数理解析研究所講究録
第 1807 巻 2012 年 21-30

ON THE COEFFICIENTS OF THE RIEMANN MAPPING FUNCTION FOR THE EXTERIOR OF THE MANDELBROT SET

HIROKAZU SHIMAUCHI

ABSTRACT. We consider the family of rational maps of the complex plane given by $P_{d,c}(z):=z^{d}+c$ where $c\in \mathbb{C}$ is a parameter and $d\in \mathbb{N}\setminus\{1\}$. The generalized Mandelbrot set is the set of all $c\in \mathbb{C}$ such that the forward orbit of 0 under $P_{d,c}$ is bounded. Let $f_{d}:\mathbb{D}\rightarrow \mathbb{C}\setminus\{1/z:z\in \mathcal{M}_{d}\}$ and $\Psi_{d}:\hat{\mathbb{C}}\setminus\overline{\mathbb{D}}\rightarrow\hat{\mathbb{C}}\setminus \mathcal{M}_{d}$ be the Riemann mapping functions and let their expansions be $f_{d}(z)=z+\sum_{m=2}^{\infty}a_{d,m}z^{m}$ and $\Psi_{d}(z)=z+\sum_{m=0}^{\infty}b_{d,m}z^{-m}$, respectively. We investigate several properties of the coefficients $a_{d,m}$ and $b_{d,m}$. In this paper, we concentrate on the zero coefficients of f_{d}. Detailed statements and proofs will be presented in [13].

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk, $\mathbb{D}^{*}$ the exterior of the closed unit disk, $\mathbb{C}$ the complex plane and $\hat{\mathbb{C}}$ the Riemann sphere. Furthermore let $G\subsetneq \mathbb{C}$ be a simply connected domain with $0\in G$ and $G'\subsetneq\hat{\mathbb{C}}$ be a simply connected domain with $\infty\in G'$ which has more than one boundary point. In particular, there exist unique conformal mappings $f:\mathbb{D}\rightarrow G$ such that $f(0)=0$, $f'(0)>0$ and $g:\mathbb{D}^{*}\rightarrow G'$ with $g(\infty)=\infty$, $\lim_{z\rightarrow\infty}g(z)/z>0$. We call f and g the normalized Riemann mapping function of G and G'.

Let $c\in \mathbb{C}$, $n\in \mathbb{N}\cup\{0\}$ and $P_{c}(z):=z^{2}+c$. We denote the n-th iteration of P_{c} by $P_{c}^{\circ n}$ which is defined inductively by $P_{c}^{\circ n+1}=P_{c}\circ P_{c}^{\circ n}$ with $P_{c}^{\circ 0}(z)=z$. For each fixed c, the *filled-in Julia set* of $P_{c}(z)$ consists of those values z, which remain bounded under iteration. The boundary of the filled-in Julia set is called the *Julia set*. The *Mandelbrot set* $\mathcal{M}$ is the set of all parameters $c\in \mathbb{C}$ for which the Julia set of $P_{c}(z)$ is connected. It is known that $\mathcal{M}=\{c\in \mathbb{C}:\{P_{c}^{\circ n}(0)\}_{n=0}^{\infty}$ is bounded$\}$ is compact and is contained in the closed disk of radius 2 with center 0. Furthermore, $\mathcal{M}$ is connected. We want to note, that there is an important conjecture which states that $\mathcal{M}$ is locally connected (see [2]).

Douady and Hubbard demonstrated the connectedness of the Mandelbrot set by constructing a conformal isomorphism $\Phi:\hat{\mathbb{C}}\setminus \mathcal{M}\rightarrow \mathbb{D}^{*}$. If the inverse map $\Phi^{-1}(z)=:\Psi(z)=z+\sum_{m=0}^{\infty}b_{d,m}z^{-m}$ extends continuously to the unit circle, then the Mandelbrot set is locally connected, according to Carathéodory's continuity theorem. This is a motivation of our study.

