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First Passage Time in Real Options *

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1 Introduction

We consider irreversible investment problems with regime switching feature under a monopoly setting. Several parameters describing the economic environment vary according to a regime switching with general number of states. A firm seeks an optimal timing to invest an irreversible project while observing the potential profits. We present a systematic procedure to derive the value function via solving a system of simultaneous ordinary differential equations with knowledge of linear algebra. It will enable us to investigate a comparative analysis of the investment problem. The contribution of this paper is a natural extension of Guo and Zhang (2004) and Jobert and Rogers (2006) to a real option problem with the general number of regime states. Furthermore, we obtain an analytical expression of an expectation of a payoff at the first passage time to the stop region by applying the Dirichlet problem and the aforementioned technique.

In the literature, the value function of a typical real option problem is usually calculated by first guessing the form and then solving the unknown coefficients as in Dixit and Pyndick (1994), Guo et al. (2005) for $S=2$, where S is the number of regimes, and Grenadier and Wang (2007) for general $S\geq 2$. For American put options, Guo and Zhang (2004) take similar approach for $S=2$. Apparently one of the drawbacks is that there are no clues why such a form is taken. Jobert and Rogers (2006), Jiang and Pistorius (2008) utilize Wiener-Hopf factorization to obtain the value function for $S\geq 2$. They present explicit forms of the value function up to exponential matrix. Similar problems with the regime switching features are discussed in Boyarchenko and Levendorski (2008, 2009) who also make use of the Wiener-Hopf factorization g in a different form. Due to the complicated form, further analysis including comparative analysis seems difficult without an insightful expression of the value function.

Our approach is simple and straightforward: solve a system of simultaneous ordinary differential equations with appropriate conditions directly on each interval of the thresholds. Due to the form, knowledge of linear algebra helps a lot in the derivation. We do not rely on a "guess functional form" nor the Wiener-Hopf factorization that is more technical. It is found that the value function is represented by the eigenvalues and the eigenvectors of a coefficient matrix. With the same technique we can carry out a calculation of the expected and discounted value of a payoff at the first passage time by applying the Dirichlet problem in the two-dimensional space. The original value function can be decomposed with these expectations. The smooth pasting conditions at the boundary between the continuation region and the stop region are recovered by the optimality conditions of the thresholds. The detailed derivation is provided for

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the case of two regimes. These results will help us to deepen our understanding of the investment problem.

2 Setup

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on infinite time horizon. Let $J = \{J(t)\}$ be a continuous-time Markov chain on a finite state space $E = \{1, 2, \cdots, S\}$. $J(t)$ is interpreted as a regime or a state of the economy at time t. The intensity matrix of the regime switching is given by $\mathbf{Q}$

$$\mathbf{Q} = (q_{ij})_{i,j \in E}, \quad q_{ii} = -\sum_{j \in E \setminus \{i\}} q_{ij}. \tag{2.1}$$

The process $X = \{X(t)\}$ satisfies

$$dX(t) = \mu_{J(t)} X(t) dt + \sigma_{J(t)} X(t) dW_t, \quad X(0) = x, \tag{2.2}$$

where $W = \{W_t\}$ is a standard Brownian motion, μ_j and $\sigma_j > 0$ are finite constants for each $j \in E$. Denote the filtration generated by (W, J) as $\{\mathcal{F}_t\}$ with $\mathcal{F}_t = \sigma(W_s, J(s), 0 \leq s \leq t)$.

The firm has a chance to start a project to make a product as a monopoly of the product whose revenue depends on the state variables $(X(t), J(t))$ of the economy. We assume that the firm has a technology to enter into the project by paying the cost K_i when the regime state is i, and after the investment the firm obtains the instant revenue of $D_{J(t)} X(t)$ at time t from the project, where $D_i K_i (i \in E)$ are positive constants. One may interpret that $X(t)$ is the unit price of the products from the project and D_i be the (potential) demand quantity for the products in the economy.

In this paper a matrix is represented in bold. $\mathbf{O}_n$ denotes the zero matrix of order n and $\mathbf{I}_n$ denotes the identity matrix of order n. An element of a matrix $\mathbf{A} = (a_{ij})$ is denoted by $a_{ij} = \{\mathbf{A}\}_{ij}$. Let us denote vectors, matrices and functions

$$\begin{aligned}
\mathbf{e}_i &= (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^S, \quad \mathbf{1}_S = (1, \cdots, 1)^\top \in \mathbb{R}^S, \\
\mathbf{D} &= \left(D_1, \cdots, D_S\right)^\top, \quad \mathbf{M} = \operatorname{diag}\left[\mu_1, \cdots, \mu_S\right], \\
\mathbf{\Lambda}(\beta) &= \begin{pmatrix} g_1(\beta) & q_{12} & \cdots & q_{1S} \\ q_{21} & g_2(\beta) & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{S1} & q_{S2} & \cdots & g_S(\beta) \end{pmatrix}, \qquad (2.3) \\
g_i(\beta) &= \frac{1}{2}\sigma_i^2 \beta^2 + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)\beta + q_{ii} - r. \qquad (2.4)
\end{aligned}$$

For each $i \in E$, consider a Lévy process $L_t^{(i)} = \left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_t$, which has the Lévy exponent $\theta^{(i)}(z) = \left(\mu_i - \frac{1}{2}\sigma_i^2 - r\right) z + \frac{1}{2}\sigma_i^2 z^2 = g_i(z) - q_{ii} + r$. Then $\ln X(t) = L_t^{(J(t))}$ evolves like a diffusion while the regime does not switch, and the matrix (2.3) is expressed as

$$\mathbf{\Lambda}(\beta) = \operatorname{diag}[\theta^{(k)}(\beta); k \in E] + \mathbf{Q} - r\mathbf{I}_S.$$

For a simple notation it is convenient to introduce a "truncating" operator $\mathbf{H}_n$ on $S \times S$ square matrices $(a_{ij})_{1 \le i,j \le S}$ defined by

$$\mathbf{H}_n((a_{ij})_{1 \le i,j \le S}) = (a_{ij})_{1 \le i,j \le n}. \tag{2.5}$$

The truncating operator $\mathbf{H}_n$ reflects our focusing on n regime states in a continuation region among S regimes as discussed later.

We assume the following properties;

Assumption 1 *1.* $\mathbf{Q}$ *has a property that* $\forall i \in E$, $\exists j \in E$, $j \neq i$, $(q_{ij}, q_{ji}) \neq (0,0)$.

2. $r > \max \left\{ \max_{i \in E} \mu_i, 0 \right\}$.

