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On Classification of Closure Spaces: Search for the Methods and Criteria

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Abstract. *This paper has as its main objective a critical review of typical classifications based on the disciplines of application (e.g. topology, algebra, geometry) and their particular needs. Special attention is given to the example of geometric closure spaces and to the question what property or properties should be used to distinguish this category. In this context, new class of closure spaces of character n is introduced. Arguments are provided that geometric closure spaces should be distinguished as closure spaces of character 2. Also, some characteristics of closure spaces of character n are given. Finally, the exchange property of closure spaces which is usually considered as defining for geometric closure spaces is associated with the issue of disjoint union decomposability of closure spaces. Some suggestions are made regarding more meaningful, comprehensive classification of closure spaces.*

Key words: *Closure space, Closure operator, Pre-closure operator, Classification of closure spaces, Disjoint unions of closure spaces.*

1. INTRODUCTION

In the hundred years of its presence in mathematics, concept of a closure space, or in other words of a set with closure operator, has found applications in many disciplines. Closure spaces are usually informally classified using additional properties which are added to the three axioms of a closure operator understood as a function f on the power set of a set S such that (1) for every subset A of S, $\mathbf{A} \subseteq \mathbf{f(A)}$; (2) for all subsets A, B of S, $\mathbf{A} \subseteq \mathbf{B} \Rightarrow \mathbf{f(A)} \subseteq \mathbf{f(B)}$; (3) for every subset A of S, $\mathbf{f(f(A))} = \mathbf{f(A)}$.

Additional conditions for classifications of closure spaces are being selected from the properties of particular examples of closure spaces which played significant roles in the domains of application of this concept. Although the identification of the properties distinguishing particular types of closure spaces (topological, algebraic, geometric, etc.) seems well motivated and meaningful, actually when we look carefully, the choice turns out to be quite arbitrary.

This paper has as its main objective to review critically typical classification based on the disciplines of application and their particular needs. Special attention is given to the example of geometric closure spaces and to the question what property or properties should be used to distinguish this category. In this context, new class of closure spaces of character n is introduced. Arguments are provided that geometric closure spaces should be distinguished as closure spaces of character 2. Also, some characteristics of closure spaces of character n are given. Finally, the exchange property of closure spaces which is usually considered as defining for geometric closure spaces is associated with the issue of decomposability of closure spaces. Some suggestions are made regarding more meaningful, comprehensive classification of closure spaces.

The text is referring to some results of the author which belong to the articles currently in preparation for publication. For this reason in this paper the proofs will be omitted as they will be presented elsewhere.

2. PARADIGMATIC PROPERTIES OF CLOSURE SPACES

The concept of a closure operator was introduced about a hundred years ago. The early work on closure operation which left trace in the literature of the subject was published in 1910 by Eliakim Hastings Moore in his "Introduction to a Form of General Analysis." [1] However, it was the formulation of the axioms for topological space in terms of a closure operator by Kazimierz Kuratowski [2] in 1922 which introduced this concept into common use.

To be more precise, Kuratowski was analyzing the operation on subsets of a topological space which assigned to each set an extension including in addition to its own elements its limit points. It turned out that the closure operation with four axioms gives an alternative definition of the original topological space.

The axioms considered by Kuratowski implied two conditions stronger than those considered by Moore. In this latter more general approach it is not necessary to assume that closure of an empty set is empty and that the operation is finitely additive with respect to union of sets (closure of the finite union of sets is the union of closures). Soon it was recognized that there are many closure operations of fundamental importance for mathematics, such as syntactic consequence used by Alfred Tarski in his algebraization of logic, closing a subset of an algebra to the least subalgebra including it, which do not require additivity.

The two examples of non-topological closures had in place of the additivity condition another property called finite character, which in the simplified latter form asserts that if an element x belongs to the closure of a set A, then it belongs to the closure of some finite subset of A (or equivalently that the closure of a set is equal to the union of closures of its all finite subsets). Since metric spaces which served as the original structures in which topological properties were studied, as well as the majority of early examples of topological spaces satisfy the condition (T_1) that one element sets are closed, and combination of conditions for such topological spaces with the finite character property produce the unique trivial closure system with all subsets closed, it was natural to conceive the classification of closure spaces into topological (additive), and algebraic (with finite character).