Jungreis presented an method to compute the coefficients b_{m} of $\Psi(z)$ in [7]. Several detailed studies of b_{m} are given in [1, 3, 4, 9]. An analysis of the dynamics of $P_{d,c}(z):=z^{d}+c$ with an integer $d\geq 2$ is presented in [15]. The generalized Mandelbrot set is defined as $\mathcal{M}_{d}:=\{c\in \mathbb{C}:\{P_{d,c}^{\circ n}(0)\}_{n=0}^{\infty}$ is bounded $\}$, which is the connected locus of the Julia set of $P_{d,c}$ (see [10]). $\mathcal{M}_{d}$ is also connected, compact and contained in the closed disk of radius $2^{1/(d-1)}$ (see [8, 15]). Constructing the normalized Riemann mapping function $\Psi_{d}(z)=z+\sum_{m=0}^{\infty}b_{d,m}z^{-m}$ of $\hat{\mathbb{C}}\setminus \mathcal{M}_{d}$, Yamashita [15] analyzed the coefficients $b_{d,m}$.

In addition, Ewing and Schober studied the coefficients a_m of the Taylor series expansion of the function $f(z) := 1/\Psi(1/z)$ at the origin in [5]. The function f is the normalized Riemann mapping function of the exterior of the reciprocal of the Mandelbrot set $\mathcal{R} := \{1/z : z \in \mathcal{M}\}$. If f has a continuous extension to the boundary, the Mandelbrot set is locally connected.

In [14], we investigated properties of the coefficients $a_{d,m}$ of the normalized Riemann mapping function $f_d(z) = z + \sum_{m=2}^{\infty} a_{d,m} z^m$ for the exterior of the reciprocal of the generalized Mandelbrot set $\mathcal{R}_d := \{1/z : z \in \mathcal{M}_d\}$ and $b_{d,m}$. In this paper, we present several properties of $a_{d,m}$. In particular, we concentrate on the zero-coefficients.

2. Computation of the Coefficients $b_{d,m}$ and $a_{d,m}$

In this section, we present a method how to compute the coefficients $a_{d,m}$ and $b_{d,m}$ with $d \geq 2$. First we recall the construction of the inverse map of the normalized Riemann mapping function of $\widehat{\mathbb{C}} \setminus \mathcal{M}_d$ (see [1, 2, 7, 15]).

Theorem 1. *The map $\Phi_d : \widehat{\mathbb{C}} \setminus \mathcal{M}_d \to \mathbb{D}^*$ defined as*

$$\Phi_d(z) := z \prod_{k=1}^{\infty} \left(1 + \frac{z}{P_{d,z}^{\circ k-1}(z)^d} \right)^{\frac{1}{d^k}}$$

is a conformal isomorphism which satisfies $\Phi_d(z)/z \to 1 (z \to \infty)$.

We set $\Psi_d := \Phi_d^{-1}$ which is the normalized Riemann mapping function of $\widehat{\mathbb{C}} \setminus \mathcal{M}_d$. It follows immediately that $f_d(z) := 1/\Psi_d(1/z)$ is the normalized Riemann mapping function of $\mathbb{C} \setminus \mathcal{R}_d$. $\Psi_d(z)$ has the following property.

Proposition 2. *Let $n \in \mathbb{N} \cup \{0\}$ and $A_{d,n}(c) := P_{d,c}^{\circ n}(c)$. Then*

$$A_{d,n}(\Psi_d(z)) = z^{d^n} + O(1/z^{d^{n+1}-d^n-1}) \text{ as } z \to \infty.$$

This proposition leads to the next method, given by Jungreis in [7], to compute $b_{d,m}$.

Let $j \in \mathbb{N}$ be fixed. Assume that the values of $b_{d,0}, b_{d,1}, \ldots, b_{d,j-1}$ are known. Set $\hat{\Psi}_d(z) := z + \sum_{i=0}^{j} b_{d,i} z^{-i}$. Take $n \in \mathbb{N}$ large enough such that $j \leq d^{n+1} - 3$ is satisfied. Considering the definition of $A_{d,m}$ and the multinominal theorem, we obtain

$$\begin{aligned} A_{d,n}(\hat{\Psi}_d(z)) &= z^{d^n} + (d^n b_{d,0} + C) z^{d^n - 1} \\ &\quad + \sum_{i=1}^{j} (d^n b_{d,i} + q_{d,n,i-1}(b_{d,0}, b_{d,1}, \ldots, b_{d,i-1})) z^{d^n - i - 1} + O(z^{d^n - j - 2}) \end{aligned}$$

as $z \to \infty$, where C is a constant, and $q_{d,n,i-1}(b_{d,1}, b_{d,2}, \cdots, b_{d,i-1})$ is a polynomial of $b_{d,1}, b_{d,2}, \cdots, b_{d,i-1}$ which has integer coefficients. According to Proposition 2, the coefficients of $z^{d^n - j - 1}$ are zero. The desired $b_{d,j}$ is the solution of the algebraic equation

$$d^n b_{d,j} + q_{d,n,i-1}(b_{d,1}, b_{d,2}, \cdots, b_{d,j-1}) = 0.$$

Considering $a_{d,m} = -b_{d,m-2} - \sum_{j=2}^{m-1} a_{d,j} b_{d,m-1-j}$ for $m \in \mathbb{N} \setminus \{1\}$, we get $a_{d,m}$. In addition, we obtain the following lemma.

