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A Note on Risk Measure Theory from a Category-Theoretic Point of View

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Abstract

We reformulate dynamic monetary value measures with the language of category theory. We show some axioms in the old setting are deduced as theorems in the new formulation, which may be one of the evidences that the *axioms* are natural. We also demonstrate a topology-as-axioms paradigm in order to give a theoretical criteria with which we can pick up appropriate sets of axioms required for monetary value measures to be *good*.

1 Introduction

The risk measure theory we are formulating is a theory of dynamic (multi-period) monetary risk measures. Since the axiomatization of monetary risk measures was initiated by [ADEH99], many axioms such as law invariance have been presented ([Kus01], [FS11]). Especially after introducing multi-period (or dynamic) versions of monetary risk measures, a lot of investigations have been made so far [ADE+07]. Those investigations are valuable in both theoretical and practical senses. However, it may be expected to have some theoretical criteria of picking appropriate sets of axioms out of them. Thinking about the recent events such as the CDS hedging failure at JP Morgan Chase, the importance of selecting appropriate axioms of monetary risk measures becomes even bigger than before. In this note, we formalize dynamic monetary risk measures in the language of category theory in order to add a new view point to the risk measure theory.

Category theory is an area of study in mathematics that examines in an abstract way the properties of maps (called *morphisms* or *arrows*) satisfying some basic conditions. It has been applied in many fields including geometry, logic, computer science and string theory. Even for measure theory, there are some attempts to apply category theory such as [Jac06] or [Bre77]. However, in finance theory, as far as we know, there has been nothing. We will use it for formulating dynamic monetary risk measures.

In this note, we will stress two points. One is how we can formulate some concepts of dynamic risk measure theory in the language of category theory and show some *axioms* in the old setting become *theorems* in our setting. Another point is to present a criteria of selecting sets of axioms required for monetary value measure theory in a sheaf-theoretic point of view.

The remainder of this paper consists of four sections.

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In Section 2, we provide brief reviews about dynamic risk measure theory and category theory.

In Section 3, we give a definition of monetary value measures as contravariant functors from a set of σ-fields as a poset. Then, we will see the resulting monetary value measures satisfy time consistency condition and dynamic programming principle that were introduced as axioms in the old version of dynamic risk measure theory.

In Section 4, we will investigate a possibility of finding an appropriate Grothendieck topology for which monetary value measures satisfying given axioms become sheaves. We also introduce the notion of complete set of axioms with which we give a method to construct a monetary value measure satisfying the axiom from any given monetary value measure.

In Section 5, we investigate the situation of monetary value measures in a quite simple case $\Omega = \{1,2,3\}$, and show that any set of axioms over Ω that accepts concave monetary value measures is not complete.

2 Review of Dynamic Risk Measures and Category Theories

In this section we give a very brief review of dynamic risk measure theory and category theory. Throughout this note, all discussions are under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.1 Dynamic Risk Measure Theory

First, we review the case of one period monetary risk measures.

Definition 2.1. A *one period monetary risk measure* is a function $\rho : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfying the following axioms

- *Cash invariance:* $(\forall X)(\forall a \in \mathbb{R})\ \rho(X + a) = \rho(X) - a$,
- *Monotonicity:* $(\forall X)(\forall Y)\ X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$,
- *Normalization:* $\rho(0) = 0$,

where $L^p(\Omega, \mathcal{F}, \mathbb{P})$ is the space of equivalence classes of $\mathbb{R}$-valued random variables which are bounded by the $\|\cdot\|_p$ norm.

Here are examples of one period risk measures.

Example 2.2. [One Period Monetary Risk Measures]

1. Value at Risk
$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \leq \alpha\}$$

2. Expected shortfall
$$\mathrm{ES}_\alpha(X) := \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)du$$

Now, we will define the notion of dynamic monetary risk measures. However, we actually adopt the way of using a *monetary value measure* φ instead of using a *monetary risk measure* ρ below by conforming the manner in recent literature such as [ADE+07] and [KM07], where we have a relation $\varphi(X) = -\rho(X)$ for any possible scenario (i.e. a random variable) X.

From now on, we think a *monetary value measure* φ instead of a monetary risk measure ρ defined by $\varphi(X) := -\rho(X)$.

Definition 2.3. For a σ-field $\mathcal{U} \subset \mathcal{F}$, $L(\mathcal{U}) := L^\infty(\Omega, \mathcal{U}, \mathbb{P}|\mathcal{U})$, is the space of all equivalence classes of bounded $\mathbb{R}$-valued random variables, equipped with the usual sup norm.