3. The matrices $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})$ *and* $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})$ *are invertible for all* $n \in E$.

The property 1 of the above assumption is to restrict our analysis to non-redundant cases of regime switching. Each regime has a (non-zero) chance to move to another regime and/or to be transferred from another regime†. The property 2 guarantees a convergence of total revenue and other properties. The property 3 is due to a technical reason to make our discussion simple. Note that the matrices mentioned in the above are expressed with $\mathbf{\Lambda}(\beta)$ as

$$r\mathbf{I}_S - \mathbf{M} - \mathbf{Q} = -\mathbf{\Lambda}(1), \quad r\mathbf{I}_S - \mathbf{Q} = -\mathbf{\Lambda}(0).$$

3 Value function

The firm seeks the optimal timing of the investment. When the current regime state is i, the value function V_i is defined by

$$V_i(x) \quad = \quad \max_{\tau} \mathbb{E} \left[\int_{\tau}^{\infty} e^{-ru} D_{J(u)} X(u) du - e^{-r\tau} K_{J(\tau)} \mid X(0) = x, J(0) = i \right].$$

The following lemma gives useful expressions for the calculation of the value function.

Lemma 1

$$(1) \quad \mathbb{E} \left[e^{-rT} X(T)^{\beta} \mid \mathcal{F}_t \right] = X(t)^{\beta} \mathbf{e}_{J(t)}^{\top} \exp \left(\mathbf{\Lambda}(\beta)(T-t) \right) \mathbf{1}_S, \quad \beta \in \mathbb{R},$$

$$(2) \quad \mathbb{E} \left[\int_t^{\infty} e^{-ru} D_{J(u)} X(u) du \mid \mathcal{F}_t \right] = e^{-rt} \alpha_{J(t)} D_{J(t)} X(t),$$

where

$$\alpha_i D_i = \mathbf{e}_i^{\top} \left(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q} \right)^{-1} \mathbf{D}.$$

† $\mathbf{Q}$ is called to be irreducible if $q_{ij} \neq 0$ for $\forall i, j \in E$, $i \neq j$, that is, each regime has a non-zero chance to move to another regime *and* to be transferred from another regime. The property 1 is weaker than the irreducibility. It follows that some reducible matrices satisfy the property 1.

By Lemma 1, the value function at the regime i is reduced to

$$V_i(x) \quad = \quad \max_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\alpha_{J(\tau)} D_{J(\tau)} X(\tau) - K_{J(\tau)}\right) \mid X(0) = x, J(0) = i\right].$$

As discussed in Jobert and Rogers (2006) and Guo and Zhang (2004), the candidate of the optimal stopping time τ^* must be in a form of

$$\tau^* = \min_{j \in E} \tau_j, \quad \tau_j = \inf\{t > 0 : X(t) \geq x_j,\ J(t) = j\}$$

with some positive x_j $(j \in E)$. We will obtain the explicit form of the value function by assuming that the order of the thresholds is

$$x_S \leq x_{S-1} \leq \cdots \leq x_2 \leq x_1 \tag{3.1}$$

in what follows. Namely the regime state S is the best and the regime state 1 is the worst for starting the project. In case that (3.1) is not satisfied, the following procedure must be carried out after the regime indices are interchanged appropriately.

On the n-th interval (x_{n+1}, x_n) the regimes $1, \cdots, n$ (continuation regimes) are in the continuation region while the regimes $n+1, \cdots, S$ (stop regimes) are in the stop region. Hence, the state will enter into a stop region either if $X(t)$ moves upward gradually beyond $x_{J(t)}$ by the Brownian motion without regime switches or if the regime $J(t)$ is suddenly switched to either of the stop regimes by a regime switch.

The value function will take a different functional form on each interval of the thresholds as

$$V_i(x) = \begin{cases} V_i^{(0)}(x) & \text{if } x \in [x_1, \infty), \\ V_i^{(n)}(x) & \text{if } x \in [x_{n+1}, x_n), \quad (n = 1, 2, \cdots, S-1), \\ V_i^{(S)}(x) & \text{if } x \in (0, x_S). \end{cases}$$

We will calculate $V_i^{(n)}(x)$ for each i on n-th interval by starting from $n = 0$ and moving on to $n = S$. When $x \in [x_1, \infty)$, the state is in the stop region at any regime and it is optimal for the firm to start the project immediately since the price $X(t)$ is high enough. Hence we have

$$V_i^{(0)}(x) = \alpha_i D_i x - K_i, \quad i \in E \tag{3.2}$$

For $x \in [x_{n+1}, x_n)$ $(n = 1, 2, \cdots, S-1)$, the firm will enter into the project if the regime is either of $n+1, \cdots, S$, otherwise she should wait. Thus, the value function $V_i^{(n)}$ $(1 \leq i \leq n)$ satisfies

$$\frac{1}{2} x^2 \sigma_i^2 \frac{d^2}{dx^2} V_i^{(n)}(x) + x\mu_i \frac{d}{dx} V_i^{(n)}(x) - rV_i^{(n)}(x) + \sum_{j \in E \setminus \{i\}} q_{ij}(V_j^{(n)}(x) - V_i^{(n)}(x)) = 0, \tag{3.3}$$

and $V_i^{(n)}(x) = \alpha_i D_i x - K_i$ for $n+1 \leq i \leq S$. Finally, for $x \in (0, x_S)$, $V_i^{(S)}$ obeys the same ODE as (3.3) with $n = S$. The first three terms of (3.3) represents a change of the value function due to a movement of Brownian motion and the last term represents a change due to a regime switching. The optimality condition requires that the sum of these changes must be zero.

In summary, we must solve simultaneous ODEs on an interval $[x_{n+1}, x_n)$, $(n = 1, 2, \cdots, S-1)$,

$$\begin{aligned}
\mathcal{A}_1 V_1^{(n)}(x) + \sum_{1\le j\le n,\, j\ne 1} q_{1j} V_j^{(n)}(x) &= -\sum_{n+1\le j\le S} q_{1j} V_j^{(n)}(x) \\
\mathcal{A}_2 V_2^{(n)}(x) + \sum_{1\le j\le n,\, j\ne 2} q_{2j} V_j^{(n)}(x) &= -\sum_{n+1\le j\le S} q_{2j} V_j^{(n)}(x) \\
&\vdots \\
\mathcal{A}_n V_n^{(n)}(x) + \sum_{1\le j\le n,\, j\ne n} q_{nj} V_j^{(n)}(x) &= -\sum_{n+1\le j\le S} q_{nj} V_j^{(n)}(x),
\end{aligned}$$

with the value matching condition and the smooth pasting conditions at $x = x_n, x_{n+1}$, where $\mathcal{A}_i$ is a differential operator defined by

$$\mathcal{A}_i f(x) = \frac{1}{2} x^2 \sigma_i^2 \frac{d^2}{dx^2} f(x) + x\mu_i \frac{d}{dx} f(x) + (q_{ii} - r) f(x).$$

