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Simple factor dressing of a minimal surface

Katsuhiro Moriya
University of Tsukuba

1 Introduction

In this paper, we announce the results in [6]. A simple factor dressing is a transform of a harmonic map from a Riemann surface. A Gauss map of a constant mean curvature surface in $\mathbb{R}^3$ is a harmonic map from a Riemann surface to S^2. It is shown that every μ-Draboux transform of a harmonic map $N\colon M\to S^2$ is given by a simple factor dressing of N ([1], Theorem 6.1). A conformal Gauss map of a Willmore conformal map from a Riemann surface to S^4 is a harmonic map from a Riemann surface to $\mathcal{Z}=\{C\in \mathrm{End}(\mathbb{H})\,|\,C^2=-\mathrm{Id}\}$ (see [7]). It is shown that the Darboux transform of a harmonic sphere congruence $\mathcal{C}$ in [2] is a μ-Darboux transform of $\mathcal{C}$ with $\mu\in(\mathbb{R}\setminus\{0\})\cup S^1$. Moreover, it is a simple factor dressing of $\mathcal{C}$ ([8]).

When we consider these transforms, there is a theory of minimal surfaces in Euclidean space in the intersection of the theory of constant mean curvature surfaces in $\mathbb{R}^3$ and that of Willmore surfaces in S^4 (see Table 1). The Gauss map of a minimal surface is a conformal harmonic map and the mean curvature sphere of a minimal surface is a harmonic map. This is an interesting point to consider μ-Darboux transforms and simple factor dressing of a minimal surface.

The definitions and propositions used in this paper are summarized in [7].

2 Minimal surfaces

We recall minimal surfaces in terms of quaternions.

map	Gauss map	conformal Gauss map
CMC	harmonic	
minimal	holomorphic	harmonic
Willmore		harmonic

Table 1: Maps and Gauss maps

2.1 One-forms with values in quaternions

We model $\mathbb{R}^4$ on $\mathbb{H}$. We denote by S^2 the two-sphere centered at the origin with radius one. Then

$$\{a \in \mathrm{End}(\mathbb{H}) \mid a^2 = -1\} = \{a \in \mathrm{Im}\,\mathbb{H} \mid |a| = 1\} = S^2.$$

Hence an element of S^2 is a quaternionic linear complex structure of $\mathbb{H}$ and a square root of -1.

Let M be a Riemann surface with complex structure J. We fix a map $N\colon M \to S^2$. Then a one-form ω on M with values in $\mathbb{H}$ is decomposed as

$$\omega = \omega_N + \omega_{-N} = \omega^N + \omega^{-N},$$
$$\omega_N := \frac{1}{2}(\omega - N * \omega), \quad \omega^N := \frac{1}{2}(\omega - *\omega N).$$

Let η be another one-form on M with values in $\mathbb{H}$. Then

$$\omega \wedge \eta = \omega^N \wedge \eta_{-N} + \omega^{-N} \wedge \eta_N.$$

2.2 Minimal surfaces

Let $f\colon M \to \mathbb{H}$ be a map. Then f is conformal if and only if there exists $N\colon M \to S^2$ and $R\colon M \to S^2$ such that $(df)_{-N} = (df)^R = 0$ ([2]). The map N is called the left normal of f and the map R is called the right normal of f. A conformal map f with $(df)_{-N} = (df)^R = 0$ is minimal with respect to the induced metric if and only if $(dN)_N = (dN)^{-N} = 0$ and, equivalently, $(dR)_R = (dR)^{-R} = 0$ ([2]). Hence if f is a minimal surface with $(df)_{-N} = (df)^R = 0$, then N and R are conformal maps. In fact, they are holomorphic map with respect to a standard complex structure of $S^2 \cong \mathbb{C}P^1$. For a minimal surface f, there exists locally a map f^* such that $df^* = -*\, df$. The map f^* is called a conjugate minimal surface of f. We see that $(df^*)_{-N} = (df^*)^R = 0$. For $(p,q) \in \mathbb{H}^2 \setminus \{(0,0)\}$, $f_{p,q} := fp + f^*q$ and $f^{p,q} := pf + qf^*$ are minimal surfaces.

Definition 1 ([6]). The family of minimal surface $\{f_{p,q}\}_{(p,q)\in\mathbb{H}^2\setminus\{(0,0)\}}$ is called the right associated family of f and the family of minimal surfaces $\{f^{p,q}\}_{(p,q)\in\mathbb{H}^2\setminus\{(0,0)\}}$ is called the left associated family of f.

If f is a minimal surface in $\mathbb{R}^3$, then $\{f_{\cos\theta,\sin\theta}\}_{\theta\in\mathbb{R}}$ is the classical associated family. The classical associated family is an isometric deformation of the original minimal surface.

Theorem 1 ([6]). $f_{p,q}$ $(p,q) \in \mathbb{H}^2\setminus\{(0,0)\}$ is isometric to f if and only if $(p,q) = (n\cos\theta, n\sin\theta)$, $n \in S^3 = \{a \in \mathbb{H} \mid |a| = 1\}$.

