\inf\{l \in \mathbb{Q} : \gamma + ld \geq n_j + lm_j$ for any $j\}$ if $\delta = d$.

Since q has only finitely many terms, we can take the minimum if $\delta \neq d$. In the case $\delta = d$, we can replace inf by min if $q(z,w) \neq b(z)w^d$, and $\alpha = -\infty$ if $q(z,w) = b(z)w^d$. Let $W_R = \{|z| > R, |w| > R|z|^\alpha\}$. We then obtain the following main lemma from the definition of α.

Lemma 2.1. *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, or $\delta \leq d$, then $q(z,w) \sim z^\gamma w^d$ on W_R for large $R > 0$; that is, the ratio of q and $z^\gamma w^d$ on W_R tends to 1 as $R \to \infty$. Moreover, f preserves W_R; that is, $f(W_R) \subset W_R$.*

Since $p(z) \sim z^\delta$ as $z \to \infty$, this lemma implies that $f(z,w) \sim (z^\delta, z^\gamma w^d)$ on W_R as $R \to \infty$. Let $A_f = \cup_{n \geq 0} f^{-n}(W_R)$. This lemma induces the existence, continuity and pluriharmonicity of the Green functions on A_f; the results are written in the next section.

We explain the importance of α and Lemma 2.1 in terms of the weighted homogeneous part of q. Let us define the weight of a monomial $z^n w^m$ as $n + \alpha m$, and let h be the weighted homogeneous part of q of highest weight. In the case $\delta > d$, the polynomial h may not contain $z^\gamma w^d$. However, if $\alpha = \gamma/(\delta - d)$, then h contains $z^\gamma w^d$. On the other hand, h always contains $z^\gamma w^d$ in the case $\delta \leq d$.

In addition, it is useful to consider the dynamics of the rational extensions of f to weighted projective spaces. Let r and s be any two positive integers. We denote a point in the weighted projective space $\mathbb{P}(r,s,1)$ by weighted homogeneous coordinates $[z : w : t]$. It follows that f extends to a rational map $\tilde{f}$ on $\mathbb{P}(r,s,1)$,

$$\tilde{f}[z : w : t] = \left[p\left(\frac{z}{t^r}\right) t^{\lambda r} : q\left(\frac{z}{t^r}, \frac{w}{t^s}\right) t^{\lambda s} : t^\lambda \right].$$

Let L_∞ be the line at infinity $\{t = 0\}$, and let $I_{\tilde{f}}$ be the indeterminacy set of $\tilde{f}$. Because $\gamma \neq 0$, the point $p_\infty = [0 : 1 : 0]$ is always an indeterminacy point. In the case $\delta > d$, the point p_∞ is the unique indeterminacy point, and $\tilde{f}$ is algebraically stable if $s/r \geq \alpha$. More precisely, if $s/r = \alpha$ then the dynamics on $L_\infty - \{p_\infty\}$ is induced by the polynomial $h(1, w)$, and if $s/r > \alpha$ then $\tilde{f}$ contracts $L_\infty - \{p_\infty\}$ to the attracting fixed point $[1 : 0 : 0]$. On the other hand, $\tilde{f}$ contracts $L_\infty - I_{\tilde{f}}$ to p_∞ in the case $\delta \leq d$. Therefore, if $\delta > d$ and $\alpha = \gamma/(\delta - d)$, or if $\delta \leq d$, then p_∞ is attracting in some sense. For these cases, W_R is included in the attracting basin of p_∞, and A_f is the restriction of the attracting basin of p_∞ to $A_p \times \mathbb{C}$.

3 Results on Green functions

Now we state our results on the existence, continuity and plurisubharmonicity of the Green functions of f. This section divides into three subsections: the cases $\delta > d$, $\delta < d$ and $\delta = d$. Since α can be negative unlike the case $\gamma = 0$, we redefine $|(z, w)|_\alpha$ as $\max\{|z|^{\max\{\alpha,0\}}, |w|\}$. See [3] for the proofs and more details.

3.1 The case $\delta > d$

Observing the dynamics of $\tilde{f}$ on $\mathbb{P}(r, s, 1)$, where $s/r > \alpha$, we obtain the following upper estimate of G_z^λ.

Proposition 3.1. *If $\delta > d$, then $\tilde{G}_z^\lambda \leq \alpha G_p$ on $A_p \times \mathbb{C}$, where $\tilde{G}_z^\lambda = \limsup_{n\to\infty} \lambda^{-n} \log^+ |Q_z^n|$.*

Corollary 3.2. *If $\delta > d$, then $G_f^\alpha = \alpha G_p$ on $\mathbb{C}^2$.*

Now we apply Lemma 2.1 to show the existence of G_z^α.

Theorem 3.3. *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then the limit G_z^α is defined, continuous and pluriharmonic on A_f. Moreover,*

$$\left|G_z^\alpha(w) - \log\left|z^{-\alpha}w\right|\right| < C_R,$$

where C_R tends to 0 as $R \to \infty$, $G_z^\alpha \sim \log|w|$ as $w \to \infty$ for fixed z in A_p, and G_z^α tends to 0 as (z, w) in A_f tends to $\partial A_f - J_p \times \mathbb{C}$.

