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INDEPENDENCE AND NON-TRIVIALITY OF RIGID SECONDARY COMPLEX CHARACTERISTIC CLASSES

TARO ASUKE

ABSTRACT. Complex secondary characteristic classes for transversely holomorphic foliations are discussed. Examining a complexification of an example of Baker [3], we will present a family of rigid, linearly independent complex secondary classes. Such a family is firstly given by Hurder [8], from which ours is different. The content of this article is based on [2], in which the details including proofs can be found. This is also based on a talk given at 'Integrated Research on Complex Dynamics' held at RIMS, Kyoto University in 2012.

Introduction

Let $B\Gamma_q$ be the classifying space for real codimension-q foliations. Given a real codimension-q foliation $\mathcal{F}$ of a manifold M, we can find a classifying mapping from M to $B\Gamma_q$. Similarly, there exists the classifying space for transversely holomorphic foliations which is denoted by $B\Gamma_q^{\mathbb{C}}$, where q denotes the complex codimension. As in the case of vector bundles, cohomology classes in $H^*(B\Gamma_q^{\mathbb{C}})$ are characteristic classes for transversely holomorphic foliations, where coefficients are chosen in $\mathbb{C}$ in this article. There is a dga (differential graded algebra) denoted by WU_q (see Section 1) and a homomorphism from $H^*(\mathrm{WU}_q)$ to $H^*(B\Gamma_q^{\mathbb{C}})$. The classes in the image other than (polynomials of) classical Chern classes are called complex secondary classes, and the homomorphism is called the characteristic homomorphism and denoted by χ. A natural question is if χ is injective, or surjective. They are still open. In this article we will focus on the injectivity. Suppose that the kernel of the characteristic homomorphism is non-trivial. Then, elements in the kernel are linear relation among secondary classes. On the other hand, it is known that some of secondary classes admit continuous deformations. Namely, they can

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vary continuously according to deformations of foliations. It is rare that fixed linear relations are kept under deformations so that the linear independence of some secondary classes can be shown. This kind of arguments is however not valid for those classes which are rigid under deformations (such classes are called rigid classes). In the case where complex normal bundles are trivial, secondary characteristic classes are constructed by the classifying space $\overline{B\Gamma_q^{\mathbb{C}}}$ and a dga $\mathrm{W}_q^{\mathbb{C}}$ (see Section 1 for a brief explanation). Hurder studied the mapping $H^*(\mathrm{W}_q^{\mathbb{C}}) \to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$ [8], and obtained a family of rigid, linearly independent classes. Hurder's method is based on homotopies. We will discuss the case where complex normal bundles are not necessarily trivial. It will be done by using an example of transversely holomorphic foliation which is a complexification of an example of Baker [3]. By evaluating secondary classes of the foliation, one can obtain a family of linearly independent classes in the image of $H^*(\mathrm{WU}_q)$ in $H^*(B\Gamma_q^{\mathbb{C}})$. Some of classes in the family are rigid (and some others are variable), and are also products of secondary classes. These classes form a family of rigid classes which is different from Hurder's one. The evaluation is based on a result of Baker [3].

This article is an announcement of [2], in which the details including proofs can be found.

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1. Preliminaries

First we introduce secondary characteristic classes.

Definition 1.1. Let $\mathbb{C}[v_1, \dots, v_q]$ be the polynomial ring generated by $v_1, \dots, v_q$, where the degree of v_j is set to be $2j$. We denote by I_q the ideal generated by monomials of degree greater than $2q$, and set $\mathbb{C}_q[v_1, \dots, v_q] = \mathbb{C}[v_1, \dots, v_q]/I_q$.

