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Riemann surface laminations generated by complex dynamical systems

— and some topics on the Type Problem—

(複素力学系が生成するリーマン面ラミネーションと型問題について)

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Abstract

We give a definition of Riemann surface laminations associated with the (backward) dynamics of rational functions on the Riemann sphere, following Lyubich and Minsky. Then we sketch some recent developments on the *Type Problems*, which mainly concerns the existence of Riemann surfaces of hyperbolic type in the space of backward orbits.

1. Riemann surface laminations. We say a Hausdorff space $\mathcal{L}$ is a *Riemann surface lamination* if there exist an open cover $\{U_i\}$ of $\mathcal{L}$ and a collection of charts $\Phi_i : U_i \to \mathbb{D} \times T$, where $\mathbb{D}$ is the open unit disk of the complex plane $\mathbb{C}$ and T a topological space, such that all the transition maps $\Phi_j \circ \Phi_i^{-1}$ are of the form

$$\Phi_j \circ \Phi_i^{-1} : (z, t) \mapsto (F_{ij}(z, t), G_{ij}(t))$$

and $z \mapsto F_{ij}(z, t)$ is conformal for any t. A topological disk in $\mathcal{L}$ of the form $\Phi_i^{-1}(\mathbb{D} \times \{t\})$ is called a *plaque*. We say two points $p,\ q \in \mathcal{L}$ are *in the same leaf* if there exists a finite chain of plaques that connects p and q. Being "in the same leaf" is an equivalence relation. We call such an equivalent class a *leaf* of $\mathcal{L}$.

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2. Sullivan's solenoidal lamination. Sullivan [S] first applied the deformation theory of Riemann surface laminations to investigate dynamical systems. For a smooth (or more generally, $C^{1+\alpha}$) self-covering map f of the unit circle of degree $d \geq 2$, we can construct an associated Riemann surface lamination $\mathcal{L}^*$ with leaves isomorphic to the upper half plane. By taking a quotient by the lifted action of f, we have *Sullivan's solenoidal Riemann surface lamination.* Sullivan developed its Teichmüller theory to establish the existence of renormalization fixed point in the space of d-fold self-covering maps of the circle.

3. Lyubich-Minsky's laminations. In 1990's, inspired by Sullivan's work, M.Lyubich and Y.Minsky [LM] introduced the theory of hyperbolic 3-laminations associated with rational functions, which is analogous to the theory of hyperbolic 3-manifolds associated with Kleinian groups. They applied some ideas of rigidity theorems for hyperbolic 3-manifolds to their hyperbolic 3-laminations to have an extended version of Thurston's rigidity theorem for critically non-recurrent dynamics without parabolic cycles.

An important thing to remark is that Lyubich-Minsky's hyperbolic 3-lamination is constructed as an $\mathbb{R}^+$-bundle of a Riemann surface lamination.

4. Natural extension and regular part. Both Sullivan's and Lyubich-Minsky's laminations (we omit "Riemann surface" for brevity) are constructed out of the inverse limit of the dynamics. Let us recall Lyubich and Minsky's version.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a rational function of degree ≥ 2. It generates a non-invertible dynamical system $(f, \overline{\mathbb{C}})$ but it also generates an invertible dynamics in the space of backward orbits (the inverse limit)

$$\mathcal{N}_f := \left\{\hat{z} = (z_{-n})_{n\geq 0} \,:\, z_0 \in \overline{\mathbb{C}},\ z_{-n} = f(z_{-n-1})\right\}$$

with action

$$\hat{f}((z_0, z_{-1}, \ldots)) := (f(z_0), f(z_{-1}), \ldots) = (f(z_0), z_0, z_{-1}, \ldots).$$

We say $\mathcal{N}_f$ (with dynamics by $\hat{f}$) is the *natural extension* of f, with topology induced by $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \cdots$. We define the *projections* $\pi_{-n} : \mathcal{N}_f \to \overline{\mathbb{C}}$ by $\pi_{-n}(\hat{z}) := z_{-n}$, the $(-n)$-th entry of $\hat{z}$. Note that π_{-n} semiconjugates $\hat{f}$ and f.

