数理解析研究所講究録
第 1811 巻 2012 年 23-41

The elliptic genus of K3 and the Mathieu group

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March, 2012

The conjecture in [1] and the background to it are reviewed. Another good overview, from a rather different emphasis, was given in [2] by Cheng and Duncan.

1 The modular J-function and the Monster

Consider the J-function[1]

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots \quad (1.1)$$

where $q = \exp(2\pi i\tau)$. This is the unique holomorphic function satisfying

$$J(\tau) = J(\tau+1), \quad J(\tau) = J(-1/\tau) \quad (1.2)$$

with a single pole of residue 1 at $q = 0$ (up to an addition of a constant).

The famous observation is that

$$\begin{aligned}
196884 &= 1 + 196883 && (1.3)\\
21493760 &= 1 + 196883 + 21296876 && (1.4)\\
864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 && (1.5)\\
20245856256 &= 1 + 1 + 196883 + 196883 + 196883 + \\
&\quad 21296876 + 21296876 + 842609326 + 19360062527 && (1.6)
\end{aligned}$$

where 1, 196883, 21296876, 842609326, 19360062527 are the dimensions of irreducible representations of the Monster simple group $\mathbb{M}$. One notices that

[1] For more on the content of this section and the next, see [3].

the same irreducible representations appear repeatedly. Let us introduce the Dedekind eta function $\eta(q)$

$$\eta(q) = q^{1/24} \prod_{n>0} (1 - q^n) \tag{1.7}$$

and decompose the J-function as

$$J(\tau) = \frac{q^{1/24}}{\eta(q)}(\frac{1}{q} - 1) + \frac{q^{1/24}}{\eta(q)}(196883q + 21296876q^2 + 842609326q^3 + 19360062527q^4 + \cdots). \tag{1.8}$$

We still have positive coefficients, and moreover, there is less repetition of the dimension of the same irreducible representation. The coefficient of q^5 in (1.8) is

$$312092484374 = 18538750076 + 293553734298, \tag{1.9}$$

for example.

2 Vertex algebras and the Monster

Let us recall how this observation is understood using the vertex algebra, in a very rough manner. For each even self-dual 24-dimensional lattice Λ, one can associate a vertex operator algebra $\mathrm{VA}(\mathbb{R}^{24}/\Lambda)$, which contains a Virasoro subalgebra of $c = 24$. Let us denote by $\mathcal{H}$ the underlying graded vector to a vertex algebra. Then $\mathcal{H}$ is a representation of the Virasoro algebra and the isometry group of Λ, and

$$q^{-1} \operatorname{tr}_{\mathcal{H}} q^{L_0} = J(\tau) + 24 + \#(\text{roots of } \Lambda). \tag{2.1}$$

A representation of the Virasoro algebra of highest weight w and central charge $c = 24$ has the character ch_w given by

$$\mathrm{ch}_0(q) = \frac{q^{1/24}}{\eta(q)}(q^{-1} - 1), \qquad \mathrm{ch}_w(q) = \frac{q^{1/24}}{\eta(q)} q^w. \tag{2.2}$$

This is why we chose to expand $J(\tau)$ as in (1.8).

Let us choose Λ to be the Leech lattice, for which there is no root. Its isometry group is denoted by Co_0, which was found by Conway. One can consider the orbifold $\mathrm{VA}(\mathbb{R}^{24}/\Lambda/\{\pm 1\})$ by the action of ± 1 on $\mathbb{R}^{24}$. Then we have

$$q^{-1}\,\mathrm{tr}_{\mathcal{H}}\,q^{L_0} = J(\tau) + \#(\text{roots of }\Lambda)/2. \tag{2.3}$$

The orbifold construction guarantees that $\mathcal{H}$ has an action of the Virasoro algebra times $2^{1+24}.\,\mathrm{Co}_1$, where Co_1 is Conway's simple group $\mathrm{Co}_0/\{\pm 1\}$. This $2^{1+24}.\,\mathrm{Co}_1$ is a stabilizer of an involution of $\mathbb{M}$. Now, the great feature of this vertex algebra is that it admits additional symmetry, so that in fact there is an action of the Virasoro algebra times $\mathbb{M}$.

3 The Jacobi form of index 1 and the Mathieu group

Let us recall some terminology: We define the subgroup $\Gamma_0(N) \subset SL(2,\mathbb{Z})$ by

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N}\}. \tag{3.1}$$

A modular form of weight k of the group $\Gamma_0(N)$ is a function $f(\tau)$ which satisfies

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{3.2}$$

A weak Jacobi form of weight k and index m of $\Gamma_0(N)$ is a function $f(\tau, z)$ on the upper half-plane times $\mathbb{C}$, satisfying

$$f(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} f(\tau, z), \tag{3.3}$$

$$f(\tau, z + a\tau + b) = e^{-2\pi i m(a^2\tau + 2az)} f(\tau, z), \tag{3.4}$$

again for $c \equiv 0 \pmod{N}$. We use $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$ below.