It is symbolic to represent them in a form of matrix as

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} V_1^{(n)}(x) \\ V_2^{(n)}(x) \\ \vdots \\ V_n^{(n)}(x) \end{pmatrix} = - \begin{pmatrix} q_{1,n+1} & q_{1,n+2} & \cdots & q_{1S} \\ q_{2,n+1} & q_{2,n+2} & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,n+1} & q_{n,n+2} & \cdots & q_{nS} \end{pmatrix} \begin{pmatrix} V_{n+1}^{(n)}(x) \\ V_{n+2}^{(n)}(x) \\ \vdots \\ V_S^{(n)}(x) \end{pmatrix}. \tag{3.4}$$

The functions on the LHS are unknown and to be solved while ones on the RHS $V_j^{(n)}(x) = \alpha_j D_j x - K_j$ $(j \in \{n+1, \ldots, S\})$ are known. As for an interval $(0, x_S)$ a similar system of ODEs must be solved

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1S} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{S1} & q_{S2} & \cdots & \mathcal{A}_S \end{pmatrix} \begin{pmatrix} V_1^{(S)}(x) \\ V_2^{(S)}(x) \\ \vdots \\ V_S^{(S)}(x) \end{pmatrix} = \mathbf{0}_S.$$

A set of ODEs to be solved is dependent of the interval of x. We study the equations (3.4) on $V_i^{(n)}(x)$ $(i = 1, 2, \cdots, n)$ defined on the interval (x_{n+1}, x_n) $(n = 1, 2, \cdots, S)$ by modifying the RHS and the interval of x appropriately in the case of $n = S$. Since we know the solution $V_i^{(n)}(x) = \alpha_i D_i x - K_i$ for $i = n+1, \cdots, S$, the equations of the remaining $V_i^{(n)}$ for $1 \le i \le n$ are reduced to simultaneous second-order linear ODEs. It follows that the solution $V_i^{(n)}$ is decomposed with the general solution $\widetilde{V}_i^{(n)}$ and the special solution $v_i^{(n)}$ for each $i = 1, 2, \cdots, n$.

The special solution $v_i^{(n)}$ is easily found to be a linear function $v_i^{(n)}(x) = a_i^{(n)} x + b_i^{(n)}$, where the coefficients $\mathbf{a}^{(n)} = (a_1^{(n)}, \cdots, a_n^{(n)})^\top, \mathbf{b}^{(n)} = (b_1^{(n)}, \cdots, b_n^{(n)})^\top$ are given by

$$\mathbf{a}^{(n)} = \mathbf{H}_n(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j}\alpha_j D_j \\ \sum_{j=n+1}^{S} q_{2j}\alpha_j D_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj}\alpha_j D_j \end{pmatrix}, \tag{3.5}$$

$$\mathbf{b}^{(n)} = -\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j} K_j \\ \sum_{j=n+1}^{S} q_{2j} K_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj} K_j \end{pmatrix},$$

where the inverse matrices are guaranteed to exist by Assumption 1.

Now we turn our eyes to the general solution $\widetilde{V}_i^{(n)}$. Let us change the variable $y = \ln x$ and introduce auxiliary functions $\overline{V}_i^{(n)}(y) = \widetilde{V}_i^{(n)}(e^y)$, $\overline{W}_i^{(n)}(y) = \frac{d}{dy}\overline{V}_i^{(n)}(y)$. Then the equations for the general solution part of (3.4) can be rewritten as a system of first-order ODEs,

$$\frac{d}{dy}\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \mathbf{\Gamma}_n \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix}, \tag{3.6}$$

where

$$\mathbf{\Gamma}_n = \begin{pmatrix} \mathbf{O}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \mathbf{\Sigma}_n = \frac{1}{2}\mathrm{diag}\,[\sigma_1^2, \cdots, \sigma_n^2] \in \mathbb{R}^{n \times n},$$

$$\mathbf{R}_n = \mathbf{\Sigma}_n^{-1}\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})) = -2 \begin{pmatrix} \frac{q_{11} - r}{\sigma_1^2} & \frac{q_{12}}{\sigma_1^2} & \cdots & \frac{q_{1n}}{\sigma_1^2} \\ \frac{q_{21}}{\sigma_2^2} & \frac{q_{22} - r}{\sigma_2^2} & \cdots & \frac{q_{2n}}{\sigma_2^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{q_{n1}}{\sigma_n^2} & \frac{q_{n2}}{\sigma_n^2} & \cdots & \frac{q_{nn} - r}{\sigma_n^2} \end{pmatrix} \in \mathbb{R}^{n \times n},$$

$$\mathbf{C}_n = \mathbf{\Sigma}_n^{-1}\mathbf{H}_n(\mathbf{\Sigma}_S - \mathbf{M}) = \mathrm{diag}\left[1 - \frac{2\mu_1}{\sigma_1^2}, \cdots, 1 - \frac{2\mu_n}{\sigma_n^2}\right] \in \mathbb{R}^{n \times n}.$$

Thus, the solution is given by

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \exp\left((y - y_0)\mathbf{\Gamma}_n\right) \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y_0) \\ \overline{\mathbf{W}}^{(n)}(y_0) \end{pmatrix}$$

with some y_0 from the boundary conditions when the exponential matrix $\exp\left((y - y_0)\mathbf{\Gamma}_n\right)$ is available. If the coefficient matrix $\mathbf{\Gamma}_n$ is diagonalizable, it is straightforward to solve and obtain an explicit representation of the solution of the system of ODEs (3.6). Otherwise, one can proceed similarly by making use of the Jordan normal form that is guaranteed to exist for any square matrix by the theory.

By the knowledge of linear algebra the characteristic function of of $\mathbf{\Gamma}_n$ is obtained as

$$\det\begin{pmatrix}\mathbf{O}_n-\beta\mathbf{I}_n & \mathbf{I}_n\\ \mathbf{R}_n & \mathbf{C}_n-\beta\mathbf{I}_n\end{pmatrix} = f_n(\beta)\prod_{j=1}^{n}\left(\frac{1}{2}\sigma_j^2\right)^{-1},$$

where

$$f_n(\beta) = \det\left(\mathbf{\Sigma}_n\left(\mathbf{I}_n\beta^2-\mathbf{C}_n\beta-\mathbf{R}_n\right)\right) = \det\mathbf{H}_n(\mathbf{\Lambda}(\beta)). \tag{3.7}$$

Thus, the eigenvalues are the solutions of $f_n(\beta) = 0$. In this paper we make the following assumption for simple and useful results.