The algebraic closure spaces were in the privileged situation. Very early, when the theory of closure spaces started to develop, Garrett Birkhoff and Orrin Frink [3] showed that whenever closure space on a set A has finite character, there exists on this set an algebra (i.e. algebraic structure), so that the closure is its subalgebra closure. In contrast, topological closures defined simply by the finite additivity condition were very far from closures defined in metric spaces, the original structures in which topologies were being introduced. This stimulated intensive studies of the conditions which have to be added to additivity to make the topological space homeomorphic with some topological space on a metric space (i.e. to find a representation in a metric space).

Interest in the metrization produced a wide range of conditions of increasing strength which were intended to bring back realization of a topological space in a metric space. They had two main forms of separation (and therefore are called separation axioms and indicated with the letter T (from German Trennungsaxiom) with an index i indicating strength of the condition. One type was based on the requirement that a pair of disjoint subsets of specific properties (viz. one element set, closed set) can be included in disjoint open subsets (i.e. in complements of closed subsets). The other is based on the existence of a continuous function from the topological space to real numbers, such that a pair of disjoint sets (as before) has distinct point images (usually 0 and 1, but this is a matter of convenience). Although metrizable topological space satisfies the strongest T_6 axiom, it turns out that this cdondition is still not sufficient for metrizability, which requires additional conditions.

Separation axioms were important for the development of topology, but they are of marginal interest in more general considerations of closure spaces. However, the large variety of examples of topological spaces very different from those metrizable which were introduced to make distinctions in this partially hierarchic classification shows that the choice of additivity as the defining property of topology is very weak and quite arbitrary.

On the other hand, some considerations related to generalizations of metric spaces led to abandoning of the third condition for closure operator (called transitivity) which assumes that closure of a set is closed, without giving up the additivity. These so called pre-closure operators remain in the margins of topology, but are of some interest for other disciplines. We will be using pre-closure operators (called simply operators) defined by only two first axioms for closure operators (as it is now a common practice) to formulate a conceptual framework for classification of closure spaces.

In spite of the representation theorem, the situation is not better for algebraic closures. Representation of algebraic closures as subalgebra closures requires algebras which may have n-ary operations of finite type, but for arbitrarily high n. It is not a problem in theoretical setting of universal algebra, but is going way beyond typical algebras with at most binary operations. Moreover, the finite character property belongs to axioms of some more specific closure spaces. Consequence operator has been mentioned above, but an example closer to the interests of topology can be found in geometry.

Using example of topology, geometry has been formulated in terms of closure spaces. The work in this direction was initiated by Reinhold Baer [4] in 1952 in his axiomatization of projective geometry. Further attempts to grasp the essence of geometry reduced the axioms of geometry to a closure operator on a set S which in addition to the conditions already known in topology that the empty set is closed and that one element set is closed (T_1), and to the condition of finite character defining algebraic closures, has apparently very "geometric" exchange property: For every subset A of S and for all x, y belonging to S, if $x \in f(A \cup \{y\})$, but $x \notin f(A)$, then $y \in f(A \cup \{x\})$.

It was a surprise that when in turn convex geometries were axiomatized in terms of closure operators, this axiom had to be replaced by another seemingly (but not exactly) contradictory condition called anti-exchange property: if $x \neq y$, $x \in f(A \cup \{y\})$ and $x \notin f(A)$, then $y \notin f(A \cup \{x\})$.

As it was in the case of topology, some additional conditions were added to the exchange property to bring back the axiomatics in terms of closure spaces to original axioms of projective and affine geometry [5]. However, the existing approach while making consistent projective and affine geometries, does not resolve inconsistency with convex geometry. In axiomatizations in terms of closure spaces they are formulated separately, as if they did not have a common root in synthetic geometry.

Before we look more carefully at the properties of closure spaces which were used to axiomatize geometry, some important aspects of the paradigm of the study of closure spaces should be presented. From the very beginning, i.e. from the work of Moore at the beginning of the 1900's, closure spaces were associated with Moore families of subsets of a set S, defined as families of subsets which include all set S and the intersections of their arbitrary subfamilies. Moore families are in bijective correspondence with closure operators. For each closure operator f on a set S, the family of its closed subsets f-Cl is a Moore family of subsets. On the other hand, given Moore family of subsets defines a closure operator assigning to a set A the intersection of all members of the family including A.