Consider a very classic function

$$Z(\tau, z) = 8 \sum_{i=2,3,4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)}\right)^2, \tag{3.5}$$

where

$$\theta_1(\tau,z) = -\mathrm{i}q^{1/8}y^{1/2}\prod_{k=1}^{\infty}(1-q^k)(1-y^{-1}q^{k-1})(1-yq^k), \tag{3.6}$$

$$\theta_2(\tau,z) = q^{1/8}y^{1/2}\prod_{k=1}^{\infty}(1-q^k)(1+y^{-1}q^{k-1})(1+yq^k), \tag{3.7}$$

$$\theta_3(\tau,z) = \prod_{k=1}^{\infty}(1-q^k)(1+y^{-1}q^{k-1/2})(1+yq^{k-1/2}), \tag{3.8}$$

$$\theta_4(\tau,z) = \prod_{k=1}^{\infty}(1-q^k)(1-y^{-1}q^{k-1/2})(1-yq^{k-1/2}) \tag{3.9}$$

are the standard theta functions. This function $Z(\tau,z)$ is a weak Jacobi form of weight 0 and index 1, which is essentially unique. It is known that the z dependence can be extracted thus [4]:

$$\begin{aligned} Z(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\Big(&24\mu(\tau,z) - 2q^{-1/8} + q^{1/8}(90q + 462q^2 \\ &+ 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + \cdots)\Big), \end{aligned} \tag{3.10}$$

where $\mu(\tau,z)$ is the Appell function

$$\mu(\tau,z) = \frac{-\mathrm{i}y^{1/2}}{\theta_1(\tau,z)}\sum_{\ell\in\mathbb{Z}}\frac{(-1)^\ell y^\ell q^{\ell(\ell+1)/2}}{1-yq^\ell}. \tag{3.11}$$

Note that $24 = 23 + 1$, and

$$90 = 45 + 45 \tag{3.12}$$
$$462 = 231 + 231 \tag{3.13}$$
$$1540 = 770 + 770 \tag{3.14}$$
$$4554 = 2277 + 2277 \tag{3.15}$$
$$11592 = 5796 + 5796 \tag{3.16}$$
$$27830 = 3520 + 3520 + 10395 + 10395 \tag{3.17}$$

where 1, 23, 45, 231, 770, 2277, 3520, 5796, 10395 are the dimensions of irreducible representations of the largest Mathieu group. For the data on M_{24}, we refer to the Atlas [5].

That this was not noticed until 2009 is somewhat surprising. Consider

$$\phi_{-2,1}(\tau,z) = -\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \tag{3.18}$$

which is the unique Jacobi form of weight -2 and index 1. Then consider the specialization of the ratio of (3.5) and (3.18) at $z=-1$:

$$\frac{Z(\tau,-1)}{\phi_{-2,1}(\tau,-1)} = -2(\theta_3(\tau)^4+\theta_4(\tau)^4) = -4-96q-96q^2-384q^3+\cdots, \tag{3.19}$$

which is the unique modular form of $\Gamma_0(2)$ of weight 2. Also, consider

$$\eta(\tau)^3\mu(\tau,-1) = \prod_{k\geq 1}\frac{(1-q^k)^2}{(1+q^k)(1+q^{k-1})}\sum_{\ell\in\mathbb{Z}}\frac{q^{\ell(\ell+1)/2}}{1+q^\ell} \tag{3.20}$$

$$= \frac{1}{4} - 4q^2 + 10q^3 - 12q^4 + 14q^5 + \cdots . \tag{3.21}$$

Then, the equation (3.10) is equivalent to

$$-(3.19) - 24\times(3.21) = -2+96q+192q^2+144q^3+384q^4+240q^4\cdots \tag{3.22}$$

and this last equation thus inherits the decomposition from (3.10), resulting in

$$96 = 6\times 1 + 2\times 45, \tag{3.23}$$

$$192 = -6\times 45 + 2\times 231, \tag{3.24}$$

$$144 = -10\times 1 - 6\times 231 + 2\times 770, \tag{3.25}$$

$$384 = 10\times 45 - 6\times 770 + 2\times 2277 \tag{3.26}$$

But this is very hard to see directly.

4 The elliptic genus

The elliptic cohomology $E^*(X)$ is[2] a generalized cohomology theory with

$$E^{2k+1}(pt) = 0, \tag{4.1}$$

$$E^{2k}(pt) = \text{space of modular forms of } \Gamma_0(2) \text{ of weight } -k. \tag{4.2}$$

[2]For the basics of elliptic genus, see [6] or [7].

This theory has an integration-along-the-fiber map $f_! : E^*(X) \to E^{*-d}(Y)$ for a map $f : X \to Y$ with the dimension of the fiber being d. Let $\pi^X : X \to pt$ be the constant map, then

$$\varphi_E(X) \equiv \pi^X_!(1) \in E^{-d}(pt) \tag{4.3}$$

is the elliptic genus, which is a modular form of $\Gamma_0(2)$ of weight $\dim_{\mathbb{R}} X/2$. So, for any manifold of dimension 4, its elliptic genus is a multiple of (3.19); the constant term is the signature of X divided by 4. Therefore, to explain the appearance of the Mathieu group, we would like to consider a four-dimensional manifold on which it acts.

Furthermore, if X of $\dim_{\mathbb{R}} X = d$ is an almost complex manifold with $c_1(X) = 0$, one can define its two-parameter elliptic genus $\varphi_{Ell}(X)$ which is a weak Jacobi form of weight 0 and index $d/4$, such that

$$\varphi_E(X)(\tau) = \frac{\varphi_{Ell}(X)(\tau, -1)}{\phi_{-2,1}(\tau, -1)^{d/4}}. \tag{4.4}$$

As the space of weak Jacobi forms of weight 0 and index 1 is one-dimensional, any almost complex manifold with $c_1(X) = 0$ gives a multiple of (3.5). A good candidate is the K3 surface, which is a compact four-dimensional hyperkähler manifold. In fact, the expression (3.5) *is* the two-parameter elliptic genus $\varphi_{Ell}(K3)$, and the expression (3.19) *is* the elliptic genus $\varphi_E(K3)$.

