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Torus Actions and the Halperin-Carlsson Conjecture

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We report some results concerning the Halperin-Carlsson conjecture. This is obtained as a joint work with Y. Kamishima.

1. Introduction

Recently the real Bott tower and its generalization have been studied by several people ([3], [10], [14], [15], [8]). A real Bott manifold is originally defined to be the set of real points in the Bott manifold [6]. Among several characterizations by group actions, the Halperin-Carlsson conjecture is true for real Bott manifolds. The Halperin-Carlsson torus conjecture says that if there is an almost free torus action T^k on a closed n-manifold M, the following inequality holds:

$$2^k \leq \sum_{j=0}^{n} b_j. \tag{1}$$

Here $b_j = \operatorname{rank} H_j(M;\mathbb{Z})$ is the j-th Betti number of M. See [16] for details and the references therein, see also [7].

Another characterization is that a real Bott manifold M is a euclidean space form (Riemannian flat manifold). It is conceivable whether the *Halperin-Carlsson conjecture* holds for compact euclidean space forms more generally.

By this motivation we revisit the Conner-Raymond's injective torus actions [5]. In this direction, we shall introduce *injective-splitting action* of a torus T^k on closed aspherical manifolds more generally. Our purpose is to prove the Halperin-Carlsson conjecture for such torus actions affirmatively.

2. Injective-splitting action

Let T^k be a k-dimensional torus ($k \geq 1$). Given an *effective* T^k-action on a closed manifold M, the orbit map at $x \in M$ is defined to be $\mathrm{ev}(t) = tx$ ($^\forall t \in T^k$). If we denote $\pi_1(T^k) = H_1(T^k;\mathbb{Z}) = \mathbb{Z}^k$ and $\pi_1(M) = \pi$, then the map ev induces a homomorphism $\mathrm{ev}_\# : \mathbb{Z}^k \to \pi$ and $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$ respectively.

According to the definition of Conner-Raymond [5], if $\mathrm{ev}_\#$ is *injective*, the action (T^k, M) is said to be *injective*. (Note that the definition is independent of the choice of the base point $x \in M$ [11, Theorem 2.4.2, also Subsection 11.1].) Classically it is known that $\mathrm{ev}_\#$ is injective for closed *aspherical* manifolds [4].

Let (T^k, M) be an injective T^k-action on a closed manifold M. We see that $\mathrm{Im}(\mathrm{ev}_\#) \leq C(\pi)$ where $C(\pi)$ is the center of π (cf. [9]). Put $\mathrm{Im}(\mathrm{ev}_\#) = \mathbb{Z}^k$. Letting $Q = \pi/\mathbb{Z}^k$, there is a central group extension:

$$1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1. \tag{2}$$

Definition 2.1. A T^k-action is said to be *injective-splitting* if there exists a finite index normal subgroup Q' of Q such that the induced extension splits;

$$\pi' = \mathbb{Z}^k \times Q'.$$

3. Statements and Results

Theorem A. *Suppose that a closed manifold M admits an injective-splitting T^k-action. Then the following holds.*

$$(3) \qquad {}_kC_j \leq b_j.$$

In particular the Halperin-Carlsson conjecture is true.

On the other hand, if $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$ is injective, then the T^k-action is said to be *homologically injective* (cf. [5]). Any homologically injective action is obviously injective.

Proposition 3.1. *Any homologically injective T^k-action on a closed manifold M is injective-splitting.*

Proof. The proof is essentially the same as [5, 2.2. Lemma]. Let $1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1$ be the central group extension. As $\mathrm{ev}_* : H_1(T^k;\mathbb{Z}) = \mathbb{Z}^k \to H_1(M;\mathbb{Z}) = \mathbb{Z}^\ell \oplus F$ is injective, $\mathrm{ev}_*(\mathbb{Z}^k) \leq \mathbb{Z}^k$ such that $\mathrm{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \leq \mathbb{Z}^\ell$. If $q : \pi \to H_1(M;\mathbb{Z})$ is a canonical projection, then $\pi' = q^{-1}(\mathrm{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \oplus F)$ is a finite index normal splitting subgroup of π. □

Theorem B. *If T^k is a homologically injective action on a closed n-manifold M, then*

$$(4) \qquad {}_kC_j \leq b_j \ \ (j = 0, \dots, k).$$

In particular the Halperin-Carlsson conjecture is true.

Corollary B. *Every effective T^k-action on a compact n-dimensional euclidean space form M is injective-splitting. Thus ${}_kC_j \leq b_j$, the Halperin-Carlsson conjecture* (1) *holds.*

We obtain a characterization of *holomorphic* torus actions originally observed by Carrell [2].