As soon as lattices appeared in mathematics, it has been recognized that Moore families are complete lattices of subsets. Thus, it is natural to investigate the properties of the lattice L_f of the closed subsets for the closure operator f. From that time lattice theory has become the main tool of the study of closure spaces, and closure spaces provided set theoretical realizations of complete lattices. Garret Birkhoff's classical book on lattice theory [6] includes almost all early significant contributions to the study of closure spaces.

When we are looking for the justification for the finite additivity of the closure operator as a criterion for topological character of the closure space, it may seem that the reason could be in the fact that when closure operator f is additive, the lattice L_f is distributive. However, finite additivity is a stronger condition, as the following simple example shows. There are closure operators which have distributive lattice of closed subsets, but which are not additive. If T, U are disjoint, but not complementary subsets of S, we can define a closure operator f by its Moore family of closed subsets consisting of the empty set, T, U, and S. Of course, the lattice of closed subsets is distributive (or even Boolean), but $f(T \cup U) = S$ and $f(T) \cup f(U) = T \cup U \neq S$.

It is easy to understand the strength of the finite additivity condition, if we recognize that it simply means that the lattice join of closed subsets, which in the general case is the closure of the union of closures (i.e. $f(A) \vee f(B) = f(f(A) \cup f(B)))$, in this case is simply the union of closures (i.e. $f(A) \vee f(B) = f(A) \cup f(B))$. Since the dual equality holds for all closure operators (i.e. $f(A) \wedge f(B) = f(A) \cap f(B))$, in finitely additive closure spaces join and meet of L_f are identical with set union and intersection. However, the strength of the condition does not help to answer the question why this particular type of closure space should be distinguished. The only answer is purely pragmatic, it is very useful in some applications, and distributivity of L_f is only one of convenient consequences.

Later, the modularity of L_f as well as its weakening to semimodularity have been associated with geometry, but the association of closure spaces with the properties of L_f remains without systematization.

Another source of the paradigmatic methods of closure space classification is in the particular example of the closure space defined by the Moore family of all subspaces of a vector space. From vector spaces we can generalize the concept of independence and generation for subsets of any closure space.

The family f-Ind of independent subsets of a closure space with closure operator f is defined by the condition that subset B is independent if for every element x in B we have $x \notin f(B \setminus \{x\})$.

Set B generates closure space, if $f(B) = S$. Finally, the independent, generating subsets can be called bases. This gives a natural distinction of closure spaces which have bases, and then which have equicardinal bases. But both these classes, especially the latter are quite narrow, and the properties are meaningful or even fundamental in some applications (e.g. in matroid theory), but marginal from the point of view of general theory of closure spaces.

One of the reasons for the original interest in the concept of a base of a closure space had its source in misunderstanding. The fact that a set generates all closure space is crucial in a vector space, because we can express every element of the vector space using only elements of this generating set. Moreover, we can express every subspace using only elements of the generating set. It does not work this way in more general closure spaces. Generating set (or base) generates all set S and only limited subset of subspaces. Thus, the analogy with the concepts in vector spaces is limited.

3. WHAT MAKES CLOSURE SPACE GEOMETRICAL?

The following part of the paper will require more formal format. I will use the concept of a pre-closure space (in the following simply operator space) $<S,f>$ defined on a set S as mapping f (operator) of its power set into itself, such that for all subsets A, B of S: (i) $A \subseteq f(A)$ and (ii) If $A \subseteq B$, then , $f(A) \subseteq f(B)$.

As in the case of closure spaces, the subsets of S satisfying $A = f(A)$, i.e. closed subsets always form a complete lattice L_f. But different operators may have the same family of closed subsets.

Closure spaces are spaces in which operator f satisfies the transitivity condition:

(I) $\mathbf{f(f(A)) = f(A)}$. In such case we can write $\mathbf{f \in I(S)}$.