In general, any genus $\varphi(X)$ of an almost complex manifold X can be expressed in the form

$$\varphi(X) = \int_X \prod_i \frac{x_i}{f(x_i)} \tag{4.5}$$

where $f(x)$ is a formal power series, and x_i are the Chern roots of the tangent bundle $T_{\mathbb{C}}X$. For our two-parameter elliptic genus, it is given by

$$f_{Ell}(x; \tau, z) = \frac{\theta_1(\tau, \frac{x}{2\pi i})}{\theta_1(\tau, \frac{x}{2\pi i} - z)}. \tag{4.6}$$

Expressing the theta function in terms of infinite products, one finds

$$\varphi_{Ell}(X)(\tau, z) = \int_X \prod_i \frac{x_i y^{-1}}{1 - e^{-x_i}} \prod_{n>0} \frac{(1 - yq^{n-1}e^{-x_i})(1 - yq^n e^{x_i})}{(1 - q^n e^{-x_i})(1 - q^n e^{x_i})} \tag{4.7}$$

$$= \chi(X, E_{q,-y}) \tag{4.8}$$

where $\chi(X, E) = \sum(-1)^i \dim H^i(X, E)$ is the index of the Dolbeault complex valued in E, and $E_{q,y}$ is the bundle

$$E_{q,y} = y^{-\dim_{\mathbb{R}} X/2} \otimes \bigotimes_{n \geq 1} \wedge_{yq^{n-1}} \bar{T} \otimes \wedge_{y^{-1}q^n} T \otimes S_{q^n} \bar{T} \otimes S_{q^n} T \tag{4.9}$$

where

$$\wedge_q V = \bigoplus_{d=0}^{\infty} q^d \wedge^d V, \quad S_q V = \bigoplus_{d=0}^{\infty} q^d S^d V \tag{4.10}$$

are the direct sum of antisymmetric and symmetric powers of V. In other words, let $\mathcal{H}$ be the vector space

$$\mathcal{H}X = \bigoplus_{n,k,i} (-1)^{k+i} Q^n Y^k \mathcal{H}^i_{n,k} X = \bigoplus_i (-1)^i H^i(X, E_{Q,-Y}) \tag{4.11}$$

where Q and Y are one-dimensional representations of $\mathbb{C}^{\times 2}$, with the generator of Lie algebra L_0 and J_0, respectively. The minus sign in front of a direct sum component should be regarded as specifying the $\mathbb{Z}/2\mathbb{Z}$ grading. We use the convention that the part with odd degree contributes negatively to the trace.

The elliptic genus is its graded dimension:

$$\varphi_{Ell}(X)(\tau, y) = \mathrm{tr}(q^{L_0} y^{J_0} | \mathcal{H}X). \tag{4.12}$$

Note that the piece of $\mathcal{H}$ with Q^0 is just the ordinary cohomology groups $\bigoplus_i (-1)^i H^i(X)$, and therefore the term of order q^0 is the Euler number.

This $\varphi_{Ell}(X)$ can be defined for any almost complex X of $d = \dim_{\mathbb{R}} X$, and the left hand side becomes a weak Jacobi form of weight 0 and index $d/4$ when X has a complex structure with $c_1(X) = 0$. In fact, under the same assumption, $\mathcal{H}X$ has the structure of a vertex algebra $\mathrm{VA}(X)$, which has $\mathcal{N} = 2$ super Virasoro subalgebra, naturally associated to X. A holomorphic map $f : X \to X$ leads to a map $f : \mathrm{VA}(X) \to \mathrm{VA}(X)$ commuting with the $\mathcal{N} = 2$ super Virasoro action. If X furthermore has a holomorphic symplectic structure, $\mathrm{VA}(X)$ has $\mathcal{N} = 4$ super Virasoro subalgebra. If a map $f : X \to X$ preserves the holomorphic symplectic form, the corresponding map $f : \mathrm{VA}(X) \to \mathrm{VA}(X)$ commutes with the $\mathcal{N} = 4$ super Virasoro action. We will come back to the construction of $\mathrm{VA}(X)$ later. For now let us accept that there is such a method.

5 $\mathcal{N} = 4$ super Virasoro algebra

The $\mathcal{N} = 4$ super Virasoro algebra in the R-sector has bosonic generators L_m, T^i_m ($m \in \mathbb{Z}$, $i = 1,2,3$) and fermionic generators G^a_r, $\bar{G}_{a,r}$ ($r \in \mathbb{Z}$, $r = 1,2$), which have the following commutation relations. Fist, the ones among bosonic operators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{k}{2}m(m^2-1)\delta_{m+n,0}, \tag{5.1}$$

$$[T^i_m, T^j_n] = \mathrm{i}\epsilon^{ijk}T^k_{m+n} + \frac{k}{2}m\delta_{m+n,0}\delta^{ij}, \tag{5.2}$$

$$[L_m, T^i_n] = -nT^i_{m+n}. \tag{5.3}$$

The ones involving fermionic operators are:

$$\{G^a_r, G^b_s\} = \{\bar{G}_{r,a}, \bar{G}_{b,s}\} = 0, \tag{5.4}$$

$$\{G^a_r, \bar{G}_{b,s}\} = 2\delta^a_b L_{r+s} - 2(r-s)\sigma^{ia}_{\ b}T^i_{r+s} + \frac{k}{2}(4r^2-1)\delta_{r+s,0}\delta^a_b, \tag{5.5}$$