Lemma 3. *The coefficients $a_{d,m}$ and $b_{d,m}$ are d-adic rational numbers.*

Building a program to compute the exact values of $b_{2,m}$ and $a_{2,m}$ by using the C programing language with multiple precision arithmetic library GMP [6], we get the first 30000 exact values of $a_{2,m}$. Some of these values (numerator, exponent of 2 for the denominator) are presented in Table 1 of Section 5.

3. Coefficient Formula

In this section, we introduce a generalization of the coefficient formula presented in [5].

Theorem 4. *Let $n \in \mathbb{N}$, $2 \le m \le d^{n+1}-1$ and r sufficiently large. Then*

$$ma_{d,m} = \frac{1}{2\pi i}\int_{|w|=r} P^{\circ n}_{d,w}(w)^{m/d^n}\,\frac{dw}{w^2}.$$

This formula shows that $a_{d,m}$ is the coefficient of degree 1 of the Laurent series expansion of $P^{\circ n}_{d,w}(w)^{m/d^n}$ at ∞. Using `Mathematica`, we calculate the exact values of $a_{3,m}$, $a_{4,m}$, $a_{5,m}$, $a_{6,m}$ and $a_{7,m}$. Part of these values (numerator, exponent of each factor for the denominator) are presented in Tables 2, 3, 4, 5 and 6 of Section 5. In these tables, we omit the zero coefficients indicated in Corollary 6.

The next lemma follows from this theorem. Let $C_j(a)$ be the general binomial coefficient, i.e. for a real number a and $|x|<1$ it is $(1+x)^a = \sum_{j=0}^{\infty} C_j(a)\,x^j$.

Lemma 5. *Let $n, N \in \mathbb{N}$, $2 \le m \le d^{n+1}-1$ and $1 \le N \le n$. We obtain that $ma_{d,m}$ is the coefficient of w in the Laurent series of the expression*

$$\sum_{j_1=0}^{\infty}\cdots\sum_{j_N=0}^{\infty} C_{j_1}\left(\frac{m}{d^n}\right) C_{j_2}\left(\frac{m}{d^{n-1}} - dj_1\right) C_{j_3}\left(\frac{m}{d^{n-2}} - d^2 j_1 - dj_2\right)$$
$$\cdots C_{j_N}\left(\frac{m}{d^{n-N+1}} - d^{N-1}j_1 - d^{N-2}j_2 - \cdots - dj_{N-1}\right)$$
$$\times w^{j_1+\cdots+j_n} P^{\circ n-N}_{d,w}(w)^{m/d^{n-N} - d^N j_1 - d^{N-1}j_2 - \cdots - dj_N}.$$

Setting $N=n$ and considering $P^{\circ 0}_{d,w}(w) = w$ leads to the next corollary.

Corollary 6. *Let $n \in \mathbb{N}$ and $2 \le m \le d^{n+1}-1$. Then*

$$ma_{d,m} = \sum C_{j_1}\left(\frac{m}{d^n}\right) C_{j_2}\left(\frac{m}{d^{n-1}} - dj_1\right) C_{j_3}\left(\frac{m}{d^{n-2}} - d^2 j_1 - dj_2\right)$$
$$\cdots C_{j_n}\left(\frac{m}{d} - d^{n-1}j_1 - d^{n-2}j_2 - \cdots - dj_{n-1}\right),$$

where the sum is over all non-negative indices $j_1, \ldots, j_n$ such that $(d^n-1)j_1 + (d^{n-1}-1)j_2 + (d^{n-2}-1)j_3 + \cdots + (d-1)j_n = m-1$.

4. Zero Coefficients

Ewing and Schober proved the following theorem concerning these coefficients for $d=2$.