Assumption 2 *1. For $n = 1, 2, \cdots, S-1$, $\mathbf{\Gamma}_n$ has $2n$ distinct eigenvalues $\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}$.*

2. $\mathbf{\Gamma}_S$ has $2S$ distinct eigenvalues such that $\beta_1^{(S)}, \cdots, \beta_S^{(S)}$ are strictly positive and $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ are strictly negative.

If the eigenvalues become complex numbers or duplicated so that the above assumption is not satisfied, the following discussion can be accordingly modified by considering the Jordan normal form as mentioned before.‡

By Assumption 2 there exist distinct eigenvalues $\beta_j^{(n)}$ $(1 \le j \le 2n)$. Since the upper right block of $\mathbf{\Gamma}_n$ is $\mathbf{I}_n$, the eigenvector for the eigenvalue $\beta_j^{(n)}$ must be in the form

$$\tilde{\mathbf{u}}_j^{(n)} = \begin{pmatrix}\mathbf{u}_j^{(n)}\\ \beta_j^{(n)}\mathbf{u}_j^{(n)}\end{pmatrix} \in \mathbb{R}^{2n},$$

with some non-zero vector $\mathbf{u}_j^{(n)} \in \mathbb{R}^n$ satisfying

$$\mathbf{H}_n(\mathbf{\Lambda}(\beta))\mathbf{u}_j^{(n)} = \mathbf{0}_n. \tag{3.8}$$

Such a vector $\mathbf{u}_j^{(n)}$ exists for each j because the determinant of the coefficient matrix on the LHS of (3.8) is equal to $f_n\left(\beta_j^{(n)}\right) = 0$ by definition of $\beta_j^{(n)}$. Thus, $\mathbf{\Gamma}_n$ is diagonalized as

$$\mathbf{\Gamma}_n = \begin{pmatrix}\mathbf{U}^{(n)}\\ \mathbf{U}^{(n)}\mathbf{B}^{(n)}\end{pmatrix}\operatorname{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right]\begin{pmatrix}\mathbf{U}^{(n)}\\ \mathbf{U}^{(n)}\mathbf{B}^{(n)}\end{pmatrix}^{-1},$$

where

$$\mathbf{U}^{(n)} = \begin{pmatrix}\mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)}\end{pmatrix} \in \mathbb{R}^{n\times 2n}, \quad \mathbf{B}^{(n)} = \operatorname{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right] \in \mathbb{R}^{2n\times 2n}.$$

‡Due to the duplicated eigenvalues, a Jordan normal form appears in the value function in Grenadier and Wang (2007) in a context of hyperbolic discounting which may be able to be modeled with a regime switching and our discussion may be applicable to.

The matrix

$$\begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)} \\ \beta_1^{(n)}\mathbf{u}_1^{(n)} & \beta_2^{(n)}\mathbf{u}_2^{(n)} & \cdots & \beta_{2n}^{(n)}\mathbf{u}_{2n}^{(n)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{u}}_1^{(n)} & \tilde{\mathbf{u}}_2^{(n)} & \cdots & \tilde{\mathbf{u}}_{2n}^{(n)} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

is invertible since the eigenvalues of $\mathbf{\Gamma}_n$ are distinct so that the corresponding eigenvectors $\tilde{\mathbf{u}}_j^{(n)}$ are linearly independent.

Then we can solve the system of ODEs (3.6) as

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \operatorname{diag}\left[e^{\beta_1^{(n)}y}, \cdots, e^{\beta_{2n}^{(n)}y}\right] \mathbf{A}^{(n)},$$

with some constant vector $\mathbf{A}^{(n)} \in \mathbb{R}^{2n}$. By adding the special solutions, we have the vector of the value functions $\mathbf{V}^{(n)}(x) = (V_1^{(n)}(x), \cdots, V_n^{(n)}(x))^\top$ at each regime on the interval $[x_{n+1}, x_n)$ given as

$$\mathbf{V}^{(n)}(x) = \mathbf{U}^{(n)}\mathbf{X}^{(n)}(x)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x), \tag{3.9}$$

where

$$\mathbf{X}^{(n)}(x) = \operatorname{diag}\left[x^{\beta_1^{(n)}}, \cdots, x^{\beta_{2n}^{(n)}}\right], \quad \mathbf{v}^{(n)}(x) = \mathbf{a}^{(n)}x + \mathbf{b}^{(n)}.$$

Unknown boundaries $x_S \leq \cdots \leq x_1$ and unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ will be determined by the value matching conditions, the smooth pasting conditions and the values at $x = 0$. We will investigate them by looking at x_1 first and moving downward to x_S as follows.

The value matching conditions at $x = x_n$, $V_i^{(n)}(x_n) = V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ are represented by n-dimensional vectors as

$$\mathbf{U}^{(n)}\mathbf{X}^{(n)}(x_n)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x_n) = \begin{pmatrix} \mathbf{U}^{(n-1)}\mathbf{X}^{(n-1)}(x_n)\mathbf{A}^{(n-1)} + \mathbf{v}^{(n-1)}(x_n) \\ \alpha_n D_n x_n - K_n \end{pmatrix}. \tag{3.10}$$

Similarly, the smooth pasting conditions $x_n \frac{d}{dx}V_i^{(n)}(x_n) = x_n \frac{d}{dx}V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ require

$$\mathbf{U}^{(n)}\mathbf{dX}^{(n)}(x_n)\mathbf{A}^{(n)} + \mathbf{a}^{(n)}x_n = \begin{pmatrix} \mathbf{U}^{(n-1)}\mathbf{dX}^{(n-1)}(x_n)\mathbf{A}^{(n-1)} + \mathbf{a}^{(n-1)}x_n \\ \alpha_n D_n x_n \end{pmatrix}, \tag{3.11}$$

where

$$\mathbf{dX}^{(n)}(x) = \operatorname{diag}\left[\beta_1^{(n)}x^{\beta_1^{(n)}}, \cdots, \beta_{2n}^{(n)}x^{\beta_{2n}^{(n)}}\right] = \mathbf{B}^{(n)}\mathbf{X}^{(n)}(x), \quad \mathbf{a}^{(S)} = \mathbf{0}_S.$$