It is commonly assumed that geometric closure spaces satisfy two additional conditions:

(N) $\mathbf{f(\varnothing) = \varnothing}$, written in short as $\mathbf{f \in N(S)}$ and

(T_1) $\mathbf{\forall a \in S\colon f(\{a\}) = a}$, in short $\mathbf{f \in T_1(S)}$.There will be also mentioned a weaker condition: **(T_0)** $\mathbf{\forall a,b \in S\colon f(\{a\}) = f(\{b\}) \Rightarrow a = b}$.

In the first, more popular of the two dominating approaches to geometry focusing on the projective or affine geometries and their generalizations, a geometry is defined as a closure space $<S,f>$ in which $\mathbf{f \in NT_1I(S)}$, and such that f satisfies two additional conditions, the "finite character" property:

(fC) $\mathbf{\forall A \subseteq S \forall x \in S\colon x \in f(A) \Rightarrow \exists\, B \in Fin(A)\colon x \in f(B)}$, where Fin(A) is a set of all finite subsets of A), and the "exchange property" (of Steinitz):

(wE) $\mathbf{\forall A \subseteq S \forall x,y \in S\colon x \notin f(A)\ \&\ x \in f(A \cup \{y\}) \Rightarrow y \in f(A \cup \{x\})}$

At this point, the formulation of projective or affine geometries in terms of closure operators splits into a wide range of different, sometimes non-equivalent theories.

A projective geometry is frequently defined by only one additional condition for a geometry called the "projective law":

(pL) $\mathbf{\forall A,B \subseteq S\ \&\ A,B \Rightarrow \varnothing \neq f(A \cup B) = \{f(\{x,y\})\colon x \in f(A)\ \&\ y \in f(B)\}}$.

However, such geometry may have very strange properties contradicting our spatial intuition (e.g. different lines intersecting in more than one point,) so other conditions are sometimes added.

In geometries defined as closure spaces ($\mathbf{f \in NT_1fCwEI(S)}$) the additional condition making such a structure consistent with our intuition of spatial relations gives a special role to the closures of pairs of points (lines): $\mathbf{\forall A \subseteq S\colon [A=f(A)\ iff\ \forall x,y \in A\colon f(\{x,y\}) \subseteq A]}$.

Thus, projective geometries are sometimes defined by the Projective Law and the condition of linearity (above).

To maintain the usual relationship between projective and affine geometries, the definition of the latter includes the usual condition of Euclid's "Fifth Postulate":

$$\mathbf{\forall x,y,z,p,q,r \in S\colon f(\{p,q\}) \subseteq f(\{x,y,x\})\ \&\ r \notin f(\{p,q\}) \Rightarrow}$$
$$\mathbf{\exists t,u \in f(\{x,y,x\})\colon t \neq u\ \&\ r \in f(\{t,u\}\ \&\ f(\{p,q\} \cap f(\{t,u\}) = \varnothing} \quad \text{and}$$

every other closure of two points satisfying this condition is identical with **f({t,u})**.

Along with the Fifth Postulate, the condition called "strong planarity," which is satisfied automatically by projective geometries, is assumed in order to maintain the relationship between the two forms of geometry, as it has to be expected from affine geometries.

Strong planarity adds to the planarity **($\forall A\subseteq S$: [$A=f(A)$ iff $\forall x,y,z\in A$: $f(\{x,y,x\})\subseteq A$])** additional condition: **(sP) $\forall A\subseteq S \forall p,q\in S \forall r\in f(A)$: $p\in f(A\cup\{q\}) \Rightarrow \exists s\in f(A)$: $p\in f(\{q,r,s\}$.**

This conceptual framework gives complete translation of projective and affine geometries into the language of closure spaces, but does not allow recovery of all geometry without going outside of it.

All earlier or recent attempts to recover either Hilbert's Axioms of Order or the concept of convexity are referring to external concepts such as for instance orientation.

Convex geometries belong to the other direction in geometry, less known and studied, but still with big volume of literature. They are (usually) defined as closure spaces <S,f> such that **$f\in NT_1fC(S)$** and that f satisfies "**anti-exchange**" condition:

(awE) $\forall A\subseteq S \forall x,y\in S$: $x\neq y$ & $x\notin f(A)$ & $x\in f(A\cup\{y\}) \Rightarrow y\notin f(A \cup \{x\})$

It is easy to see that the anti-exchange condition is a generalization of the basic property of Hilbert's "betweenness," which also is related to exchange property. However, the connection of such convex geometries with projective and affine geometries on one hand, and synthetic geometry on the other is not as simple as could be expected, unless we assume some additional strong conditions.