$$[T^i_m, G^a_r] = -\frac{1}{2}\sum_b \sigma^{ia}_{\ b}G^b_{m+r}, \tag{5.6}$$

$$[T^i_m, \bar{G}_{a,r}] = \frac{1}{2}\sum_b \sigma^{ib}_{\ a}\bar{G}_{b,m+r}, \tag{5.7}$$

$$[L_m, G^a_r] = (\frac{m}{2} - r)G^a_{m+r}, \tag{5.8}$$

$$[L_m, \bar{G}_{a,r}] = (\frac{m}{2} - r)\bar{G}_{a,m+r}. \tag{5.9}$$

Here $\{A, B\} = AB + BA$ is the anti-commutator, and ϵ^{ijk} is the completely-antisymmetric tensor such that $\epsilon^{123} = 1$, and $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the representation matrices of $SU(2)$. We also introduce $J_0 = 2T^3_0$. Note that the relation (5.2) is the affine $SU(2)$ algebra at level k, and the relation (5.1) is the Virasoro algebra with central charge $c = 6k$. When X is a d-dimensional hyperkähler manifold, $\mathrm{VA}(X)$ contains the $\mathcal{N} = 4$ super Virasoro algebra with $k = d/4$.

We are interested in the case $d = 4$, and physical consideration says that $\mathrm{VA}(X)$ should form a unitary lowest-weight representation. With $k = d/4 = 1$, the affine $SU(2)$ algebra can have two types of irreducible unitary representation, which severely constrains the super Virasoro representation theory too. Let an irreducible unitary lowest-weight representation $\mathcal{V}$ have the decomposition

$$\mathcal{V} = \bigoplus_{k=0}^{\infty} V_{k+h} \tag{5.10}$$

where $k+h$ is the eigenvalue of L_0. It is known that $h \geq 0$. Furthermore, when $h > 0$, we have

$$V_h = X_0 \oplus -X_{1/2} \oplus X_0 \tag{5.11}$$

as a representation of $SU(2)$ generated by T_0^i, where X_0 is the one-dimensional representation and $X_{1/2}$ is the defining two-dimensional representation of $SU(2)$. The character is

$$\mathrm{ch}_h(\tau, z) = \mathrm{tr}(q^{L_0 - c/24} y^{J_0} | \mathcal{V}_h) = q^{h-1/8} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}. \tag{5.12}$$

When $h = 0$, V_h can either be X_0 or $-X_{1/2}$. We denote corresponding irreducible representations by $\mathcal{V}_{0,0}$ and $\mathcal{V}_{0,1/2}$ respectively. They have characters

$$\mathrm{ch}_{0,0}(\tau, z) = \mathrm{tr}(q^{L_0 - c/24} y^{J_0} | \mathcal{V}_{0,0}) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \mu(\tau, z), \tag{5.13}$$

$$\mathrm{ch}_{0,1/2}(\tau, z) = \mathrm{tr}(q^{L_0 - c/24} y^{J_0} | \mathcal{V}_{0,1/2}) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} (q^{-1/8} - 2\mu(\tau, z)). \tag{5.14}$$

Then $\mathcal{V}_0 = \mathcal{V}_{0,0}{}^{\oplus 2} \oplus \mathcal{V}_{0,1/2}$ has the character $\mathrm{ch}_0(\tau, z)$. These characters were first determined in the physics literature in [8, 9]. Mathematical analysis was done in [10].

Then, the expansion of the weak Jacobi form (3.10) means

$$\mathrm{VA}(K3) = W_{0,0} \otimes \mathcal{V}_{0,0} - W_0 \otimes \mathcal{V}_0 + W_1 \otimes \mathcal{V}_1 + W_2 \otimes \mathcal{V}_2 + W_3 \otimes \mathcal{V}_3 + \cdots \tag{5.15}$$

as the representation of the $\mathcal{N} = 4$ super Virasoro algebra, with

$$W_{0,0} = \mathbb{C}^{24}, \quad W_0 = \mathbb{C}^2, \quad W_1 = \mathbb{C}^{90}, \quad W_2 = \mathbb{C}^{462}, \ldots. \tag{5.16}$$

Then, the observation of the agreement of the coefficients 24, 90, 462, and the dimensions of the irreducible representations of the largest Mathieu group M_{24} suggests that W_d are representations of M_{24}, so that there is a commuting action of M_{24} and the $\mathcal{N} = 4$ super Virasoro algebra on $\mathrm{VA}(K3)$. This will be automatic if there is a K3 surface with the action of M_{24} preserving its $Sp(1)$ structure, or in other words, a K3 whose group of holomorphic symplectic automorphisms is M_{24}. In the following by an automorphism of K3 we mean a holomorphic symplectic one.

6 K3 and the Mathieu group

The geometry of $K3$ comes close to admitting an action of M_{24}. Let G be the group of automorphisms of a $K3$. It is known [11, 12] that it is naturally a subgroup of $M_{23} \subset M_{24}$, the second largest Mathieu group, and that the action of G on 24 points naturally induced from it has at least five orbits. Furthermore, any such subgroup of M_{23} can act on a $K3$ as holomorphic symplectic automorphisms.