Theorem 7 (see [5]). *For any integers k and ν satisfying $k \ge 1$ and $2^\nu \ge k+1$, let $m = (2k+1)2^\nu$. Then $a_{2,m} = 0$.*

It is unknown whether the converse is true. They reported that their computation of 1000 terms of $a_{2,m}$ has not produced a zero-coefficient besides those indicated in the theorem [5]. The next statement is a generalization of the above.

Theorem 8. *Suppose the positive integers k, ν satisfy $\nu \ge 1, 2 \le k \le d^{\nu+1}-1$ and $k \not\equiv 0 (mod\ d)$. Then $a_{d,m} = 0$ for $m = kd^\nu$.*

For $d = 3$, if m is even, then $a_{d,m} = 0$. In addition, when $d = 4$, if $m \not\equiv 1 (\mathrm{mod}\, 3)$, then $a_{d,m} = 0$. This phenomena is caused by the rotation symmetry of the generalized Mandelbrot set (see [8, 15]). We gave a short proof in [13].

Corollary 9. *Suppose $d \geq 3$ and $m \not\equiv 1(\mathrm{mod}\, d-1)$. Then $a_{d,m} = 0$.*

Furthermore there are other zero-coefficients for $d \geq 3$. For example, $d = 3$ and $m = 39$. Some of these can be determined as follows:

Theorem 10. *Suppose $d \geq 3$ and the positive integers k, ν satisfy $\nu \geq 1, 2 \leq k \leq 2(d^{\nu+1} - 1), k \not\equiv 0(mod\ d)$ and $k \not\equiv -1(mod\ d)$. Then $a_{d,m} = 0$ for $m = kd^{\mu}$.*

5. Tables

m	Numerator	Exponent of 2
2	1	1
3	1	3
4	1	2
5	15	7
6	0	0
7	81	10
8	1	3
9	1499	15
10	1	5
11	16551	18
12	0	0
13	-19557	22
14	7	8
15	1026129	25
16	1	4
17	78558483	31
18	7	9
19	496067595	34
20	0	0
21	-506111055	38
22	135	12
23	66414150615	41
24	0	0
25	402782136143	46
26	683	16
27	-7661205650557	49
28	0	0
29	159606082621811	53
30	159	14
31	1420861495703249	56
32	1	5
33	118802466511637251	63
34	6147	20
35	978823547108164723	66
36	7	10
37	11679916854812498869	70
38	136987	23
39	87928513714596704251	73
40	0	0

TABLE 1. The coefficients of f_2

m	Numerator	Exponent of 3
3	1	1
5	1	2
7	2	4
9	1	2
11	52	6
13	155	8
15	0	0
17	2657	10
19	29533	13
21	0	0
23	-69655	15
25	2969930	17
27	1	3
29	23095973	19
31	56696777	21
33	10	6
35	2343898963	23
37	24995524274	26
39	0	0
41	115000492832	28
43	3201040250650	30
45	0	0
47	-6747874422283	32
49	27156979500091	34
51	206	9
53	1754740271356126	36
55	39359185743143624	40
57	104	10
59	664202023454689654	42
61	8022885267816295453	44
63	0	0
65	-7391510296706161637	46
67	221780172965492286820	48
69	998	11
71	-2686651941493059666679	50
73	-32087457055180397296552	53
75	8788	14
77	746925320310443260300229	55
79	6876851947180179910150669	57
81	1	4
83	124855798180021255239446495	59
85	637437763117857269357937478	61
87	39127	15
89	9942473917721354152195660708	63
91	120356314540026798358102260334	66
93	17849	15
95	238821046435703298297129023039	68
97	10637737798335560537468828132786	70
99	0	0
101	-10370735200491148482774112789591	72
103	111719030172930182970912859124588	74
105	9614018	19
107	14868303604623474006298195379693026	76
109	432892231404754050837137676654921275	80
111	808906	19