There is a natural question about properties common for both types of geometries. Of course, in both cases we have **$f\in NT_1fC(S)$**.

Also, it is obvious that in both cases we have:

(linearity) $\forall A\subseteq S$: [$A=f(A)$ iff $\forall x,y\in A$: $f(\{x,y\})\subseteq A$], or at least

(planarity) $\forall A\subseteq S$: [$A=f(A)$ iff $\forall x,y,z\in A$: $f(\{x,y,x\})\subseteq A$].

Notice that Hilbert's Axioms of Connection are related to the first of the conditions when f({x,y}) is interpreted as a line, and at the same time his Axioms of Order are used to define convexity by using the same condition when f({x,y}) is interpreted as a segment.

The conditions above have some affinity with the second of the equivalent formulations of the finite character property (**fC**):

i) $\forall A\subseteq S \forall x\in S$: $x\in f(A) \exists \Rightarrow B\in Fin(A)$: $x\in f(B)$,

ii) $\forall A\subseteq S$: $A=f(A)$ iff $\forall B\in Fin(A)$: $f(B)\subseteq A$.

iii) $\forall A\subseteq S$: $f(A) = \cup\{ f(B)$: $B\in Fin(A)\}$.

However, the equivalence is lost when instead of assuming finiteness of set B, we assume some particular finite number of elements, as in the conditions of linearity or planarity.

DEFINITION 3.1 *An operator f on set S is of character n if:*

(C_n) $\forall A\subseteq S$: $A=f(A)$ iff $\forall B \subseteq A$: $|B|\leq n \Rightarrow f(B)\subseteq A$.

There is a straightforward relationship between different levels of character n property and finite character property:

PROPOSITION 3.1 $f\in C_n(S) \Rightarrow f\in C_{n+1}(S) \Rightarrow f\in fC(S)$.

Thus, when we define geometry using C_2 (or C_n for any n) the finite character property becomes redundant.

The n character property for lowest values of n is relating closure operators (i.e. transitive operators) to binary relations.

PROPOSITION 3.2

i) $f \in \mathbf{K}C_0(S)$ *iff* $\exists T \subseteq S$: $f(A) = A$ *for* $T \subseteq A$ *and* $f(A) = A \cup T$ *otherwise.*

ii) $f \in \mathbf{IN}C_1(S)$ *iff there exists a reflexive and transitive relation (quasiorder) R on S, such that* $\forall A \subseteq S$: $f(A) = R^e(A) := \{y \in S: \exists x \in S: xRy\}$.

iii) $f \in \mathbf{INT}_0 C_1(S)$ *iff there exists partial order R, such that* $f(A) = R^e(A)$

iv) $f(A) = R^e(A)$ *and R is an equivalence relation iff* $f \in \mathbf{IN}C_1(S)$ *and f satisfies:* $\forall x,y \in S$: $x \in f(\{y\}) \Rightarrow y \in f(\{x\})$.

For closure spaces of character n higher than one we can easily get an analogue of the Birkhoff-Fring theorem for finite character closure spaces using the same idea for the proof.

PROPOSITION 3.3 *For every closure space <S,f> of character n there exists an algebra with operations of n-arity not exceeding n, such that f is its subalgebra closure operator.*

Thus, closure operators of character 2 are associated with algebras equipped only with unary and binary operations.

As I mentioned before, the equivalence between the three formulations of the finite character property is lost for character n. In this case the first condition is obviously equivalent to the third:

$$(\mathbf{sC_n})\forall A \subseteq S: \mathbf{f(A)} = \cup\{\mathbf{f(B)}: B \subseteq A \ \& \ |B| \leq n\},$$

but they are stronger than $\mathbf{C_n}$ itself, i.e. $\mathbf{sC_n(S)} \subseteq \mathbf{C_n(S)}$. Surprisingly, the Projective Law defining projective geometries

$$(\mathbf{pL})\ \forall A,B \subseteq S \ \& \ A,B \Rightarrow \varnothing \neq \mathbf{f}(A \cup B) = \{\mathbf{f}(\{x,y\}): x \in \mathbf{f(A)} \ \& \ y \in \mathbf{f(B)}\}$$

turns out to be a weakening of this stronger condition for n = 2, which places projective geometries between $\mathbf{C_2}$ and $\mathbf{sC_2}$.