The point $\hat{z} = (z_0, z_{-1}, \ldots)$ is *regular* if there exists a neighborhood U_0 of z_0 whose pull-back $\cdots \to U_{-1} \to U_0$ along $\hat{z}$ (*i.e.*, U_{-n} is the connected component of

$f^{-1}(U_{-n+1})$ containing z_{-n}) is eventually univalent. The *regular part* (or the *regular leaf space*) $\mathcal{R}_f$ of $\mathcal{N}_f$ is the set of all regular points, and we say each point in $\mathcal{N}_f - \mathcal{R}_f$ is *irregular*. The regular part is invariant under $\hat{f}$, and each path-connected component ("leaf") of the regular part possesses a Riemann surface structure isomorphic to $\mathbb{C}$, $\mathbb{D}$, or an annulus. (The annulus appears only when f has a Herman ring.)

5. Affine part and the affine lamination. We take the union of all leaves isomorphic $\mathbb{C}$ in $\mathcal{R}_f$ and call it the *affine part* $\mathcal{A}_f^{\mathrm{n}}$ of f. For each leaf L of $\mathcal{A}_f^{\mathrm{n}}$, we take a uniformization $\phi : \mathbb{C} \to L$. Then the sequence of maps $\{\psi_k = \pi_k \circ \phi : \mathbb{C} \to \overline{\mathbb{C}}\}_{k \le 0}$ are all non-constant and meromorphic satisfying $\psi_{k+1} = f \circ \psi_k$. So we regard it as an element of $\widehat{\mathcal{U}} = \mathcal{U} \times \mathcal{U} \times \cdots$, where $\mathcal{U}$ is the space of non-constant meromorphic functions on $\mathbb{C}$.

We say two elements $(\psi_k)_{k \le 0}$ and $(\psi'_k)_{k \le 0}$ in $\widehat{\mathcal{U}}$ are *equivalent* ($\sim$) if there exists an $a \neq 0$ such that $\psi_k(aw) = \psi'_k(w)$ for any $k \le 0$ and $w \in \mathbb{C}$. For a given $\hat{z} \in \mathcal{A}_f^{\mathrm{n}}$ in the leaf $L(\hat{z})$, we may choose a uniformization $\phi : \mathbb{C} \to L(\hat{z})$ so that $\phi(0) = \hat{z}$. Such a uniformization is determined up to pre-composition of rescaling $w \mapsto aw$ $(a \neq 0)$, hence $\hat{z}$ determines an equivalent class $\iota(\hat{z}) = [(\psi_k)_{k \le 0}]$ in $\widehat{\mathcal{U}}/\sim$.

Finally we define *Lyubich-Minsky's affine lamination* by

$$\mathcal{A}_f := \overline{\iota(\mathcal{A}_f^{\mathrm{n}})} \subset \widehat{\mathcal{U}}/\sim .$$

Remark. There is a bypass to construct $\mathcal{A}_f$ without using the regular part and the uniformizations: we may use the class of meromorphic functions generated by *Zalcman's lemma* instead.

6. The type problem. When the critical orbits of f behave nicely, we may regard $\mathcal{R}_f$ as a Riemann surface lamination with all leaves isomorphic to $\mathbb{C}$. Such a situation yields some nice properties of dynamics, like rigidity, or existence of conformal invariant measures on the lamination. For example, this is the case when f has no recurrent critical points in the Julia set [LM, Prop.4.5]. Another intriguing case is when f is an infinitely renormalizable quadratic map with a persistently recurrent critical point [KL, Lem.3.18].

For general cases, the following problem is addressed in [LM, §4, §10]:

Type problem. *When does $\mathcal{R}_f$ have leaves of hyperbolic type, especially leaves isomorphic to $\mathbb{D}$?*

(The counterpart, leaves isomorphic to $\mathbb{C}$, are conventionally called *parabolic.*) This question is closely related to the topology of $\mathcal{A}_f$:

Theorem 1 (Thm.1.3 of [KLR]) *If there exists a hyperbolic leaf L in the regular part $\mathcal{R}_f$ such that $\pi_0(L)$ intersects the Julia set, then $\mathcal{A}_f$ is not locally compact.*

Easy examples of hyperbolic leaves are provided by the invariant lifts of rotation domains, *i.e.,* Siegel disks and Herman rings. Non-rotational hyperbolic leaves (that are rather non-trivial) are constructed in the paper by J.Kahn, M.Lyubich, and L.Rempe [KLR, §3], that can be summarized as follows:

Theorem 2 (Thm.3.1 of [KLR]) *If the Julia set is contained in the postcritical set, then the regular part contains uncountably many hyperbolic leaves.*