Corollary C. *Every holomorphic action of the complex torus $T^k_{\mathbb{C}}$ on a compact Kähler manifold is homologically injective. In particular, ${}_{2k}C_j \leq b_j$, the Halperin-Carlsson conjecture holds.*

4. Preliminaries for a proof of Theorem A

Suppose (T^k, M) is an *injective action* on a closed manifold M. Let $\tilde{M}$ be the universal covering space of M. Since $\mathbb{Z}^k \leq C(\pi)$, letting $Q = \pi/\mathbb{Z}^k$, there is a central group extension:

$$(5) \qquad 1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1.$$

Now the universal covering group $\mathbb{R}^k$ of T^k acts properly and freely on $\tilde{M}$ such that $\tilde{M} = \mathbb{R}^k \times W$ where $W = \tilde{M}/\mathbb{R}^k$ is a simply connected smooth manifold. The central group extension (5) represents a 2-cocycle f in $H^2(Q;\mathbb{Z}^k)$ in which π is viewed as the product $\mathbb{Z}^k \times Q$ with group law:

$$(n, \alpha)(m, \beta) = (n + m + f(\alpha, \beta), \alpha\beta).$$

Let $Map(W,\mathbb{R}^k)$ (respectively $Map(W,T^k)$) be the set of smooth maps of W into $\mathbb{R}^k$ (respectively T^k) endowed with a Q-module structure in which there is an exact sequence of Q-modules [4]:

$$1 \to \mathbb{Z}^k \to Map(W,\mathbb{R}^k) \xrightarrow{\exp} Map(W,T^k) \to 1.$$

When Q acts properly discontinuously on W with compact quotient, we have the vanishing theorem from [4, Lemma 8.5], [11]:

$$H^i(Q, Map(W,\mathbb{R}^k)) = 0 \ (i \geq 1). \tag{6}$$

By (6), the connected homomorphism induces an isomorphism :

$$\delta : H^1(Q; Map(W,T^k)) \to H^2(Q;\mathbb{Z}^k).$$

From this, there exists a map $\chi : Q \to Map(W,\mathbb{R}^k)$ such that $\delta^1\chi = f$. Then the action of π on $\tilde{M}$ can be described as

$$\begin{aligned}(n,\alpha)(x,w) = (n + x + \chi(\alpha)(\alpha w), \alpha w)\\ (^\forall(n,\alpha) \in \pi, {}^\forall(x,w) \in \mathbb{R}^k \times W).\end{aligned} \tag{7}$$

The π-action may depend on the choice of χ' such that $\delta^1\chi' = f$. However, the vanishing cohomology group (6) shows that

Proposition 4.1. *Such π-actions are equivalent to each other.*

5. Proof of Theorem A

Proof. **Algebraic part.** (5) induces a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota'} & & \\ 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi' & \longrightarrow & Q' & \longrightarrow & 1 \end{array} \tag{8}$$

Here Q/Q' is a finite group by Definition 2.1. For the cocycle f representing the upper group extension, it follows $\iota'^*[f] = 0 \in H^2(Q';\mathbb{Z}^k)$ by the hypothesis. We may assume

$$f|_{Q'} = 0. \tag{9}$$

On the other hand, if $\tau : H^2(Q';\mathbb{Z}^k) \to H^2(Q;\mathbb{Z}^k)$ is the transfer homomorphism, then $\tau \circ \iota'^* = |Q : Q'| : H^2(Q;\mathbb{Z}^k) \to H^2(Q;\mathbb{Z}^k)$ so that $[f]$ is a torsion in $H^2(Q;\mathbb{Z}^k)$. There exists an integer ℓ such that $\ell \cdot f = \delta^1\tilde{\lambda}$ for some function $\tilde{\lambda} : Q \to \mathbb{Z}^k$. Put $\lambda = \dfrac{\tilde{\lambda}}{\ell} : Q \to \mathbb{R}^k$. Then

$$f = \delta^1\lambda. \tag{10}$$

The equation (9) shows $[\lambda|_{Q'}] \in H^1(Q;\mathbb{R}^k)$. Viewed $\mathbb{R}^k \leq Map(W,\mathbb{R}^k)$ as constant maps, $[\lambda|_{Q'}] \in H^1(Q; Map(W,\mathbb{R}^k)) = 0$ by (6). So there is an element $h \in Map(W,\mathbb{R}^k)$ such that $\lambda|_{Q'} = \delta^0 h$. The equality $\lambda(\alpha') = \delta^0 h(\alpha')(w)$ $(^\forall \alpha' \in Q', {}^\forall w \in W)$ implies

$$h(w) = h(\alpha' w) + \lambda(\alpha'). \tag{11}$$

Geometric part. Noting Proposition 4.1, the π-action (7) on $\tilde{M}$ is equivalent with

$$(n,\alpha)(x,w)=(n+x+\lambda(\alpha),\alpha w)\ \ ({}^{\forall}(x,w)\in\mathbb{R}^k\times W). \tag{12}$$