By coupling these conditions (3.10), (3.11) into one vector and making use of a relationship

$$\begin{pmatrix} \mathbf{U}^{(n)}\mathbf{X}^{(n)}(x) \\ \mathbf{U}^{(n)}\mathbf{dX}^{(n)}(x) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathbf{X}^{(n)}(x),$$

$\mathbf{A}^{(n)}$ is represented with a function of x_n and $\mathbf{A}^{(n-1)}$ as

$$\mathbf{A}^{(n)} = \mathbf{X}^{(n)}\left(x_n^{-1}\right)\begin{pmatrix}\mathbf{U}^{(n)}\\ \mathbf{U}^{(n)}\mathbf{B}^{(n)}\end{pmatrix}^{-1} \times \left[\begin{pmatrix}\mathbf{U}^{(n-1)}\mathbf{X}^{(n-1)}(x_n)\mathbf{A}^{(n-1)}+\mathbf{v}^{(n-1)}(x_n)\\ \alpha_n D_n x_n - K_n\\ \mathbf{U}^{(n-1)}\mathbf{dX}^{(n-1)}(x_n)\mathbf{A}^{(n-1)}+\mathbf{a}^{(n-1)}x_n\\ \alpha_n D_n x_n\end{pmatrix} - \begin{pmatrix}\mathbf{v}^{(n)}(x_n)\\ \mathbf{a}^{(n)}x_n\end{pmatrix}\right]. \tag{3.12}$$

for $n = 2, \cdots, S-1$, and

$$\mathbf{A}^{(1)} = \mathbf{X}^{(1)}\left(x_1^{-1}\right)\begin{pmatrix}\mathbf{U}^{(1)}\\ \mathbf{U}^{(1)}\mathbf{B}^{(1)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha_1 D_1 x_1 - K_1 - v^{(1)}(x_1)\\ \alpha_1 D_1 x_1 - a^{(1)}x_1\end{pmatrix}, \tag{3.13}$$

$$\mathbf{A}^{(S)} = \mathbf{X}^{(S)}\left(x_S^{-1}\right)\begin{pmatrix}\mathbf{U}^{(S)}\\ \mathbf{U}^{(S)}\mathbf{B}^{(S)}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{U}^{(S-1)}\mathbf{X}^{(S-1)}(x_S)\mathbf{A}^{(S-1)}+\mathbf{v}^{(S-1)}(x_S)\\ \alpha_S D_S x_S - K_S\\ \mathbf{U}^{(S-1)}\mathbf{dX}^{(S-1)}(x_S)\mathbf{A}^{(S-1)}+\mathbf{a}^{(S-1)}x_S\\ \alpha_S D_S x_S\end{pmatrix}. \tag{3.14}$$

Therefore, we can represent unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ as functions of $x_1, \cdots, x_S$ recursively.

Furthermore, on $(0, x_S]$, we want to impose another condition $\lim_{x\to 0} V_i^{(S)}(x) = 0$ for all i in order to make the value function finite. It implies that the coefficient of $\mathbf{A}^{(S)}$ corresponding to negative eigenvalues $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ must be zero,

$$\begin{pmatrix}\mathbf{O}_S & \mathbf{I}_S\end{pmatrix}\mathbf{A}^{(S)} = \mathbf{0}_S. \tag{3.15}$$

This is a set of S equations that S unknown constants $x_1, \cdots, x_S$ must satisfy. Apparently (3.15) is a system of complicated algebraic equations, hence they must be solved numerically. In case that the numerical solution doesn't satisfy the order condition (3.1), the indices of the regimes must be interchanged.

Then, by noting that $\mathbf{v}^{(S)}(x) = 0$, the value function on $(0, x_S)$ can be expressed with terms with positive eigenvalues as

$$\mathbf{V}^{(S)}(x) = \mathbf{U}_S\mathbf{X}_S(x)\mathbf{A}_S, \tag{3.16}$$

where

$$\mathbf{U}_S = (\mathbf{u}_1^{(S)} \cdots \mathbf{u}_S^{(S)}), \quad \mathbf{X}_S(x) = \mathrm{diag}\left[x^{\beta_1^{(S)}}, \cdots, x^{\beta_S^{(S)}}\right], \quad \mathbf{A}_S = \begin{pmatrix}\mathbf{I}_S & \mathbf{O}_S\end{pmatrix}\mathbf{A}^{(S)}.$$

As a summary, we obtain the main result.

Proposition 1 *Suppose that Assumption 1 and 2 hold, and $x_1, \cdots, x_S$ satisfy (3.1) and (3.15). Then the value function is given by (3.9) and (3.16).*

4 First passage time

With the same technique we can carry out a calculation of the expected and discounted value of a payoff at the first passage time by applying the Dirichlet problem in the two-dimensional space.

Once the optimal stopping time τ^* is determined and the thresholds $x_S \leq \cdots \leq x_1$ are fixed, the value function can be decomposed by functions related to the regime at the first passage time as

$$\begin{aligned}
V_i(x) &= \max_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\alpha_{J(\tau)}D_{J(\tau)}X(\tau) - K_{J(\tau)}\right) \mid X(0)=x, J(0)=i\right] \\
&= \sum_{k\in E} \mathbb{E}\left[1_{\{\tau^*=\tau_k\}}e^{-r\tau^*}\left(\alpha_{J(\tau^*)}D_{J(\tau^*)}X_{\tau^*} - K_{J(\tau^*)}\right) \mid X(0)=x, J(0)=i\right] \\
&= \sum_{k\in E}\left[\alpha_k D_k F_i^k(x) - K_k G_i^k(x)\right], \qquad (4.1)
\end{aligned}$$

where $\tau^* = \min_{k\in E}\tau_k$, $\tau_k = \inf\{t>0 : X(t)\geq x_k,\ J(t)=k\}$ and

$$F_i^k(x) = \mathbb{E}\left[e^{-r\tau^*}M_F^k(X_{\tau^*}, J(\tau^*)) \mid X(0)=x, J(0)=i\right], \quad M_F^k(x,j) = x\delta_{jk}, \qquad (4.2)$$

$$G_i^k(x) = \mathbb{E}\left[e^{-r\tau^*}M_G^k(X_{\tau^*}, J(\tau^*)) \mid X(0)=x, J(0)=i\right], \quad M_G^k(x,j) = \delta_{jk}. \qquad (4.3)$$

$F_i^k(x)$ is the discounted expected value of the payoff of $X_{\tau^*}\mathbf{1}_{\{J(\tau^*)=k\}}$ after starting from $(X(0),J(0)) = (x,i)$. Similarly, $G_i^k(x)$ is one of the payoff of $\mathbf{1}_{\{J(\tau^*)=k\}}$.