PROPOSITION 3.4 $\quad sC_2(S) \subseteq pL(S) \subseteq C_2(S)$.

Finally, the condition $\mathbf{C_2}$ with the direct translations of the Axioms of Order in terms of closure operators and one additional condition (third condition below) allow formulating axioms of synthetic geometry in terms of closure spaces. The closure here is a generalization of the convex hull operation, which is consequently used to define the concept of a line. If in addition to these three conditions we assume the projective law, we can recover the anti-exchange property and the closure operator becomes a familiar convex hull operator.

More specifically we need the following three conditions:

i) $\forall \mathbf{x,y,z} \in \mathbf{S}: \mathbf{y} \in \mathbf{f}(\{\mathbf{x,z}\}) \ \& \ \mathbf{x} \in \mathbf{f}(\{\mathbf{y,z}\}) \Rightarrow \mathbf{x=y}$.

ii) $\forall \mathbf{p,q,r,s,t,u} \in \mathbf{S}: \mathbf{t \neq u} \ \& \ \{\mathbf{t,u}\} \subseteq \mathbf{f}(\{\mathbf{p,r}\}) \cap \mathbf{f}(\{\mathbf{q,s}\}) \Rightarrow$

$$\exists \mathbf{x,y}\{\in \mathbf{p,q,r,s}\}: \mathbf{x \neq y} \ \& \ \mathbf{f}(\{\mathbf{p,q,r,s}\}) \subseteq \mathbf{f}(\{\mathbf{x,y}\}.$$

iii) $\forall \mathbf{x,y,z} \in \mathbf{S}: \mathbf{z} \in \mathbf{f}(\{\mathbf{x,y}\}) \Rightarrow \mathbf{f}(\{\mathbf{x,y}\}) = \mathbf{f}(\{\mathbf{x,z}\}) \cup \mathbf{f}(\{\mathbf{z,y}\})$

Then we can derive:

iv) $\forall \mathbf{p,q,r,s} \in \mathbf{S}: \mathbf{q} \in \mathbf{f}(\{\mathbf{p,r}\}) \ \& \ \mathbf{r} \in \mathbf{f}(\{\mathbf{q,s}\}) \Rightarrow \quad \{\mathbf{q,r}\} \subseteq \mathbf{f}(\{\mathbf{p,s}\})$.

v) $\forall \mathbf{A} \subseteq \mathbf{S} \forall \mathbf{x,y} \in \mathbf{S}: \mathbf{A} \subseteq \mathbf{f}(\{\mathbf{x,y}\}) \ \& \ \mathbf{A} \in \mathbf{Fin(S)} \Rightarrow \exists \ \mathbf{r,s} \in \mathbf{A}: \mathbf{A} \subseteq \mathbf{f}(\{\mathbf{r,s}\})$.

Then, we can define a line passing through two different points w,z∈ S as the set:

$$\mathbf{L_{wz}} = \cup\{\mathbf{f}(\{\mathbf{r,s}\}): \{\mathbf{w,z}\} \subseteq \mathbf{f}(\{\mathbf{r,s}\})\}.$$

It can be shown that this definition is consistent with the general definition of a line in geometry <S,f> as a set L such that:

a) $\forall x,y \in L$: $f(\{x,y\}) \subseteq L$. (redundant, can be omitted)

b) $\forall K \subseteq L \forall x,y \in S$: $|K| \geq 2$ & $K \subseteq f(\{x,y\}) \Rightarrow f(\{x,y\}) \subseteq L$.

c) $\forall K \subseteq L$: $A \in Fin(S)$ $\exists \Rightarrow r,sK\in$: $K \subseteq f(\{r,s\})$ & $r \neq s$.