For example, take a K3 X with an order-2 automorphism g. We can consider its twisted elliptic genus

$$\varphi_{Ell}(X,g)(\tau,z) = \mathrm{tr}(g q^{L_0} y^{J_0} |\, \mathrm{VA}(X)). \tag{6.1}$$

From general argument, this is a weak Jacobi form of $\Gamma_0(2)$. This was calculated, and can be expanded as

$$\begin{aligned}\varphi_{Ell}(X,g)(\tau,z) &= \frac{1}{3} Z(\tau,z) + \frac{4}{3}\phi_2^{(2)}(\tau)\phi_{-2,1}(\tau,z) &(6.2)\\ &= 8\,\mathrm{ch}_{0,0} - 2\,\mathrm{ch}_0 - 6\,\mathrm{ch}_1 + 14\,\mathrm{ch}_2 - 28\,\mathrm{ch}_3 + 42\,\mathrm{ch}_4 + \cdots . &(6.3)\end{aligned}$$

Here, $\phi_2^{(N)}$ is a modular form of $\Gamma_0(N)$ of weight 2 given by

$$\phi_2^{(N)} = \frac{24}{N-1} q \frac{\partial}{\partial q} \log \frac{\eta(N\tau)}{\eta(\tau)} \tag{6.4}$$

Take the corresponding element g in M_{24}, called $2A$ in the atlas. We can indeed check

$$\begin{aligned} &\mathrm{tr}(g|W_{0,0}) = 8, && \mathrm{tr}(g|W_1) = -6, && \mathrm{tr}(g|W_2) = 14, && (6.5)\\ &\mathrm{tr}(g|W_3) = -28, && \mathrm{tr}(g|W_4) = 42, && \cdots && (6.6)\end{aligned}$$

where

$$\begin{aligned} W_{0,0} &= R_1 + R_{23}, & W_0 &= R_1 + R_1, & (6.7)\\ W_1 &= R_{45} + R_{\overline{45}} & W_2 &= R_{231} + R_{\overline{231}}, & (6.8)\\ W_3 &= R_{770} + R_{\overline{770}}, & W_4 &= R_{2277} + R_{\overline{2277}}, \ldots & (6.9)\end{aligned}$$

where R_d is a irreducible representation of dimension d; we distinguish a complex conjugate pair by R_d and $R_{\overline{d}}$. Recall the leading piece $W_{0,0}$ of

VA($K3$) can be naturally identified with $H^*(K3)$. So, it behaves as if it is the representation associated to the natural permutation presentation on 24 points.

There are K3 surfaces X with automorphism g of order 3, 4, 5, 6, 7, 8. The corresponding conjugacy classes in M_{24} are respectively called $3A$, $4B$, $5A$, $6A$, $7A$, $8A$ in the atlas. The corresponding twisted elliptic genus can be also calculated, and gives

$$\begin{aligned}\varphi_{Ell}(X,g)(\tau,z) &= \mathrm{tr}(gq^{L_0}y^{J_0}|\,\mathrm{VA}(X)) \\ &= \frac{\mathrm{tr}(g|H^*(K3))}{24}Z(\tau,z) + T_g(\tau)\phi_{-2,1}(\tau,z) \qquad (6.10)\end{aligned}$$

where $T_g(\tau)$ is tabulated in Appendix A, and indeed we find agreement

$$\varphi_{Ell}(X,g)(\tau,z) = \frac{\mathrm{tr}(g|W_{0,0})}{24}Z(\tau,z)\,\mathrm{ch}_{0,0} - 2\,\mathrm{ch}_0 + \sum_k \mathrm{tr}(g|W_k)\,\mathrm{ch}_k\,. \quad (6.11)$$

Encouraged by these observations, people tried to find $\varphi_{Ell}(X,g)(\tau,z)$ of the form (6.10) for other g of order n_g in M_{23} or in M_{24}, which would be a weak Jacobi form of weight 0 and index 1 of $\Gamma_0(n_g)$, so that the expansion (6.11) holds. They indeed succeeded and the results are again tabulated in Appendix A, with an important caveat: when g is not in M_{23}, the corresponding $\varphi_{Ell}(X,g)$ was not strictly in $\Gamma_0(n_g)$ but with a multiplier system in (3.3), i.e. a factor with absolute value 1 depending on the element in $SL(2,\mathbb{Z})$ and $\mathbb{Z}^2$. Furthermore, it was found that it is in $\Gamma_0(N_g)$, where N_g/n_g is the length of the shortest cycle in the cycle shape of g, as acting on 24 points.

Now, it is a simple matter to re-construct the irreducible decomposition of W_d by computer. They are found to be always a genuine representation of M_{24}.[3] It is important that what is guaranteed from the construction is only the action of certain subgroups of M_{23}, and we need to understand how there can be 'additional symmetry elements' which makes it to M_{24}. This sounds familiar: in the Monster vertex algebra, the action of $2^{1+24}.\mathrm{Co}_1$ was guaranteed by construction, but we needed an 'additional symmetry element' which makes it to $\mathbb{M}$.

[3]The author checked it up to $d = 500$.

7 Construction of the vertex algebra

The action on M_{24} on VA($K3$) remains a conjecture. Before closing, let us discuss the vertex algebra VA($K3$) associated to a K3. Physically, for any compact Calabi-Yau manifold X of complex dimension d, we expect to have a two-dimensional $\mathcal{N} = (2, 2)$ superconformal theory CFT(X). This has an underlying Hilbert space $\mathcal{H}(\mathrm{CFT}(X))$, on which two copies of $\mathcal{N} = 2$ super Virasoro algebra act, corresponding to the holomorphic and the anti-holomorphic sides of the world sheet. This Hilbert space is unitary, and the spectrum of the primary states is discrete. The central charge of both of the Virasoro algebras is $3d$. When X is hyperkähler, CFT(X) is a $\mathcal{N} = (4, 4)$ superconformal theory, and has the action of two copies of small $\mathcal{N} = 4$ super Virasoro algebras. CFT(X) depends on the metric on X. Thanks to Yau's theorem, this is equivalent to the dependence on the Kähler class and the complex structure of X.