3 Holomorphic Gauss maps

We explain transforms of the Gauss map of a minimal surface. These are similar to the transforms of a harmonic map into S^2.

3.1 The associated family of a harmonic map into S^2

We set $I\phi := \phi i$. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by the complex structure I. A map $R\colon M \to S^2$ is harmonic if and only if $d(R * dR) = 0$. We define a family of connections on $\underline{\mathbb{H}}$ by setting $d_\mu := d + (\mu - 1)Q^{(1,0)} + (\mu^{-1} - 1)Q^{(0,1)}$, where $\mu \in \mathbb{C} \setminus \{0\}$ and $Q = -\frac{1}{2}(*\,dR)_R$.

Lemma 1 ([3]). A map $R\colon M \to S^2$ is harmonic if and only if d_μ is flat for any $\mu \in \mathbb{C} \setminus \{0\}$

3.2 The SFD of a holomorphic Gauss map

Let $r_\lambda\colon M \to \mathrm{GL}(2, \mathbb{C})$ be a map such that it is meromorphic on $\mathbb{C}P^1$ with respect to λ with simple pole away from $\{0, \infty\}$ and $r_1 = \mathrm{Id}$. Set $\widehat{d_\lambda} := r_\lambda \circ d_\lambda \circ r_\lambda^{-1}$.

Definition 2. A map $\widehat{R}\colon M \to S^2$ is called a simple factor dressing of R if there exists r_λ such that $\hat{d}_\lambda$ is the associated family of $\widehat{R}$.

A simple factor dressing is a harmonic map. In fact, it is written as follows.

Theorem 2 ([1]). If $\widehat{R}$ is a simple factor dressing of R, then $\widehat{R} := \widehat{T}^{-1}R\widehat{T}$, where $\widehat{T} := \frac{1}{2}(-R\beta(a - 1)\beta^{-1} + \beta b\beta^{-1})$, $a := \frac{\lambda+\lambda^{-1}}{2}$, $b := i\frac{\lambda^{-1}-\lambda}{2}$, and $d_\lambda\beta = 0$.

We consider the case where R is holomorphic. If f is minimal, then R is holomorphic. The simple factor dressing $\hat{R}$ is harmonic. In fact, we have the following:

Theorem 3 ([6]). Let $f\colon M \to \mathbb{H}$ be minimal with $(df)^R = 0$. Then a simple factor dressing $\widehat{R}$ is a right normal of $f_{p,q}$ for some $(p, q) \in \mathbb{H}^2 \setminus \{(0, 0)\}$.

$$\begin{array}{ccc} R & \longleftrightarrow & f \\ \downarrow & & \downarrow \\ \widehat{R} & \longleftrightarrow & f_{p,q} \end{array}$$

4 Conformal Gauss maps

We explain transforms of a conformal Gauss map of a minimal surface.

4.1 The SFD of a conformal Gauss map

Let $f\colon M \to \mathbb{H}$ be a minimal surface with $(df)_{-N} = (df)^R = 0$. It is known that f is Willmore, too. We set the sections e and ψ of $\underline{\mathbb{H}^2}$ as

$$e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi := \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

We define a line bundle L by $L := \underline{\psi\mathbb{H}}$. Then L is associated with a Willmore conformal map with mean curvature sphere

$$S\begin{pmatrix} e & \psi \end{pmatrix} := \begin{pmatrix} e & \psi \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & -R \end{pmatrix}.$$

Let $d_\mu^{\mathcal{S}}$ be the associated family of d with $\mathcal{S}$ (see [7]). Set $a := \frac{\mu+\mu^{-1}}{2}$, $b := i\frac{\mu^{-1}-\mu}{2}$. Then, we have the following:

Theorem 4 ([6]). If $\widehat{\mathcal{S}}$ is a simple factor dressing of a conformal Gauss map $\mathcal{S}$ of a minimal surface f, then $\widehat{\mathcal{S}}$ is the conformal Gauss map of a minimal surface $\widehat{f} = h^{n\frac{b}{a-1}n^{-1},-1}$, $h := -f_{m\frac{b}{2}m^{-1},m\frac{a-1}{2}m^{-1}}$.

$$\begin{array}{ccc} \mathcal{S} & \longleftrightarrow & f \\ \downarrow & & \downarrow \\ \widehat{\mathcal{S}} & \longleftrightarrow & \hat{f} \end{array}$$

4.2 μ-Darboux transform of a conformal Gauss map

Let $f\colon M \to \mathbb{H}$ be a minimal with $(df)^R = 0$. Then, the map $g := fR - f^*$ is a super-conformal map. A super-conformal map is a conformal map with vanishing Willmore energy.

Definition 3 ([6]). The super-conformal map g is called an associated Willmore surface of f.

The following is a relation between μ-Darboux transform, associated family, and associated Willmore surface.

Theorem 5 ([6]). Every non-constant μ-Darboux transform of a minimal surface f is an associated Willmore surface of an associated minimal surface $f^{p,q}$.

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