The purpose of this section is to obtain the explicit form of these decomposing functions F_i^k, G_i^k. Our plan and idea are as follows. The functions defined by the expectation in (4.2) and (4.3) are applicable to the Dirichlet problem thanks to the functional form in the definition. However, the domain must be set appropriately in the two-dimensional space. Then we obtain a set of ODEs which can be solved with the same technique as in the previous section. Finally, we verify the decomposition (4.1) and the smooth pasting conditions at the boundary, which are not imposed in the Dirichlet problem.

Since the stopping time τ^* has the regime-dependent thresholds, we need to rename them on each relevant interval as

$$F_i^k(x) = \begin{cases} F_i^{(k,0)}(x), & x\in[x_1,\infty) \\ F_i^{(k,n)}(x), & x\in[x_{n+1},x_n), \quad 1\leq n\leq S-1 \\ F_i^{(k,S)}(x), & x\in(0,x_S) \end{cases}$$

$$G_i^k(x) = \begin{cases} G_i^{(k,0)}(x), & x\in[x_1,\infty) \\ G_i^{(k,n)}(x), & x\in[x_{n+1},x_n), \quad 1\leq n\leq S-1 \\ G_i^{(k,S)}(x), & x\in(0,x_S) \end{cases}$$

On the n-th interval, the regimes $i>n$ are in the stop region so that

$$F_i^{(k,n)}(x) = \delta_{ik}x \equiv M_F^k(x,i), \quad G_i^{(k,n)}(x) = \delta_{ik} \equiv M_G^k(x,i), \quad (i>n). \qquad (4.4)$$

In order to obtain the explicit expression of the functions $F_i^{(k,n)}$ and $G_i^{(k,n)}$, we apply the following Dirichlet problem.

Lemma 2 *Consider an open set $\mathcal{C} \subset \mathbb{R}^d$ and $\mathcal{D} = \mathbb{R}^d \setminus \mathcal{C}$. Define the first passage time $\tau_{\mathcal{D}} = \inf\{t : Y_t \in \mathcal{D}\}$ of a Markov process Y on $\mathbb{R}^d$ with $Y_0 = x \in \mathcal{C}$, and define $F(x) = \mathbb{E}\left[e^{-r\tau_{\mathcal{D}}} M\left(Y_{\tau_{\mathcal{D}}}\right)\right]$ for a given continuous function $M : \partial\mathcal{C} \to \mathbb{R}$. Then F solves the Dirichlet problem*

$$\mathbb{A}_Y F = rF \quad in \quad \mathcal{C}, \qquad F|_{\partial\mathcal{C}} = M$$

where $\mathbb{A}_Y$ is the infinitesimal operator $\mathbb{A}_Y F(x) = \lim_{t \downarrow 0} \dfrac{\mathbb{E}\left[F(Y_t) \mid Y_0 = x\right] - F(x)}{t}$.

Proof. See, for example, Peskir and Shiraev (2006).

For a regime switching diffusion $dX(t) = \mu_{J(t)} X(t)dt + \sigma_{J(t)} X(t) dW_t$, it is known that

$$\mathbb{A}_X F_i(x) = \frac{1}{2}\sigma_i^2 x^2 \frac{d^2 F_i}{dx^2}(x) + \mu_i x \frac{dF_i}{dx}(x) + \sum_{j=1, j\neq i}^{S} q_{ij}\left(F_j(x) - F_i(x)\right).$$

The continuation region of our problem and the relevant sets are given by

$$\begin{aligned} \mathcal{B} &= (0, x_1) \times (0, S+1) \subset \mathbb{R}^2_{++}, \quad \mathcal{C} = \mathcal{B} \setminus \bigcup_{i=1}^{S} \{(x,i) \in \mathbb{R}^2_{++} \mid x_i \le x < x_1\}, \\ \mathcal{D} &= \mathbb{R}^2_{++} \setminus \mathcal{C}, \quad \partial\mathcal{C} = \partial\mathcal{B} \cup \left[\bigcup_{i=1}^{S} \{(x,i) \in \mathbb{R}^2_{++} \mid x_i \le x < x_1\}\right]. \end{aligned}$$

When we fix k and consider F_i^k $(i \in E)$, the continuous function M on the boundary is defined with $M_F^k(x,i)$ (or $M_G^k(x,i)$)

$$M = \begin{cases} M_F^k(x,i), & y = i \in E,\ x_i \le x < x_1, \\ ([y]+1-y) M_F^k(x_1,[y]) & \\ +(y-[y]) M_F^k(x_1,[y]+1), & [y] \in E,\ x = x_1, \\ 0, & \text{otherwise}. \end{cases}$$

The function M when considering G_i^k $(i \in E)$ can be constructed similarly by replacing $M_F^k(x,i)$ with $M_G^k(x,i)$ in the above definition. Note that the domain is expanded to a dense subset in $\mathbb{R}^2$, though our interest is in a subset in $\mathbb{R} \times \mathbb{Z}$, in order to apply the Dirichlet problem directly. The regimes take values in $\mathbb{Z}$ only so that we don't need to pay attention to the function values in the non-integer area in the regimes. Thus, the interpolated values of M at (x_1, y) $(y \notin E)$ are sufficient to make M continuous on the boundary $\partial\mathcal{C}$.

Then by Dirichlet problem, each of F_i^k and G_i^k satisfies a set of certain ODEs. We must solve simultaneous ODEs on each interval such as

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = - \begin{pmatrix} q_{1,n+1} & q_{1,n+2} & \cdots & q_{1S} \\ q_{2,n+1} & q_{2,n+2} & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,n+1} & q_{n,n+2} & \cdots & q_{nS} \end{pmatrix} \begin{pmatrix} F_{n+1}^{(k,n)}(x) \\ F_{n+2}^{(k,n)}(x) \\ \vdots \\ F_S^{(k,n)}(x) \end{pmatrix} \tag{4.5}$$

on $x \in [x_{n+1}, x_n)$, $(n = 1, 2, \cdots, S-1)$ for each $k = 1, 2$, where

$$\mathcal{A}_i f(x) = \frac{1}{2} x^2 \sigma_i^2 \frac{d^2}{dx^2} f(x) + x\mu_i \frac{d}{dx} f(x) + (q_{ii} - r) f(x).$$

The value matching conditions are imposed at x_j in both $\mathcal{C}$ and $\partial\mathcal{C}$. The smooth pasting conditions, however, are imposed at x_j in $\mathcal{C}$ only since a smooth pasting condition at the boundary is meaningless in the Dirichlet problem.