VA(X) is obtained from CFT(X) by keeping only the vacuum states on the anti-holomorphic side. In other words, two copies of Virasoro generators L_m and $\bar{L}_m$ act on $\mathcal{H}(\mathrm{CFT}(X))$, and we keep only the states with $\bar{L}_0 = 0$. This VA(X) is a vertex algebra, with $\mathcal{N} = 2$ ($\mathcal{N} = 4$) super Virasoro subalgebra when X is Calabi-Yau (hyperkähler). The central charge is given by $c = 3d$. VA(X) still depends on the Kähler class and the complex structure.

Physicists have many constructions of $\mathcal{N} = (2, 2)$ conformal field theories with central charge 6, some based on geometry and some based on representation theory. For an overview, see [13]. So far, all known examples automatically have $\mathcal{N} = (4, 4)$ conformal symmetry, and their elliptic genera are either zero (when the conformal field theory comes from T^4) or are equal to (3.5). Moreover, they can always be modified continuously so that they become CFT(X) for $X = T^4$ or $X = K3$ with large radius.

This motivates the following conjecture: under mild assumptions,

- Vertex algebras with $\mathcal{N} = 2$ super Virasoro subalgebra with central charge $c = 6$ automatically have $\mathcal{N} = 4$ super Virasoro subalgebra.

- The moduli space of such objects consists of two pieces, one associated to T^4 and another associated to $K3$.

The large radius limit of VA(X) for a Calabi-Yau manifold X was constructed mathematically by Malikov, Schechtman and Vaintrob [14], and its relevance to the elliptic genus is explained by Borisov and Libgober in [15].

See also [16, 17]. Let us denote it by MSV(X). This depends on the complex structure of X, but is independent of its Kähler structure. The construction is fairly straightforward. Let us recall the prototypical vertex algebra with $\mathcal{N} = 2$ super Virasoro symmetry with central charge $3d$, which are given by free bosonic fields $\phi^a(z)$, $\bar{\phi}_a(z)$ and free fermionic fields $\psi^a(z)$, $\bar{\psi}_a(z)$, $(a = 1, \ldots, d)$, with the operator product expansion

$$\partial\phi^a(z)\partial\bar{\phi}_b(z) \sim -\frac{\delta^a_b}{(z-w)^2}, \qquad \psi^a(z)\bar{\psi}_b(z) \sim \frac{\delta^a_b}{z-w}. \tag{7.1}$$

Then

$$L(z) = \sum_a \left[\partial\phi^a\partial\bar{\phi}_a(z) + \frac{1}{2}\bar{\psi}_a\partial\psi^a(z) + \frac{1}{2}\psi^a\partial\bar{\psi}_a(z)\right], \tag{7.2}$$

$$G^-(z) = \sum_a \sqrt{2}\bar{\psi}_a\partial\phi^a(z), \tag{7.3}$$

$$G^+(z) = \sum_a \sqrt{2}\psi^a\partial\bar{\phi}_a(z), \tag{7.4}$$

$$J(z) = \sum_a \bar{\psi}_a\psi^a(z) \tag{7.5}$$

gives the $\mathcal{N} = 2$ Virasoro subalgebra of central charge $3d$. When $d = 2k$ is even, we can consider the holomorphic symplectic two-form $\omega_{a,b} = -\omega_{b,a}$ such that $\omega_{i,i+k} = 1$ and zero otherwise. Then

$$T^+(z) = \sum_{a,b} \omega_{a,b}\psi^a\psi^b(z), \quad T^-(z) = \sum_{a,b} \omega_{a,b}\bar{\psi}_a\bar{\psi}_b(z), \tag{7.6}$$

together with $J(z)$ generate the affine $SU(2)$ subalgebra of level k. G^a and $\bar{G}^a$ can similarly be defined, and we then have the $\mathcal{N} = 4$ Virasoro subalgebra.

Malikov, Schechtman and Vaintrob took $\phi^a(z)$ and $p_a(z) = \partial\bar{\phi}_a(z)$ as the basic fields. Then we have

$$\phi^a(z)p_b(w) \sim \frac{\delta^a_b}{z-w}. \tag{7.7}$$

$L(z)$, $G^+(z)$, $G^-(z)$ and $J(z)$ can be written in terms of $\phi^a(z)$ and $p_a(z)$.