We also define $\mathbb{C}_q[\bar{v}_1, \ldots, \bar{v}_q]$ by replacing v_j by $\bar{v}_j$. We set

$$\begin{aligned}
\mathrm{WU}_q &= \bigwedge[\widetilde{u}_1, \ldots, \widetilde{u}_q] \otimes \mathbb{C}_q[v_1, \ldots, v_q] \otimes \mathbb{C}_q[\bar{v}_1, \ldots, \bar{v}_q], \\
\mathrm{W}_q &= \bigwedge[u_1, \ldots, u_q] \otimes \mathbb{C}_q[v_1, \ldots, v_q], \\
\mathrm{W}_q^{\mathbb{C}} &= \bigwedge[u_1, \ldots, u_q, \bar{u}_1, \ldots, \bar{u}_q] \otimes \mathbb{C}_q[v_1, \ldots, v_q] \otimes \mathbb{C}_q[\bar{v}_1, \ldots, \bar{v}_q].
\end{aligned}$$

These algebras are equipped with derivatives such that $d\widetilde{u}_i = v_i - \bar{v}_i$, $du_i = v_i$, $d\bar{u}_i = \bar{v}_i$ and $dv_i = d\bar{v}_i = 0$. The degree of $\widetilde{u}_i$, u_i and $\bar{u}_i$ are set to be $2i-1$. We define $\overline{\mathrm{W}_q}$ by replacing u_i and v_j by $\bar{u}_i$ and $\bar{v}_j$ in W_q. Then, $\mathrm{W}_q^{\mathbb{C}} = \mathrm{W}_q \wedge \overline{\mathrm{W}_q}$.

It is easy to see that $H^*(\mathrm{W}_q^{\mathbb{C}})$ is isomorphic to $H^*(\mathrm{W}_q) \otimes H^*(\overline{\mathrm{W}_q})$, and that there is a natural inclusion from $H^*(\mathrm{W}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$. There is also a natural mapping from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ which maps $\widetilde{u}_i$ to $u_i - \bar{u}_i$, v_i to v_i and $\bar{v}_i$ to $\bar{v}_i$, respectively. Indeed, this mapping corresponds to the natural mapping from $\overline{B\Gamma_q^{\mathbb{C}}}$ to $B\Gamma_q^{\mathbb{C}}$, which is a part of the homotopy fibration $\overline{B\Gamma_q^{\mathbb{C}}} \to B\Gamma_q^{\mathbb{C}} \to \mathrm{BGL}(q;\mathbb{C})$ and also is the classifying map of the $\Gamma_q^{\mathbb{C}}$-structure of $\overline{B\Gamma_q^{\mathbb{C}}}$, namely, the map which forgets the triviality of the complex normal bundle.

The following result is classical.

Theorem 1.2. *Let $B\Gamma_q^{\mathbb{C}}$ be the classifying space of transversely holomorphic foliations of complex codimension q, and let $\overline{B\Gamma_q^{\mathbb{C}}}$ the classifying space of transversely holomorphic foliations of complex codimension q with trivialized complex normal bundles. Then, there is a well-defined homomorphism $H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$ and $H^*(\mathrm{W}_q^{\mathbb{C}}) \to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$.*

The above homomorphisms are called the characteristic homomorphisms. Let M be a manifold and $\mathcal{F}$ a transversely holomorphic foliation of M, of complex codimension q. Then, the classifying map induces a homomorphism from $H^*(\mathrm{WU}_q)$ to $H^*(M)$. If the complex normal bundle of $\mathcal{F}$ is trivial, then by fixing the homotopy type of a trivialization, we obtain a homomorphism from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M)$. Elements of $H^*(\mathrm{WU}_q)$, $H^*(\mathrm{W}_q^{\mathbb{C}})$ and $H^*(\mathrm{W}_q)$ or their image under the characteristic homomorphisms which involve $\widetilde{u}_i$, u_i or $\bar{u}_i$ are called complex secondary characteristic classes.

Let ρ be a mapping from WU_{q+1} to WU_q such that if $i \leq q$ then $\rho(\widetilde{u}_i) = \widetilde{u}_i$, $\rho(v_i) = v_i$, $\rho(\bar{v}_i) = \bar{v}_i$ and if $i = q+1$ then $\rho(\widetilde{u}_i) = 0$, $\rho(v_i) = 0$ and $\rho(\bar{v}_i) = 0$, respectively. Then, ρ induces a homomorphism $\rho_*\colon H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$. It is known that classes in the image of ρ_* are rigid under deformations of foliations. Such classes are called rigid classes. Similar homomorphisms from W_{q+1} to W_q and from $\mathrm{W}^{\mathbb{C}}_{q+1}$ to $\mathrm{W}^{\mathbb{C}}_q$ are also defined. The classes in the image are also rigid under deformations, and called rigid classes. On the other hand, a class $\omega \in H^*(\mathrm{WU}_q)$ is said to be variable if ω varies continuously with respect to a family of foliations.