When $n \le k-1$, $F_k^{(k,n)}(x) = x$ but other $F_i^{(k,n)}$ on the RHS of (4.5) are zero by (4.4). On the other hand, when $n \ge k$, all of $F_i^{(k,n)}$ on the RHS of (4.5) are zero. Hence, when $n \le k-1$, (4.5) becomes

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = -x \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix},$$

and when $n \ge k$, (4.5) is reduced to

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = \mathbf{0}_n.$$

In summary, (4.5) is equivalent to

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} F_1^{(k,n)}(x) \\ F_2^{(k,n)}(x) \\ \vdots \\ F_n^{(k,n)}(x) \end{pmatrix} = -x 1_{\{n<k\}} \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix}.$$

Similarly we have equations for G_i^k,

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} G_1^{(k,n)}(x) \\ G_2^{(k,n)}(x) \\ \vdots \\ G_n^{(k,n)}(x) \end{pmatrix} = -1_{\{n<k\}} \begin{pmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{pmatrix}.$$

For the following calculation, it is worth of noting that a special solution appears only when $n \le k-1$, especially $(k,n,i) = (2,1,1)$ if $S = 2$.

We can obtain explicit representations for the case of $S = 2$ by applying the same technique as discussed in the previous section.

Let us change notations as $-q_{11} = q_{12} = q_1 \ge 0$, $-q_{22} = q_{21} = q_2 \ge 0$ and define the following quantities,

$$p_1 = \frac{q_1}{r - \mu_1 + q_1}, \quad p_2 = \frac{q_1}{r + q_1}, \quad l_1 = -g_2(\beta_1^{(2)})/q_2, \quad l_2 = -g_2(\beta_2^{(2)})/q_2,$$
$$d_1 = q_2\left[-\left(-\beta_2^{(1)} + \beta_1^{(2)}\right) l_1 - \left(\beta_2^{(1)} - \beta_2^{(2)}\right) l_2\right], \quad d_2 = q_2\left[-\left(\beta_1^{(1)} - \beta_1^{(2)}\right) l_1 + \left(\beta_1^{(1)} - \beta_2^{(2)}\right) l_2\right].$$

The following proposition shows the results in case of $q_1 q_2 \ne 0$. One can obtain the results in case of $q_1 q_2 = 0$.

Proposition 2 *Suppose that $q_1 q_2 \neq 0$. Then $F_i^{(k,n)}, G_i^{(k,n)}$ are given as follows.*

$$
\begin{aligned}
F_1^{(1,0)}(x) &= x, \quad F_1^{(1,1)}(x) = x_1 \frac{d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
F_1^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[g_2\left(\beta_1^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2\left(\beta_2^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_2^{(1,0)}(x) &= 0, \quad F_2^{(1,1)}(x) = 0, \\
F_2^{(1,2)}(x) &= x_1 \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[-q_2\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
F_1^{(2,0)}(x) &= 0, \quad F_1^{(2,1)}(x) = B_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + B_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_1 x, \\
F_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right) B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
F_2^{(2,0)}(x) &= x, \quad F_2^{(2,1)}(x) = x, \quad F_2^{(2,2)}(x) = -q_2 B_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 B_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},
\end{aligned}
$$

and

$$
\begin{aligned}
G_1^{(1,0)}(x) &= 1, \quad G_1^{(1,1)}(x) = \frac{d_1\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}}, \\
G_1^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[g_2\left(\beta_1^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} - g_2\left(\beta_2^{(2)}\right)\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
G_2^{(1,0)}(x) &= 0, \quad G_2^{(1,1)}(x) = 0, \\
G_2^{(1,2)}(x) &= \frac{\beta_1^{(1)} - \beta_2^{(1)}}{d_1\left(\frac{x_1}{x_2}\right)^{\beta_1^{(1)}} + d_2\left(\frac{x_1}{x_2}\right)^{\beta_2^{(1)}}} \left[-q_2\left(\frac{x}{x_2}\right)^{\beta_1^{(1)}} + q_2\left(\frac{x}{x_2}\right)^{\beta_2^{(1)}} \right], \\
G_1^{(2,0)}(x) &= 0, \quad G_1^{(2,1)}(x) = b_1\left(\frac{x}{x_1}\right)^{\beta_1^{(1)}} + b_2\left(\frac{x}{x_1}\right)^{\beta_2^{(1)}} + p_2, \\
G_1^{(2,2)}(x) &= g_2\left(\beta_1^{(2)}\right) b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} + g_2\left(\beta_2^{(2)}\right) b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}}, \\
G_2^{(2,0)}(x) &= 1, \quad G_2^{(2,1)}(x) = 1, \quad G_2^{(2,2)}(x) = -q_2 b_3\left(\frac{x}{x_2}\right)^{\beta_1^{(2)}} - q_2 b_4\left(\frac{x}{x_2}\right)^{\beta_2^{(2)}},
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= -p_1x_1\frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}+q_2x_2\frac{l_1l_2\left(\beta_1^{(2)}-\beta_2^{(2)}\right)-p_1\left(l_1\left(\beta_1^{(2)}-1\right)-l_2\left(\beta_2^{(2)}-1\right)\right)}{d_1\lambda^{\beta_2^{(1)}}+d_2\lambda^{\beta_1^{(1)}}},\\
B_3 &= \frac{-p_1x_1\left(\beta_1^{(1)}-\beta_2^{(1)}\right)+l_2x_2\left(\left(\beta_2^{(1)}-\beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}}-\left(\beta_1^{(1)}-\beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}\\
&\quad -p_1x_1\frac{\left(\beta_2^{(1)}-1\right)\lambda^{-\beta_1^{(1)}}-\left(\beta_1^{(1)}-1\right)\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}\\
B_2 &= -p_1x_1-B_1,\quad B_4=-\frac{x_2}{q_2}-B_3,\\
b_1 &= -p_2\frac{d_1\lambda^{-\beta_1^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}+q_2\frac{l_1l_2\left(\beta_1^{(2)}-\beta_2^{(2)}\right)-p_2\left(l_1\beta_1^{(2)}-l_2\beta_2^{(2)}\right)}{d_1\lambda^{\beta_2^{(1)}}+d_2\lambda^{\beta_1^{(1)}}}\\
b_3 &= \frac{-p_2\left(\beta_1^{(1)}-\beta_2^{(1)}\right)+l_2\left(\left(\beta_2^{(1)}-\beta_2^{(2)}\right)\lambda^{-\beta_1^{(1)}}-\left(\beta_1^{(1)}-\beta_2^{(2)}\right)\lambda^{-\beta_2^{(1)}}\right)}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}-p_2\frac{\beta_2^{(1)}\lambda^{-\beta_1^{(1)}}-\beta_1^{(1)}\lambda^{-\beta_2^{(1)}}}{d_1\lambda^{-\beta_1^{(1)}}+d_2\lambda^{-\beta_2^{(1)}}}\\
b_2 &= -p_2-b_1,\quad b_4=-\frac{1}{q_2}-b_3.
\end{aligned}
$$