Now, consider a complex manifold with dimension d, with two patches U and $\hat{U}$, with coordinates $(x^1, \ldots, x^d)$ and $(\hat{x}^1, \ldots, \hat{x}^d)$. The functions

$\hat{x}^a(x^1, \ldots, x^n)$ are holomorphic. Define

$$\hat{\phi}^a(z) = \hat{x}^a, \tag{7.8}$$

$$\hat{\psi}^a(z) = \sum_b \psi^b(z) \frac{\partial \hat{x}^a}{\partial x^b}, \tag{7.9}$$

$$\hat{\bar{\psi}}_a(z) = \sum_b \bar{\psi}_b(z) \frac{\partial x^a}{\partial \hat{x}^b}, \tag{7.10}$$

$$\hat{p}_a(z) = \sum_b \frac{\partial x^a}{\partial \hat{x}^b} p_b(z) + \sum_{b,c} \frac{\partial^2 x^b}{\partial \hat{x}^a \partial \hat{x}^c} \frac{\partial \hat{x}^c}{\partial x^d} \bar{\psi}_b \psi^d \tag{7.11}$$

where, in the right hand side, the partial derivatives of x^a and $\hat{x}$ are regarded as functions of $(x^1, \ldots, x^n)$ and then we let $x^a = \phi^a(z)$; this is a consistent procedure because the fields $\phi^a(z)$ do not have nontrivial operator product expansions among themselves. They showed that the hatted fields $\hat{\phi}^a(z)$, $\hat{p}_a(z)$, $\hat{\psi}^a(z)$ and $\hat{\bar{\psi}}_a(z)$ have the same operator product expansions as the original fields $\phi^a(z)$, $p_a(z)$ $\psi^a(z)$ and $\bar{\psi}_a(z)$.

Now, let us define $\hat{L}(z)$, $\hat{G}^{\pm}(z)$ and $\hat{J}(z)$ as in (7.2) from hatted fields. They showed that

$$\hat{L}_{\text{top}}(z) = L_{\text{top}}(z), \qquad \hat{G}^+(z) = G^+(z) \tag{7.12}$$

while

$$\hat{J}(z) - J(z) \propto \log\det \frac{\partial \hat{x}^a}{\partial x^b}, \quad \hat{G}^-(z) - G^-(z) \propto \sum_c \psi^c \frac{\partial}{\partial \hat{x}^c} \log\det \frac{\partial x^a}{\partial \hat{x}^b}. \tag{7.13}$$

Here $L_{\text{top}}(z) = L(z) - \partial J(z)/2$ is another Virasoro element in the vertex algebra, with central charge 0. Therefore, $\hat{J}(z) = J(z)$ and $\hat{G}^-(z) = G^-(z)$ if and only if $c_1(X) = 0$. Similarly, when one defines $\hat{T}^{\pm}(z)$ in terms of fields with hatted fields, $\hat{T}^{\pm}(z) = T^{\pm}(z)$ if and only if the coordinate transformation preserves the holomorphic symplectic form ω_{ab}.

This means that,

1. for any complex manifold X, there is a bundle of vertex operator algebras $\mathcal{MSV}(X)$ with central charge 0.

2. If $c_1(X) = 0$, it is a bundle of vertex algebras with $\mathcal{N} = 2$ super Virasoro subalgebra, with central charge $3d$.

3. If X is holomorphic symplectic, it is a bundle of vertex algebras with $\mathcal{N}=4$ super Virasoro subalgebra, again with central charge $3d$.

Then we let

$$\mathrm{MSV}(X)=\bigoplus_i(-1)^i H^i(X,\mathcal{MSV}(X)) \tag{7.14}$$

in a suitable sense; this is a vertex operator algebra for any X, which has $\mathcal{N}=2$ super Virasoro subalgebra if $c_1(X)=0$ and has $\mathcal{N}=4$ super Virasoro subalgebra if X is holomorphic symplectic. The underlying graded vector bundle to $\mathcal{MSV}(X)$ and the underlying graded vector space to $\mathrm{MSV}(X)$ are naturally identified with the bundle (4.9) and the graded vector space (4.11) which was in the definition of the elliptic genus.

This natural appearance of the bundle (4.9) is the reason why we spent some time here to review its construction; it is possible that there is a suitable K3 X such that $\mathrm{MSV}(X)$ thus constructed has the symmetry M_{24}. However, $\mathrm{MSV}(X)$ is physically speaking the large-radius limit of a more general $\mathrm{VA}(X)$ which depends on the Kähler class of X. So, it might also be possible that M_{24} can only act on $\mathrm{VA}(X)$ with suitably chosen complex structure and the Kähler class. It might even be the case that the moduli space of vertex algebras with $\mathcal{N}=4$ super Virasoro subalgebra with $c=6$ contains a few exceptional objects, with the same elliptic genus (3.5), and that M_{24} only acts on those exceptional things. Vertex algebras with $\mathcal{N}=4$ super Virasoro subalgebra with $c=6$ can also be approached purely representation-theoretically, and that might give us *the* required object.

One completely baseless speculation is the following. Consider the vertex algebra associated to an orbifold of a torus $\mathrm{VA}(\mathbb{R}^6/\Lambda/\Gamma)$ where Λ is a lattice and Γ is a subgroup of the automorphism group of Λ. It has central charge $c=6$. It is known that by choosing Λ and Γ carefully, the vertex algebra can have $\mathcal{N}=4$ super Virasoro subalgebra. Note also that the largest Mathieu group has a subgroup $2^{1+6}.L_3(2)$ as the centralizer of 2A, and $2\times L_3(2)$ is the automorphism group of a six-dimensional lattice, as explained in the atlas [5]. All this sounds very similar to the situation for the monster.