TABLE 2. The coefficients of f_3

m	Numerator	Exponent of 2
4	1	2
7	3	5
10	1	5
13	15	11
16	1	4
19	2995	16
22	93	12
25	59451	23
28	0	0
31	7405653	28
34	17127	20
37	102177851	34
40	0	0
43	-1017988077	39
46	2092125	27
49	716781072211	47
52	0	0
55	-8057836991135	52
58	-107583317	36
61	2910453741726705	58
64	1	6
67	91893393031048069	63
70	37808167947	43
73	1318087272305007215	70
76	231	15
79	444913124772728735913	75
82	15183120823331	51
85	5638826034225284751059	81
88	0	0
91	313435297799410921771475	86
94	2446791012271421	58
97	11845011179826719081484019 5	95
100	0	0
103	-1301193230791636493236184615	100
106	664048285923294771	68
109	51211345120475652876034366059 7	106
112	0	0
115	-3520423070490219326949797654607	111
118	-4572788779264571040 1	75
121	5062121757224909856951071869050 45	118
124	165891	25
127	587968416430716271654494224879163 63	123
130	2309263552422315210245 7	83
133	397055491958203159505945410084345677	129
136	715	19
139	78431329910398805770642975112640575077	134
142	5449594290814991549012715	90
145	798062417388638756928318972873439646543 1	142
148	0	0
151	91152299800810756756837172530825935597981	147
154	1498244827443611355653020543	99
157	3910680316997805881817069699983414104638528 5	153
160	0	0
163	-27213280152884716837462079140894594129957111 1	158
166	-5015072798157096341615114953	106

TABLE 3. The coefficients of f_4

m	Numerator	Exponent of 5
5	1	1
9	2	2
13	4	3
17	7	4
21	44	6
25	1	2
29	12272	8
33	36603	9
37	85256	10
41	669768	12
45	0	0
49	112321771	14
53	388257398	15
57	1032056524	16
61	9125770814	18
65	0	0
69	-81246358698	20
73	5215736042762	21
77	13061209292514	22
81	120874136987029	24
85	0	0
89	-1223557557246132	26
93	-6414828347025054	27
97	27497953655115 5328	28
101	896352130005917605 1	31
105	0	0
109	-89389483729234487652	33
113	-486246831892374376053	34
117	-1649151316991870622151	35
121	3914832540358666804 50124	37
125	1	3
129	1024311536213325470470 1388	39
133	2749155792596475256381 3559	40
137	5601189126322627686242 0782	41
141	36614819556139508710954 0258	43
145	1596	8
149	1824445993614563142695330 21049	45
153	6140136238112930375081755 96984	46
157	15663401715499055627209962 54608	47
161	131290248689017867660160220 08219	49
165	0	0
169	124065133010123770994353183 8913108	51
173	110903124662408197355804023 03782236	52
177	294830031134105108029518272 13615633	53
181	2724575608965032078286464589 48743729	55
185	0	0
189	-2621807240948417080067307939 236241968	57
193	6502836915359106963033502715 3700993403	58
197	6841982971801964491537396413 57729566871	59
201	2573564530241233016561193336 3120519759738	62
205	0	0
209	-2649760146482089323388601417 90274161278099	64
213	-1425266395615462618015450915 405906612862477	65
217	1821041180863951401284710902 1885181438394584	66
221	1091692220547900332047427749 01098359004201 7202	68

TABLE 4. The coefficients of f_5

m	Numerator	Exponent of 2	3
6	1	1	1
11	5	3	2
16	5	1	4
21	5	7	1
26	7	1	6
31	8645	10	8
36	1	2	2
41	44166115	15	10
46	96545	2	13
51	20051	18	2
56	224695	2	15
61	682050153785	22	17
66	0	0	0
71	510189065505655	25	19
76	412426453	2	21
81	120083275	31	2
86	2394396445	3	23
91	49144739612524327415	34	26
96	0	0	0
101	-2801171227435232984071	38	28
106	52774878534565	5	30
111	2597391412505	41	5
116	19211930633005	2	32
121	1137781778131315301990813815	46	34
126	0	0	0
131	-36348749336652649096486356745	49	36
136	-12198493242631315	1	40
141	36377488424879315	53	6
146	65241041542982158265	8	42
151	60530806118279000681493768465284389	56	44
156	0	0	0
161	-32627838290042061648075762005265809365	63	46
166	-4934686005225577895375	7	48
171	-260083422502506625	66	2
176	2963646870280316029705	0	50
181	20230431803670558980492779907280064385147543	70	53
186	0	0	0
191	-615519080443710081786835807335058560652341635	73	55
196	-2875148465039005768533865	2	57
201	-250734186268353826949737	78	7
206	-1714693662743917675142615131	9	59