Let $B_f = A_p \times \mathbb{C} - A_f$. Since $G_z^\alpha = 0$ on B_f, this theorem implies the existence of G_z^α on $A_p \times \mathbb{C}$.

Corollary 3.4. *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then G_z^α is defined, continuous and plurisubharmonic on $A_p \times \mathbb{C}$. Moreover, it is pluriharmonic on A_f and $\mathrm{int}B_f$.*

Theorem 3.3 also implies the existence of G_z^λ and G_f on A_f.

Corollary 3.5. *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then $G_z^\lambda = \alpha G_p$ and $G_f = \max\{\alpha, 1\}G_p$ on $(K_p \times \mathbb{C}) \cup A_f$.*

We end this subsection with a claim on the uniform convergence to G_z^α and the asymptotics of G_z^α near infinity. Let $h(c) := h(1,c)$. Because h and G_h have some symmetries related to the denominator of α, the Green function $G_h(z^{-\alpha}w)$ is well defined.

Proposition 3.6. *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then the convergence to G_z^α is uniform on $V \times \mathbb{C}$, where $\overline{V} \subset A_p$, and $G_z^\alpha(w) = G_h(z^{-\alpha}w) + o(1)$ as $z \to \infty$.*

3.2 The case $\delta < d$

Lemma 2.1 induces the existence of G_z.

Theorem 3.7. *If $\delta < d$, then the limit G_z is defined, continuous and pluriharmonic on A_f. Moreover,*

$$\left|G_z(w) - \log\left|z^{\gamma/(d-\delta)}w\right|\right| < C_R \text{ on } W_R,$$

where C_R tends to 0 as $R \to \infty$, $G_z \sim \log|w|$ as $w \to \infty$ for fixed z in A_p, and G_z tends to 0 as (z,w) in A_f tends to $\partial A_f - J_p \times \mathbb{C}$.

Since $G_z = 0$ on B_f, this theorem implies the following corollary.

Corollary 3.8. *If $\delta < d$, then G_z is defined on $\mathbb{C}^2$, which is continuous and plurisubharmonic on $A_p \times \mathbb{C}$. Moreover, $G_z = G_f = G_f^\alpha$ on $\mathbb{C}^2$, and $G_z = G_z^\alpha$ on $A_p \times \mathbb{C}$.*

The convergence to G_z seems not to be uniform on W_R. However, one can prove that the convergence to G_z^α is uniform on W_R if $\alpha = \gamma/(\delta - d)$.

Proposition 3.9. *If $\delta < d$ and $\alpha = \gamma/(\delta - d)$, then the convergence to G_z^α is uniform on $V \times \mathbb{C}$, where $\overline{V} \subset A_p$, and $G_z^\alpha(w) = G_h(z^{-\alpha}w) + o(1)$ as $z \to \infty$.*

3.3 The case $\delta = d$

The dynamics of f differs depending on whether $\gamma = 0$ or $\gamma \neq 0$. See [2] for the case $\gamma = 0$; if $\delta = d$ and $\gamma = 0$ then f extends to holomorphic maps on weighted projective spaces. Assuming $\gamma \neq 0$, we obtain the following three theorems from Lemma 2.1.

Theorem 3.10. *If $\delta = d$ and $\gamma \neq 0$, then $G_z = \infty$ on A_f and $\tilde{G}_z \leq \max\{\alpha, 0\} G_p$ on B_f, where $\tilde{G}_z = \limsup_{n\to\infty} d^{-n} \log^+ |Q_z^n|$.*

Corollary 3.11. *If $\delta = d$ and $\gamma \neq 0$, then*

$$G_f^\alpha(z, w) = \begin{cases} \infty & \text{on } A_f \\ \max\{\alpha, 0\} G_p(z) & \text{on } B_f. \end{cases}$$

Theorem 3.12. *If $\delta = d$ and $\gamma \neq 0$, then*

$$\lim_{n\to\infty} \frac{1}{n\gamma d^{n-1}} \log^+ |f^n(z, w)| = \lim_{n\to\infty} \frac{1}{n\gamma d^{n-1}} \log^+ |Q_z^n(w)|$$

$$= \begin{cases} G_p(z) & \text{on } (K_p \times \mathbb{C}) \cup A_f \\ 0 & \text{on } B_f. \end{cases}$$

Theorem 3.13. *If $\delta = d$ and $\gamma \neq 0$, then the limit G is defined, continuous and pluriharmonic on A_f. Moreover,*

$$|G(z, w) - \log |w|| < C_R \text{ on } W_R,$$

where the constant $C_R > 0$ tends to 0 as $R \to \infty$, $G \sim \log |w|$ as $w \to \infty$ for fixed z in A_p, and G tends to $-\infty$ as (z, w) in A_f tends to any point in $\partial A_f - J_p \times \mathbb{C}$.

Since $G = -\infty$ on B_f, this theorem implies the following corollary.

Corollary 3.14. *If $\delta = d$ and $\gamma \neq 0$, then G is defined and plurisubharmonic on $A_p \times \mathbb{C}$ if we admit minus infinity.*

References

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