Acknowledgments

The author thanks Eguchi-sensei and Ooguri-sensei for the collaboration leading to the paper [1]. Eguchi-sensei has been a big fan of four-dimensional hyperkähler manifolds (he is one of the men who constructed the first explicit metric for a noncompact four-dimensional hyperkähler manifold). Ooguri-sensei already had the decomposition (3.10) in his PhD thesis, published as [4] in 1989; see the equation (3.16) there. Eguchi-sensei knows Mukai-sensei and Mukai-sensei's work on the subgroups of M_{23} acting on $K3$ surfaces very well, and when the author was a graduate student he often told the author about his belief that string theory should somehow enhance M_{23} to M_{24}. After the author became a postdoc, Eughi-sensei, Ooguri-sensei and he gathered in the summer of 2009 in Aspen, and decided to revisit this question. The author was in a rather optimistic mood around that time, and suggested that they might see M_{24} directly in the elliptic genus. They happened to have the PDF of the Encyclopedic Dictionary of Mathematics by Iwanami-shoten at hand, in whose appendix the character table of M_{24} could be found. And indeed they found a nice decomposition. The only contribution by the author in this whole business was the suggestion that they should look the character table, and therefore he felt a bit awkward when he was asked to give a talk on it in front of mathematicians.

He also thanks Scott Carnahan for carefully reading the draft and for making many useful comments. He is currently supported in part by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan through the Institute for the Physics and Mathematics of the Universe, the University of Tokyo. When he was working on the paper [1], he was in the Institute for Advanced Study, where he was supported in part by NSF grant PHY-0969448 and by the Marvin L. Goldberger membership.

A Table of $\varphi_{Ell}(K3, g)$

Here we tabulate the twisted elliptic genus

$$\begin{aligned}\varphi_{Ell}(X,g)(\tau,z) &= \mathrm{tr}(gq^{L_0}y^{J_0}|\,\mathrm{VA}(X)) \\ &= \frac{\mathrm{tr}(g|H^*(K3))}{24}Z(\tau,z) + T_g(\tau)\phi_{-2,1}(\tau,z). \qquad \text{(A.1)}\end{aligned}$$

for all conjugacy classes in M_{24}, by listing $T_g(\tau)$ for each g. Those which act on $K3$ are:

$$T_{2A} = \frac{4}{3}\phi_2^{(2)}(\tau), \qquad T_{3A} = \frac{3}{2}\phi_2^{(3)}(\tau),$$

$$T_{4B} = -\frac{1}{3}\phi_2^{(2)}(\tau) + 2\,\phi_2^{(4)}(\tau), \qquad T_{5A} = \frac{5}{3}\phi_2^{(5)}(\tau),$$

$$T_{6A} = -\frac{1}{6}\phi_2^{(2)}(\tau) - \frac{1}{2}\phi_2^{(3)}(\tau) + \frac{5}{2}\phi_2^{(6)}(\tau), \qquad T_{7A,7B} = \frac{7}{4}\phi_2^{(7)}(\tau),$$

$$T_{8A} = -\frac{1}{2}\phi_2^{(4)}(\tau) + \frac{7}{3}\phi_2^{(8)}(\tau).$$

Those which do not act on $K3$ but in M_{23} are:

$$T_{11A} = \frac{11}{6}\phi_2^{(11)}(\tau) - \frac{22}{5}\,[\eta(\tau)\,\eta(11\,\tau)]^2\,,$$

$$T_{14A,14B} = -\frac{1}{36}\phi_2^{(2)}(\tau) - \frac{7}{12}\phi_2^{(7)}(\tau) + \frac{91}{36}\phi_2^{(14)}(\tau) - \frac{14}{3}\,\eta(\tau)\,\eta(2\,\tau)\,\eta(7\,\tau)\,\eta(14\,\tau),$$

$$T_{15A,15B} = -\frac{1}{16}\phi_2^{(3)}(\tau) - \frac{5}{24}\phi_2^{(5)}(\tau) + \frac{35}{16}\phi_2^{(15)}(\tau) - \frac{15}{4}\,\eta(\tau)\,\eta(3\,\tau)\,\eta(5\,\tau)\,\eta(15\,\tau),$$

$$T_{23A,23B} = \frac{23}{12}\phi_2^{(23)}(\tau) - \frac{188}{11}\eta(\tau)^2\eta(23\tau)^2 - \frac{23}{11}\left[\frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2\right].$$

Those which are not in M_{23}:

$$T_{2B} = 2\,\frac{\eta(\tau)^8}{\eta(2\tau)^4}, \qquad T_{4A} = 2\,\frac{\eta(2\tau)^8}{\eta(4\tau)^4},$$

$$T_{4C} = 2\,\frac{\eta(\tau)^4\,\eta(2\tau)^2}{\eta(4\tau)^2}, \qquad T_{3B} = 2\,\frac{\eta(\tau)^6}{\eta(3\tau)^2},$$

$$T_{6B} = 2\,\frac{\eta(\tau)^2\,\eta(2\tau)^2\,\eta(3\tau)^2}{\eta(6\tau)^2}, \qquad T_{12B} = 2\,\frac{\eta(\tau)^4\,\eta(4\tau)\,\eta(6\tau)}{\eta(2\tau)\,\eta(12\tau)},$$

$$T_{10A} = 2\,\frac{\eta(\tau)^3\,\eta(2\tau)\,\eta(5\tau)}{\eta(10\tau)} \qquad T_{12A} = 2\,\frac{\eta(\tau)^3\,\eta(4\tau)^2\,\eta(6\tau)^3}{\eta(2\tau)\,\eta(3\tau)\,\eta(12\tau)^2},$$

$$T_{21A,21B} = \frac{7}{3}\,\frac{\eta(\tau)^3\,\eta(7\,\tau)^3}{\eta(3\,\tau)\,\eta(21\,\tau)} - \frac{1}{3}\,\frac{\eta(\tau)^6}{\eta(3\,\tau)^2}.$$

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