TABLE 5. The coefficients of f_6

m	Numerator	Exponent of 7
7	1	1
13	3	2
19	10	3
25	33	4
31	102	5
37	276	6
43	3828	8
49	1	2
55	4886892	10
61	24323193	11
67	106806036	12
73	412326959	13
79	1338614628	14
85	21663929508	16
91	0	0
97	15881452467207	18
103	88458814741695	19
109	430684532453670	20
115	1827731386205295	21
121	6473496513847509	22
127	113543753471947366	24
133	0	0
139	-2397245366530485621	26
145	399981602588647434254	27
151	1896566348461207903576	28
157	8353830782381410698123	29
163	31069977330243000729086	30
169	573571363151516572431860	32
175	0	0
181	-13382662008145137285729374	34
187	-117342656009013997691001647	35
193	11347420366217549394264329589	36
199	41815800807425247397362906544	37
205	153373107599284780595599688311	38
211	2885412977789314774479022653216	40
217	0	0
223	-71706622150248358307439522992655	42
229	-651008699388212278045484479088121	43
235	-4098721239127187521462669622469906	44
241	345927006962224035750738278739011930	45
247	866140781977152053621693275086815604	46
253	15245220943627210103007062650905532493	48
259	0	0
265	-380306789601016566873834419097402398703	50
271	-3523387908136747198365040697162766512823	51
277	-22724723781272885854678843859430404875092	52
283	-117794971375196444565305062061434092520623	53
289	11051541173662752530186346243508710914942760	54
295	684816309855195490889404171649532724071846121	57
301	0	0
307	-14547727532901231765679437172717656067150007195	59
313	-134126312225852378807562286473816429601253045251	60
319	-872404263082910091131629789120394545746805945905	61
325	-4590488276809319231550940780999336662495098259853	62
331	-19440010418342626606123045335588177346374583978436	63

TABLE 6. The coefficients of f_7

ACKNOWLEDGEMENT

The author expresses his gratitude to Prof. Yohei Komori and Osamu Yamashita for their helpful comments and suggestions. He is also deeply grateful to Prof. Takehiko Morita and Prof. Hiroki Sumi for their generous support and encouragement.

REFERENCES

[1] B. Bielefeld , Y. Fisher and F. V. Haeseler, *Computing the Laurent series of the map* $\Psi : C \setminus \overline{D} \to C \setminus M$, Adv. in Appl. Math. **14** (1993), 25-38.
[2] A. Douady and J. H. Hubbard, *Exploring the Mandelbrot set*, The Orsay Notes, (1985).
[3] J. Ewing and G. Schober, *On the coefficients of the mapping to the exterior of the Mandelbrot set*, Michigan Math. J. **37** (1990), 315-320.
[4] J. Ewing and G. Schober, *The area of the Mandelbrot set*, Numerische Mathematik **61** (1992), 59-72.
[5] J. Ewing and G. Schober, *Coefficients associated with the reciprocal of the Mandelbrot set*, J. Math. Anal. Appl. **170** (1992), no. 1, 104-114.
[6] The GNU Multiple Precision Arithmetic Library, http://gmplib.org/.
[7] I. Jungreis, *The uniformization of the complement of the Mandelbrot set*, Duke Math. J. **52** (1985), no. 4, 935-938.
[8] E. Lau and D. Schleicher, *Symmetries of fractals revisited*, The Mathematical Intelligencer **18** (1996), no. 1, 45-51.
[9] G. M. Levin, *On the arithmetic properties of a certain sequence of polynomials*, Russian Math. Surveys 43 (1988), 245-246.
[10] S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda, *Holomorphic Dynamics*, Cambridge University Press, (2000).
[11] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, (1992).
[12] H. Shimauchi, *On the coefficients of the Riemann mapping function for the complement of the Mandelbrot set*, RIMS Kôkyûroku **1772** "Conditions for Univalency of Functions and Applications" (2011), 109-113.
[13] H. Shimauchi, *Coefficients associated with the reciprocal of the Generalized Mandelbrot set, proceedings of the 19-th ICFIDCAA*, Tohoku University press, submitted.
[14] H. Shimauchi, *On the coefficients of the Riemann mapping function for the exterior of the Mandelbrot set*, Master thesis (2012), Graduate School of Information Sciences, Tohoku University.
[15] O. Yamashita, *On the coefficients of the mapping to the exterior of the Mandelbrot set*, Master thesis (1998), Graduate School of Information Science, Nagoya University, http://www.math.human.nagoya-u.ac.jp/master.thesis/1997.html.

GRADUATE SCHOOL OF INFORMATION SCIENCES,
TOHOKU UNIVERSITY, SENDAI, 980-8579, JAPAN.
E-mail address: shimauchi@ims.is.tohoku.ac.jp