There are some significant classes.

Definition 1.3. 1) The class $u_1 v_1^q \in H^{2q+1}(\mathrm{W}_q)$ or its image in $H^{2q+1}(\mathrm{W}^{\mathbb{C}}_q)$ is called the Bott class.

2) The class $\widetilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \cdots + \bar{v}_1^q) \in H^{2q+1}(\mathrm{WU}_q)$ is called the imaginary part of the Bott class.

3) The class $\sqrt{-1}\widetilde{u}_1 v_1^q \bar{v}_1^q \in H^{4q+1}(\mathrm{WU}_q)$ is called the Godbillon-Vey class.

These classes are known to be non-trivial. We have the following.

Theorem 1.4. 1) *The image of the imaginary part of the Bott class in* $H^*(\mathrm{W}^{\mathbb{C}}_q)$ *is equal to* $u_1 v_1^q - \bar{u}_1 \bar{v}_1^q$.

2) *The image of the Godbillon-Vey class in the sense of Definition 1.3 under the characteristic homomorphism coincides with the classical Godbillon-Vey class (for real foliations). Moreover, the Godbillon-Vey class belongs to the image of* $\rho_*\colon H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$ *so that it is a rigid class.*

Remark 1.5. It is known that real secondary classes for transversely holomorphic foliations are complex secondary classes defined in terms of $H^*(\mathrm{WU}_q)$. It is in particular the case for the Godbillon-Vey class which is the most significant real secondary class. The second claim of Theorem 1.4 implies that the image of the classical Godbillon-Vey class in $H^{4q+1}(\mathrm{WU}_q)$ is equal to $\sqrt{-1}\widetilde{u}_1 v_1^q \bar{v}_1^q$. It is well-known that the Godbillon-Vey class admits continuous deformations in the category of real foliations [11], [12], nevertheless, it is rigid in the category of transversely holomorphic foliations (see Theorem 2.3).

2. Independent classes of $H^*(\mathrm{WU}_q)$

We begin with a classical example.

Example 2.1 ([4]). Let X be a holomorphic vector field on $\mathbb{C}^{n+1}$ defined by

$$X(z) = \sum_{i=0}^{n} \lambda_i z_i \frac{\partial}{\partial z_i},$$

where each λ_i is a non-zero complex number. Then, the integral curves of X form a holomorphic foliation of $\mathbb{C}^{n+1} \setminus \{0\}$. This foliation is invariant under homothecies so that a holomorphic foliation $\mathcal{F}_\lambda$ is induced on $(\mathbb{C}^{n+1} \setminus \{0\})/\times 2 \cong S^1 \times S^{2n+1}$. It is well-known that the complex normal bundle of $\mathcal{F}_\lambda$ is trivial, and $u_1 v_J(\mathcal{F}_\lambda) = \dfrac{v_1 v_J(\lambda_0, \ldots, \lambda_n)}{\lambda_0 \cdots \lambda_n} \mathrm{vol}_{S^{2n+1}}$, where v_J is a Chern polynomial of degree n [4] (see also [7]). It follows that the classes $u_1 v_J$ are variable and linearly independent.

We study the following example, which is a complexification of Baker's example [3, Example 1(a)].

Example 2.2. Let $G = \mathrm{SL}(k+n; \mathbb{C})$, where $n > k > 0$ or $n = k = 1$, $H = \{(a_{ij}) \in G \mid a_{ij} = 0 \text{ if } i > k \text{ and } j \leq k\}$, $K = \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n))$ and $T = T^{k+n-1}$ the maximal torus in G realized as diagonal matrices. Then, the left cosets of H induce transversely holomorphic foliations of complex codimension kn on $\varGamma \backslash G/T$ and $\varGamma \backslash G/K$, where $\varGamma$ is a discrete subgroup of G such that $\varGamma \backslash G/K$ is a closed manifold.