Such hyperbolic leaves do not intersect the Julia set, hence we cannot apply Theorem 1. However, by using the tuning technique, they also showed:

Theorem 3 (Thm.1.1 and Prop.3.2 of [KLR]) *There exists a quadratic function $f(z) = z^2 + c$ whose regular part $\mathcal{R}_f$ contains hyperbolic leaves L such that $\pi_0(L)$ intersects the Julia set, In particular, $\mathcal{A}_f$ is not locally compact in this case.*

7. The Gross criterion. Here we sketch the idea of the proof of Theorem 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial of the form $f(z) = z^2 + c$. Let P and J denote the postcritical set and the Julia set. (Conventionally we remove ∞ from quadratic postcritical sets.) For the natural extension $\mathcal{N} = \mathcal{N}_f$, let $\pi = \pi_0 : \mathcal{N} \to \overline{\mathbb{C}}$ denote the projection.

Fix any $z_0 \in \mathbb{C} - P$. Then each $\hat{z} \in \pi^{-1}(z_0)$ is regular in $\mathcal{N}$. In particular, the projection $\pi : L(\hat{z}) \to \overline{\mathbb{C}}$ is locally univalent near $\pi : \hat{z} \mapsto z_0$.

Let $\ell(\theta)$ ($\theta \in [0, 2\pi)$) denote the half-line given by $\ell(\theta) := \{z_0 + re^{i\theta} : r \geq 0\}$. By using the Gross star theorem, *if $L(\hat{z})$ is isomorphic to $\mathbb{C}$, then for almost every angle $\theta \in [0, 2\pi)$ the locally univalent inverse $\pi^{-1} : z_0 \mapsto \hat{z}$ has an analytic continuation along the whole half-line $\ell(\theta)$* [KLR, Lem.3.3]. Hence the leaf $L(\hat{z})$ is hyperbolic if:

> ($*$): *There exist a $\hat{w} \in \pi^{-1}(z_0) \cap L(\hat{z})$ and a set $\Theta_0 \subset [0, 2\pi)$ of positive length such that for any $\theta \in \Theta_0$ the analytic continuation of $\pi^{-1} : z_0 \mapsto \hat{w}$ along $\ell(\theta)$ hits an irregular point $\hat{\imath}(\theta)$ at some $z = z_0 + re^{i\theta}$ ($r > 0$).*

To show Theorem 3, we first take a quadratic map g with $J_g = P_g$. By Theorem 2, such g has uncountably many hyperbolic leaves that are isomorphic to $\mathbb{D}$, but they do not intersect the Julia set. Now we apply the tuning technique. Let f be any tuned quadratic map of g. Roughly put, we first choose a small copy of the Mandelbrot set and we may take the parameter c in the small copy as the parameter corresponding to g . Then the postcritical set $P = P_f$ is still a union of continuum, and the backward orbits remaining in P provide continuums of irregular points. Then we can check the $(*)$-condition.

We say a hyperbolic leaf $L(\hat{z})$ that can be guaranteed by the condition $(*)$ is a *hyperbolic leaf of Gross type.*

8. Some results on Siegel, Feigenbaum and Cremer quadratic functions. In the quest of new non-rotational hyperbolic leaves, it is natural to ask the following question: *Is there any non-rotational hyperbolic leaf when f has an irrationally indifferent fixed point?* Because existence of such a fixed point implies existence of a recurrent critical point whose postcritical set is a continuum, and it seems really close to the situations in [KLR]. Let me present some results following a joint work [CK] with C.Cabrera (UNAM, Cuernavaca).

Siegel disk of bounded type. $f(z) = e^{2\pi i\theta}z + z^2$ with irrational θ of bounded type has a Siegel disk Δ centered at the origin, whose boundary $\partial\Delta$ is a quasicircle. In this case we have:

Theorem 4 (C-K) *In the regular part of the natural extension $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the only hyperbolic leaf is the invariant lift $\widehat{\Delta}$ of the Siegel disk.*

In the proof we use Lyubich and Minsky's criteria for parabolic leaves, *uniform deepness* of the postcritical set, and one of McMullen's results on bounded type Siegel disks. (In Paragraph 9 we will give a sketch the proof.)