With the set of lines defined this way we can recover all synthetic geometry. This way we can see that it is the property of character 2 for closure spaces which makes them geometrical, not exchange property. Thus the question is what is the role of the exchange property? To answer this question, we have to consider how closure spaces can be combined using disjoint sum.

4. DISJOINT SUMS OF CLOSURE SPACES

Relatively little has been done in the past in the study of the constructions combining general spaces, or even closure spaces defined on different sets. In this paper we will consider only disjoint sums of spaces and their relationship to the direct product of their lattices of closed subsets.

DEFINITION 4.1 *Let f be an operator on a set S, g an operator on set T, and φ be a function from S to T. The function φ is (f,g)-continuous if $\forall A \subseteq S$: $\varphi f(A) \subseteq g\varphi(A)$. We will write continuous, if no confusion is likely.*

PROPOSITION 4.1 *Continuity of the function φ as defined above is equivalent to each of the following statements:*
(1) $\forall A \subseteq S$: $f(A) \subseteq \varphi^{-1} g\varphi(A)$, (2) $\forall B \subseteq T$: $f\varphi^{-1}(B) \subseteq \varphi^{-1} g(B)$,(3) $\forall B \subseteq T$: $\varphi f \varphi^{-1}(B) \subseteq g(B)$.
If both operators f and g are transitive, continuity of the function φ is equivalent to the condition:(4) $\forall B \in g\text{-}Cl$: $\varphi^{-1}(B) \in f\text{-}Cl$.

DEFINITION 4.2 *Let f be an operator on a set S, g an operator on set T, and φ be a function from S to T. The function φ is (f,g)-isomorphism if it is bijective and $\forall A \subseteq S$: $\varphi f(A) = g\varphi(A)$. We will write isomorphism, if no confusion is likely.*

DEFINITION 4.3 *Disjoint sum $\langle \oplus S_I, g \rangle$ of the indexed family S_I of sets equipped with operators is defined as the disjoint sum of sets $\oplus S_I$ with its family of canonical injections $\{\theta_i : S_i \rightarrow \oplus S_I,\ i \in I\}$ equipped with the operator g defined by $\forall A \subseteq \oplus S_I$: $g(A) = \cup\{\theta_i f_i \theta_i^{-1}(A): i \in I\}$. If no confusion is likely we will use the symbol $\oplus S_I$ for $\langle \oplus S_I, g \rangle$.*
When the operators involved are transitive, we will call $\langle \oplus S_I, g \rangle$ a disjoint sum of closure spaces. Evidently, when the operator g is defined as above, all canonical injections become (f_i, g)-continuous.

PROPOSITION 4.2 *The disjoint sum of arbitrary family of sets with transitive operators has its operator transitive, i.e. the disjoint sum of a family of closure spaces is a closure space.*

PROPOSITION 4.3 *The lattice of closed subsets of the disjoint sum of arbitrary family of closure spaces is isomorphic to the direct product of lattices of closed subsets for the component closure spaces, i.e. $\langle \mathcal{L}_g, \leq \rangle \approx \langle \otimes \mathcal{L}_I, \leq \rangle$, where $\langle \mathcal{L}_g, \leq \rangle$ is a lattice of g-closed sets in the disjoint union of closure spaces $\langle \oplus S_I, g \rangle$ whose components are closure spaces from the family $S_I = \{\langle S_i, f_i \rangle: i \in I\}$, $\langle \otimes \mathcal{L}_I, \leq \rangle$ is the cardinal product of lattices from the family $\{\langle L_i, \leq_i \rangle: i \in I\}$, where each lattice $\langle L_i, \leq_i \rangle$ is a lattice of closed subsets in the closure space $\langle S_i, f_i \rangle$, and $\approx$ is the order isomorphism.*

The isomorphism of the lattice of the closed subsets of the disjoint sum of closure spaces with the direct product of the lattices of closed subsets in the factors opens a rich toolbox of the methods developed in lattice theory, which can be used for the study of decomposition of closure spaces. It is obviously of great interest for the study of classification of closure spaces.

There is a natural question about the conditions for simple (irreducible) closure spaces. Of course, as a consequence of the preceding proposition it is equivalent to the question of direct product irreducibility of the lattice of closed subsets. It is rather a surprise, that the exchange property appears at this point.