Secondary characteristic classes of this kind of foliations, so-called locally homogeneous foliations, are well-studied, e.g. [9], [3], [1]. In particular, if $n = k = 1$, then it is a complexification of an example of Roussarie [6]. If $k = 1$, then the Godbillon-Vey class is studied in [1]. The case where $n = 3$ and $k = 2$ is studied by Enatsu [5].

We have a complexification of [3, Theorem 5.3] as follows. If $I = \{i_1, \ldots, i_l\}$ then we set $\widetilde{u}_I = \widetilde{u}_{i_1} \cdots \widetilde{u}_{i_l}$. We denote $\widetilde{u}_I$ also by $u_{i_1, \ldots, i_l}$. If $I = \varnothing$ then we set $\widetilde{u}_I = 1$ and regard $i_1 = +\infty$.

Theorem 2.3 ([2, Theorem 2.2]). *Let $I = \{i_1, \ldots, i_l\}$ and suppose that $k < i_1 < \cdots < i_l \leq n$. The classes of the form $\widetilde{u}_{1,\ldots,k} \widetilde{u}_I v_1^{kn} \bar{v}_1^{kn}$ are non-trivial and linearly independent in $H^*(\varGamma \backslash \mathrm{SL}(k+n; \mathbb{C})/T)$. These classes are rigid classes, namely, rigid under deformations of foliations if $k = 1$ and $i_1 > 2$.*

The last claim can be seen as follows. The cochain $\widetilde{u}_{1,\ldots,k} \widetilde{u}_I (v_1^{kn+1} \bar{v}_1^{kn-1} + v_1^{kn} \bar{v}_1^{kn} + v_1^{kn-1} \bar{v}_1^{kn+1})$ is mapped to the cocycle $\widetilde{u}_{1,\ldots,k} \widetilde{u}_I v_1^{kn} \bar{v}_1^{kn}$ under $\rho \colon \mathrm{WU}_{q+1} \to \mathrm{WU}_q$, and is closed if $k = 1$ and $i_1 > 2$.

Let $n = 3$ and $k = 2$ in Example 2.2. Then, we obtain a transversely holomorphic foliation of complex codimension 6. We set $\eta_{2,6} = \widetilde{u}_2 v_1^6 + \widetilde{u}_1 (v_1^5 + v_1^4 \bar{v}_1 +$

$\cdots + \bar{v}_1^5)\bar{v}_2$. Then ξ_6 and $\eta_{2,6}$ are non-trivial, because we have $\xi_6\eta_{2,6} = \widetilde{u}_1\widetilde{u}_2 v_1^6\bar{v}_1^6$ and it is non-trivial by Theorem 2.3. Hence we have a non-trivial class which is a product of secondary classes. On the other hand, if we set $\mu = \widetilde{u}_1\widetilde{u}_2(v_1^6\bar{v}_1^5 + v_1^5\bar{v}_1^6)$, then μ is closed and $\mu v_1 = \widetilde{u}_1\widetilde{u}_2 v_1^6\bar{v}_1^6$. Hence μ and v_1 are also non-trivial. Actually v_1^6 is a generator of $H^{12}(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)))$. On the other hand, by examining the degree of generators of $H^*(\mathrm{SU}(5))\otimes H^*(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)))$, we see that the Godbillon-Vey class $\widetilde{u}_1 v_1^6\bar{v}_1^6$ is trivial. We have $\widetilde{u}_1 v_1^6\bar{v}_1^6 = \xi_6 v_1^6$. Hence, unlike the above cases, a product of non-trivial classes yields a trivial class.

We remark finally that using a result of Kamber-Tondeur [10], one can show an analogue of Theorem 2.3 [2, Theorem 2.3].

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Graduate School of Mathematical Sciences, University of Tokyo, 3–8–1, Komaba, Meguro-ku, Tokyo 153–8914, Japan

E-mail address: asuke@ms.u-tokyo.ac.jp