Feigenbaum maps. It would be worth mentioning that the same method as the proof of Theorem 4 can be applied to a class of infinitely renormalizable quadratic maps, called *Feigenbaum maps.* We will have an alternative proof of:

Theorem 5 (Lyubich-Minsky) *The regular part $\mathcal{R}_f$ of a Feigenbaum map f has only parabolic leaves.*

Cremer points and hedgehogs. The situation for Cremer case looks more complicated. For any small neighborhood of Cremer fixed point ζ_0 of a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, there exists an invariant continuum H (a "hedgehog") containing ζ_0, equipped with invertible "sub-dynamics" $f|\ H \to H$.

According to an idea by A.Chéritat, we have

Theorem 6 (Lifted hedgehogs are irregular) *The invariant lift $\widehat{H}$ of H is a continuum contained in the irregular part of the natural extension.*

Since this natural extension has a continuum of irregular points, one may expect to apply the Gross criterion to find a hyperbolic leaf, as in [KLR]. However, the actual situation is not that good. It is still difficult to show the existence or non-existence of hyperbolic leaves without assuming the same conditions as [KLR]. Indeed, we can show that the irregular points in the hedgehogs are not big enough to apply the Gross criterion [CK, Thm 4.3]. In other words, by only the lifted hedgehogs we cannot construct hyperbolic leaves of Gross type: we need more irregular points!

9. Sketch of the proof of Theorem 4. Here we give a brief sketch of the proof of Theorem 4. In this case we have $\partial\Delta = P_f$.

Deep points and uniform deepness of the postcritical set. Let K be a compact set in $\mathbb{C}$. For $x \in K$, let $\delta_x(r)$ denote the radius of the largest open disk contained in $\mathbb{D}(x,r) - K$. (When $\mathbb{D}(x,r) \subset K$, we define $\delta_x(r) := 0$.) Then it is not difficult to check that the function $(x,r) \mapsto \delta_x(r)$ is continuous.

We say $x \in K$ is a *deep point* of K if $\delta_x(r)/r \to 0$ as $r \to 0$. For a subset P of K, we say P is *uniformly deep* in K if for any $\epsilon > 0$ there exists an r_0 such that for any $x \in P$ and $r < r_0$, we have $\delta_x(r)/r < \epsilon$.

We will use the following result by C.McMullen [Mc2, §4]:

Theorem 7 (Uniform deepness of $P_f = \partial\Delta$) *The postcritical set $P_f = \partial\Delta$ is uniformly deep in K_f, the filled Julia set of f.*

Here *the filled Julia set K_f* is defined by

$$K(f) := \{z \in \mathbb{C} : \{f^n(z)\}_{n\geq 0} \text{ is bounded}\}.$$

We take P as the postcritical set P_f of f.

Let $\mathcal{R} = \mathcal{R}_f$ be the regular part of $\mathcal{N}_f$, and $\widehat{\Delta}$ be the invariant lift of the Siegel disk Δ. We will show that any leaf L of $\mathcal{R} - \widehat{\Delta}$ is parabolic.

- We first show that any leaf L of $\mathcal{R} - \widehat{\Delta}$ contains a backward orbit $\hat{z} = \{z_{-n}\}_{n \geq 0}$ that stays in the basin at infinity. Let us fix such an orbit.

- When $\hat{z} = \{z_{-n}\}_{n \geq 0}$ does not accumulate on $P_f = \partial\Delta$, the leaf $L = L(\hat{z})$ is parabolic by a criterion of parabolicity by Lyubich and Minsky [LM, Cor.4.2].

- Now let us assume that $\hat{z} = \{z_{-n}\}$ accumulates on $P_f = \partial\Delta$. By another criterion of parabolicity by Lyubich and Minsky [LM, Lem 4.4], it is enough to show: *by taking n in a subsequence of $\mathbb{N}$, we have $\|Df^{-n}(z_0)\| \to 0$ $(n \to \infty)$, where Df^{-n} is the derivative of the branch of f^{-n} sending z_0 to z_{-n}, and the norm is measured in the hyperbolic metric of $\mathbb{C} - \partial\Delta$.*

- Now set $\Omega := \mathbb{C} - \overline{\Delta}$. Then z_{-n} is contained in Ω for all n. Since Ω is topologically a punctured disk, it has a unique hyperbolic metric $\rho = \rho(z)|dz|$ induced by the metric $|dz|/(1-|z|^2)$ of constant curvature -4 on the unit disk. To show the claim, it is enough to show

$$\|Df^n(z_{-n})\|_\rho = \frac{\rho(z_0)|Df^n(z_{-n})|}{\rho(z_{-n})} \to \infty \quad (n \to \infty),$$

where the norm in the left is measured in the hyperbolic metric ρ.