Note that Proposition 2 is valid for any $x_2 \le x_1$. We don't impose "smooth pasting conditions" of F_i^k, G_i^k at the boundary of the continuation region when solving the Dirichlet problem since "smooth pasting conditions at the boundary" are meaningless in the problem. It follows that, for arbitrary given x_1, x_2, a function constructed with these obtained F_i^k, G_i^k

$$W_i^{(n)}(x)=\sum_{k\in E}\left(\alpha_kD_kF_i^{(k,n)}(x)-K_kG_i^{(k,n)}(x)\right)$$

does not necessary satisfy the smooth pasting condition $\frac{d}{dx}W_n^{(n)}(x)=\alpha_nD_n$ at the boundary x_n of the continuation region. The optimality condition involved in $\lambda = x_1/x_2$ plays an important role in veryifying (4.1) and the smooth pasting conditions at the boundaries.

Lemma 3 *Suppose that* $q_2 \neq 0$ *and*

$$r>\max(\mu_1,\mu_2,0),\quad (r-\mu_1+q_1)(r-\mu_2+q_2)-q_1q_2\neq 0,\quad (r+q_1)(r+q_2)-q_1q_2\neq 0. \quad (4.6)$$

Then,

(1) Assumption 1 and 2 hold.

(2) There exist threshholds $0 < x_2 \le x_1$ *if and only if there exists unique solution* λ *satisfying* $0 < \lambda \le 1$ *and*

$$\frac{-d_{11}\lambda^{\beta_1^{(1)}}+k_Kd_{21}}{c_{11}\lambda^{\beta_1^{(1)}-1}-k_Kk_\alpha c_{21}}=\frac{-d_{12}\lambda^{\beta_2^{(1)}}+k_Kd_{22}}{c_{12}\lambda^{\beta_2^{(1)}-1}-k_Kk_\alpha c_{22}}>0, \quad (4.7)$$

where

$$\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = (1 - k_K p_1 k_\alpha)\begin{pmatrix} 1-\beta_2^{(1)} \\ \beta_1^{(1)} - 1 \end{pmatrix}, \quad \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix} = (1 - k_K p_2)\begin{pmatrix} \beta_2^{(1)} \\ -\beta_1^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = -\begin{pmatrix} \beta_2^{(1)} & -1 \\ -\beta_1^{(1)} & 1 \end{pmatrix}\left[\frac{1}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)} \end{pmatrix}\begin{pmatrix} 1-\beta_2^{(2)} \\ \beta_1^{(2)} - 1 \end{pmatrix} - p_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right],$$

$$\begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix} = -\begin{pmatrix} \beta_2^{(1)} & -1 \\ -\beta_1^{(1)} & 1 \end{pmatrix}\left[\frac{1}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} l_1 & l_2 \\ l_1\beta_1^{(2)} & l_2\beta_2^{(2)} \end{pmatrix}\begin{pmatrix} \beta_2^{(2)} \\ -\beta_1^{(2)} \end{pmatrix} + p_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right].$$

When the above condition (4.7) is satisfied, the solution (x_1, x_2) *is given by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\widetilde{\alpha}_1\lambda}\frac{-d_{11}\lambda^{\beta_1^{(1)}} + k_K d_{21}}{c_{11}\lambda^{\beta_1^{(1)}-1} - k_K k_\alpha c_{21}}\begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \tag{4.8}$$

The following lemma shows that λ is given by a solution of equations.

Lemma 4 *λ must satisfy the following vector equation,*

$$\begin{aligned} &\frac{\left(1-\beta_2^{(1)}\right)(\alpha_1 D_1 - p_1\alpha_2 D_2)x_1 + \beta_2^{(1)}(K_1 - p_2K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_1^{(1)}}\begin{pmatrix} 1 \\ \beta_1^{(1)} \end{pmatrix} \\ &-\frac{\left(1-\beta_1^{(1)}\right)(\alpha_1 D_1 - p_1\alpha_2 D_2)x_1 + \beta_1^{(1)}(K_1 - p_2K_2)}{\beta_1^{(1)} - \beta_2^{(1)}}\lambda^{\beta_2^{(1)}}\begin{pmatrix} 1 \\ \beta_2^{(1)} \end{pmatrix} \\ = {} & l_1\frac{\left(1-\beta_2^{(2)}\right)\alpha_2 D_2 x_2 + \beta_2^{(2)}K_2}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} 1 \\ \beta_1^{(2)} \end{pmatrix} - l_2\frac{\left(1-\beta_1^{(2)}\right)\alpha_2 D_2 x_2 + \beta_1^{(2)}K_2}{\beta_1^{(2)} - \beta_2^{(2)}}\begin{pmatrix} 1 \\ \beta_2^{(2)} \end{pmatrix} \\ &-p_1\alpha_2 D_2 x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + p_2K_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

By making use of Lemma 3 and 4, we take the optimality condition into consideration so that we obtain the following result which is seemingly trivial but non-trivial.

Proposition 3 *Suppose that $q_1q_2 \neq 0$ and x_1, x_2 are given by (4.8). Then it holds that*

$$V_i^{(n)}(x) = W_i^{(n)}(x), \quad n = 0, 1, 2,\ i = 1, 2,$$

where $V_i^{(n)}$ are given in Proposition 1.

5 Concluding remarks

Our technique with linear algebra provides an insight of the functional form of the value function and it would help us in analysis of the value function more explicitly under many regime states. Especially, the results on the first passage time are remarkable for futher research. Numerical calculation is relatively easy due to the expression with eigenvalues and eigenvectors.

As remarks on parameters, we made explicit assumptions on $r, \mathbf{M}, \mathbf{\Sigma}$ and $\mathbf{Q}$ in order to obtain appropriate eigenvalues $\beta_i^{(n)}$ and convergence of income flow multiplier α_i. However, conditions on $\mathbf{D}$ and $\mathbf{K}$ are implicitly involved in calculation of thresholds x_n satisfying the order of $x_S \leq x_{S-1} \leq \cdots \leq x_2 \leq x_1$. Our assumptions assure the distinct eigenvalues though they can be relaxed. The cases of duplicate eigenvalues and an entry-exit problem are left for future research.

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