Recall that π has the splitting subgroup $\pi'=\mathbb{Z}^k\times Q'$. Obviously we have the product action of $\mathbb{Z}^k\times Q'$ on $\mathbb{R}^k\times W$ such that $\mathbb{R}^k\times W/\mathbb{Z}^k\times Q'=T^k\times W/Q'$. Define a diffeomorphism $\tilde{G}:\mathbb{R}^k\times W\to\mathbb{R}^k\times W$ to be $\tilde{G}(x,w)=(x+h(w),w)$. Using (11), it is easy to check that $\tilde{G}:(\pi',\mathbb{R}^k\times W)\to(\mathbb{Z}^k\times Q',\mathbb{R}^k\times W)$ is an equivariant diffeomorphism with respect to the action (12) and the product action. Putting $\mathbb{R}^k\times W/\pi'=T^k\underset{Q'}{\times}W$ as a quotient space, $\tilde{G}$ induces a diffeomorphism $G:T^k\underset{Q'}{\times}W\to T^k\times W/Q'$. Let $q:T^k\times W\to T^k\underset{Q'}{\times}W$ be the covering map ($q(t,w)=[t,w]$). Then

$$G\circ q(t,w)=G([t,w])=(t\exp 2\pi\mathbf{i}h(w),[w]). \tag{13}$$

Noting (12), π induces an action of Q on $\tilde{M}/\mathbb{Z}^k=T^k\times W$ such that

$$\alpha(t,w)=(t\exp 2\pi\mathbf{i}\lambda(\alpha),\alpha w)\ \ ({}^{\forall}\alpha\in Q). \tag{14}$$

$F=Q/Q'$ has an induced action on $T^k\underset{Q'}{\times}W$ by $\hat{\alpha}[t,w]=[t\exp 2\pi\mathbf{i}\lambda(\alpha),\alpha w]$ (${}^{\forall}\hat{\alpha}\in F$) which gives rise to a covering map:

$$F\to T^k\underset{Q'}{\times}W\stackrel{\nu}{\longrightarrow}T^k\underset{Q}{\times}W=M. \tag{15}$$

For any $\alpha\in Q$, consider the commutative diagram:

$$\begin{array}{ccc} H_j(T^k\times W) & \xrightarrow{\alpha_*} & H_j(T^k\times W)\\ \big\downarrow{\scriptstyle q_*} & & \big\downarrow{\scriptstyle q_*}\\ H_j(T^k\times_{Q'}W) & \xrightarrow{\hat{\alpha}_*} & H_j(T^k\times_{Q'}W)\end{array} \tag{16}$$

in which $H_j(T^k)\otimes H_0(W)\le H_j(T^k\times W)$. By the formula (14), the Q-action on the T^k-summand is a translation by $\exp 2\pi\mathbf{i}\lambda(\alpha)\in T^k$ so the homology action α_* on $H_j(T^k)\otimes H_0(W)$ is trivial. If $H_j(T^k\underset{Q'}{\times}W)^F$ denotes the subgroup left fixed under the homology action for every element $\hat{\alpha}\in F$, it follows

$$q_*(H_j(T^k)\otimes H_0(W))\le H_j(T^k\underset{Q'}{\times}W)^F. \tag{17}$$

Using the transfer homomorphism, ν of (15) induces an isomorphism:

$$\nu_*:H_j(T^k\underset{Q'}{\times}W;\mathbb{Q})^F\longrightarrow H_j(M;\mathbb{Q}).$$

In particular, $\nu_*:q_*(H_j(T^k;\mathbb{Q})\otimes H_0(W;\mathbb{Q}))\to H_j(M;\mathbb{Q})$ is injective.

On the other hand, let $q':W\to W/Q'$ be the projection $q'(w)=[w]$. Define a homotopy $\Psi_\theta:T^k\times W\to T^k\times W/Q'$ ($\theta\in[0,1]$) to be

$$\Psi_\theta(t,w)=(t\exp 2\pi\mathbf{i}(\theta\cdot h(w)),[w]).$$

Then $\Psi_0=\mathrm{id}\times q'\simeq G\circ q$ from (13). As $G_*\circ q_*=\mathrm{id}\times q'_*:H_j(T^k;\mathbb{Q})\otimes H_0(W;\mathbb{Q})\to H_j(T^k;\mathbb{Q})\otimes H_0(W/Q';\mathbb{Q})$ is obviously isomorphic, it implies that $q_*:H_j(T^k;\mathbb{Q})\otimes$

$H_0(W;\mathbb{Q}) \longrightarrow H_j(T^k \underset{Q'}{\times} W;\mathbb{Q})$ is injective. If $p = \nu \circ q : T^k \times W \to M$ is the projection, then $p_* : H_j(T^k;\mathbb{Q}) \otimes H_0(W;\mathbb{Q}) \longrightarrow H_j(M;\mathbb{Q})$ becomes injective. This shows Theorem A. □

6. Application to Euclidean space forms

Let M be a compact euclidean space form $\mathbb{R}^n/\pi$ with $\operatorname{rank} H_1(M) = k$, and set $s = \operatorname{rank} C(\pi)$. In [5, § 7], Conner and Raymond stated (without proof) that Calabi's theorem [1] shows the existence of a T^k-action. From this, we see that $k \leq s$ because $\mathbb{Z}^k \leq C(\pi)$. On the other hand, using the algebraic hull argument, it is easy to see that M admits an effective T^s-action, so by Corollary B, $s \leq k$. Therefore, we obtain:

Theorem E. *A compact n-dimensional euclidean space form M admits an action of T^k, where $k = \operatorname{rank} H_1(M)$, in which* $\operatorname{rank} C(\pi) = \operatorname{rank} H_1(M)$.

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