5. DECOMPOSITION INTO DISJOINT SUM AND EXCHANGE PROPERTY

Non-transitive operators cannot be defined by their families of closed (or open) subsets, but some other cryptomorphic descriptions remain valid. For instance the derived set operator defined as a function mapping a subset A of S to $A^d = \{x \in S: x \in f(A\setminus\{x\})$, has a set of properties which can be used for an alternative definition of a not necessarily transitive closure space.

The derived set operator can be used to define the concept of duality of operators. For an operator f on S, the dual operator f* is defined by $f^*(A) = A \cup A^{cdc}$, where A^c is the complement of A in S. Of course, f* is actually an operator, but the dual operator of a transitive operator may not be transitive.

Consequently, we can define dual properties of operators. If f* has some property xY, then we can say that f has property xY*. Victor Klee [7] showed that I* is a strengthening of the weak exchange property (**wE**), i.e. **I* = E**, where **E** is as follows.

(E) $\forall A,B \subseteq S \forall x \in S$**:** $x \notin f(A\setminus B)$ **&** $x \in f(A) \Rightarrow \exists y \in B$**:** $y \in f((A\setminus\{y\}) \cup \{x\})$**.**

In his paper Klee also considered an additional property **C**:

(C) $\forall A \subseteq S \forall x \in S$**:** $x \in f(A) \Rightarrow \exists B \subseteq A$**: B is minimal such that** $B \in$ **f-Ind &** $y \in f(B)$**.**

He showed that **IfC** $\subseteq$ **IC** & **wEC** $\subseteq$ **E**. From that we can get easily:

fCwEI $\subseteq$ **EI** and therefore **NT$_1$fCwEI(S)= NT$_1$fCEI(S)**

Since in all contexts of traditionally defined geometric closure spaces, as well as in the context of closure spaces defined in vector spaces we have the combination **fCwE** of properties, it is not just **wE** weak exchange property which is involved, but actually its stronger version **E**. It is also interesting that **IE** operators can be characterized in general as closure operators whose dual operators are transitive, i.e. **IE = II*.**

However, it is even more surprising that from the properties **IE** combined with another property follows irreducibility of the closure space into a disjoint union of closure spaces.

PROPOSITION 5.1 *Let $f \in EI(S)$. Then <S,f> is disjoint-sum-irreducible if closure operator f satisfies the condition:*

$\forall x,y \in S$ $x \notin f(\{y\})$ & $y \notin f(\{x\}) \Rightarrow \exists z \in S$: $z \notin f(\{y\})$ & $z \notin f(\{x\})$ & $z \in f(\{x,y\})$.

We can see here, that the exchange property appeared in the study of geometric closure spaces not because it is related to any geometric characteristics, but because it is related to irreducibility into disjoint sum.

6. CONCLUSIONS

The paradigm (or rather paradigms) in the study of closure space classification is based on properties which are quite arbitrary. They have sources in the particular interests of the disciplines of mathematics where they served as tools for making generalizations, but their selection was guided more by convenience, than by deeper methodological reflection.

Although the correspondence between closure spaces and lattices provided a great opportunity to enrich methodology of closure spaces generating a large number of deep results, it was not exploited sufficiently for the purpose of classifications at the level of generality beyond disciplinary divisions. Topological, algebraic and geometric closure spaces were distinguished by properties which are too strong (topology) or too weak (algebra or geometry) to have direct interpretation in terms of lattice theoretic analysis meaningful for the study of all closure spaces. In case of geometric closure spaces, the property (weak exchange property) used to distinguish this class turns out to have other important consequences (irreducibility of the closure space) rather than introduction of geometric characteristics, which are more a matter of introduced here property of being of character 2.

Correspondence between disjoint unions of the closure spaces and direct products of the lattices of closed subsets opens new direction in the study of classification. First step in this direction should be characterization of irreducibility of closure spaces with respect to disjoint sums (a sufficient, but not necessarily necessary conditions were given here), and the next step should be comprehensive classification of such irreducible closure spaces. Similarly, the study should establish the relationship between direct products of closure spaces and appropriate constructions on lattices of closed subsets, followed by the analogous classification of the direct product irreducible closure spaces.

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