- By using $1/d$-metric (see for example, [Ah, Thm. 1-11]), we have $\rho(z) \leq \dfrac{1}{d(z, \partial\Omega)} = d(z, \partial\Delta)^{-1}$ for any $z \in \Omega$. Hence it is enough to show:

$$\|Df^n(z_{-n})\|_\rho \asymp \frac{|Df^n(z_{-n})|}{\rho(z_{-n})} \geq d(z_{-n}, \partial\Delta)|Df^n(z_{-n})| \to \infty. \tag{1}$$

- Set $R_n := d(z_{-n}, \partial\Delta)$. By assumption, R_n tends to 0 by taking n in a suitable subsequence. Let D_0 denote the disk of radius R_0 centered at z_0, and let U_n denote the connected component of $f^{-n}(D_0)$ containing z_{-n}. Since $D_0 \subset \Omega$, we have a univalent branch $g_n : D_0 \to U_n$ of f^{-n} with $g_n(z_0) = z_{-n}$. Set $v_n := |Dg_n(z_0)| = |Df^n(z_{-n})|^{-1} > 0$. By the Koebe 1/4 theorem, $g_n(D_0) = U_n$ contains the disk of radius $R_0 v_n/4$ centered at z_{-n}, and since $U_n \subset f^{-n}(\Omega) \subset \Omega$ we have $R_0 v_n/4 \leq R_n$.

- First assume that $\liminf v_n/R_n = 0$. If n ranges over a suitable subsequence, we have $v_n/R_n \to 0$ and thus (1) holds.

- Next consider the case when $\liminf v_n/R_n = q > 0$. We may assume that n ranges over a subsequence with $\lim v_n/R_n = q$.

  For $t > 0$, let tD_0 denote the disk $\mathbb{D}(z_0, tR_0)$. Since $D_0 = \mathbb{D}(z_0, R_0)$ is centered at a point in $\mathbb{C} - K$, we can choose an $s < 1$ such that $sD_0 \subset \mathbb{C} - K$. By the Koebe 1/4 theorem, $|g_n(sD_0)|$ contains $\mathbb{D}(z_{-n}, sR_0v_n/4) \subset \mathbb{C} - K$.

- Let us take a point x_n in $\partial\Delta$ such that $|x_n - z_{-n}| = R_n$. Then we have

$$\mathbb{D}(z_{-n}, sR_0v_n/4) \subset \mathbb{D}(x_n, 2R_n)$$

  and thus $\delta_{x_n}(2R_n) \geq sR_0v_n/4$. Recall the assumption $v_n/R_n \sim q > 0$ for $n \gg 0$. This implies that the ratio $\delta_{x_n}(2R_n)/2R_n$ is bounded by a positive constant from below. However, $R_n = d(z_{-n}, \partial\Delta) \to 0$ by assumption and it contradicts to the uniform deepness of P_f (Theorem 7). ■

According to the technique of Theorem 4, it seems reasonable to conjecture the following

Conjecture. *There exists a Cremer quadratic polynomial whose regular part has no hyperbolic leaf.*

References

[Ah] L.V. Ahrlfors. *Conformal Invariants.* McGraw-Hill, 1973.

[CK] C. Cabrera and T. Kawahira. On the natural extensions of dynamics with a Siegel or Cremer point. *To appear in J. Difference Equ. Appl.*. (arXiv:1103.2905v1 [Math.DS])

[KL] V.A. Kaimanovich and M. Lyubich. Conformal and harmonic measures on laminations associated with rational maps. *Mem. Amer. Math. Soc.*, **820**, 2005.

[KLR] J. Kahn, M. Lyubich, and L. Rempe. A note on hyperbolic leaves and wild laminations of rational functions. *J. Difference Equ. Appl.*, **16** (2010), no. 5–6, 655–665.

[LM] M. Lyubich and Y. Minsky. Laminations in holomorphic dynamics. *J. Diff. Geom.* **49** (1997), 17–94.

[S] D. Sullivan. Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. *Topological Methods in Modern Mathematics*, L. Goldberg and Phillips, editor, Publish or Perish, 1993

[Mc2] C. McMullen. Self-similarity of Siegel disks and Hausdorff dimension of Julia sets. *Acta Math.* **180** (1998), no.2, 247–292.