Definition 2.4. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration. A *dynamic monetary value measure* is a collection of functions $\varphi = \{\varphi_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)\}_{t\in[0,T]}$ satisfying

- *Cash invariance:* $(\forall X \in L(\mathcal{F}_T))(\forall Z \in L(\mathcal{F}_t))\ \varphi_t(X+Z) = \varphi_t(X) + Z$,
- *Monotonicity:* $(\forall X \in L(\mathcal{F}_T))(\forall X \in L(\mathcal{F}_T))\ X \leq Y \Rightarrow \varphi_t(X) \leq \varphi_t(Y)$,
- *Normalization:* $\varphi_t(0) = 0$.

Note that the directions of some inequalities in Definition 2.1 are different from those of Definition 2.4 because we now monetary value measures instead of monetary risk measures.

Since dynamic monetary value measures treat multi-period situations, we may require some extra axioms to regulate them toward the time dimension. Here are two possible such axioms.

Axiom 2.5. [*Dynamic programming principle*] For $0 \leq s \leq t \leq T$, $(\forall X \in L(\mathcal{F}_T))\ \varphi_s(X) = \varphi_s(\varphi_t(X))$.

Axiom 2.6. [*Time consistency*] For $0 \leq s \leq t \leq T$, $(\forall X, \forall Y \in L(\mathcal{F}_T))\ \varphi_t(X) \leq \varphi_t(Y) \Rightarrow \varphi_s(X) \leq \varphi_s(Y)$.

2.2 Category Theory

The description about category theory presented in this subsection is very limited. For those who are interested in more detail about category theory, please consult [Mac97].

Definition 2.7. [Categories] A *category* $\mathcal{C}$ consists of a collection $\mathcal{O}_\mathcal{C}$ of *objects* and a collection $\mathcal{M}_\mathcal{C}$ of *arrows* or *morphisms* such that

1. there are two functions $\mathcal{M}_\mathcal{C} \overset{dom}{\underset{cod}{\rightrightarrows}} \mathcal{O}_\mathcal{C}$.

   When $dom(f) = A$ and $cod(f) = B$, we write $f : A \to B$.

   We define a so-called *hom-set* of given objects A and B by $\mathrm{Hom}_\mathcal{C}(A,B) := \{f \in \mathcal{M}_\mathcal{C} \mid f : A \to B\}$. We sometimes write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_\mathcal{C}(A,B)$.

2. for $f : A \to B$ and $g : B \to C$, there is an arrow $g \circ f : A \to C$, called the *composition* of g and f.

3. every object A is associated with an *identity arrow* $1_A : A \to A$ satisfying $f \circ 1_A = f$ and $1_A \circ g = g$
   where $dom(f) = A$ and $cod(g) = A$.

Example 2.8. [Examples of Categories]

1. **Set** : the category of small sets
   - $\mathcal{O}_{\mathbf{Set}}$:= collection of all small sets,
   - $\mathcal{M}_{\mathbf{Set}}$:= collection of all functions between small sets.
2. **Top** : the category of topological spaces
   - $\mathcal{O}_{\mathbf{Top}}$:= collection of all topological spaces,
   - $\mathcal{M}_{\mathbf{Top}}$:= collection of all continuous functions between topological spaces.
3. *Opposite category* $\mathcal{C}^{op}$
   Let $\mathcal{C}$ be a given category. Then we define its opposite category $\mathcal{C}^{op}$ by the following way.
   - $\mathcal{O}_{\mathcal{C}^{op}} := \mathcal{O}_{\mathcal{C}}$,
   - for $A, B \in \mathcal{O}_{\mathcal{C}}$, $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Example 2.9. [Partial Ordered Sets as Categories]
A partial ordered set (sometimes we call it *poset*) $(S, \leq)$ can be considered as a category defined in the following way.

- $\mathcal{O}_S := S$,
- for $a, b \in S$, $\mathrm{Hom}_S(a, b) := \begin{cases} \{i_b^a\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$

We see the correspondence between definitions of posets and categories below.

1. Reflexivity vs. identity arrows: $a \leq a$

$$a \xrightarrow{1_a = i_a^a} a$$

2. Transitivity vs. composition arrows: $a \leq b$ and $b \leq c \implies a \leq c$

$$\begin{array}{ccc} a & \xrightarrow{i_b^a} & b \\ & {\scriptstyle i_c^a = i_c^b \circ i_b^a} \searrow & \downarrow {\scriptstyle i_c^b} \\ & & c \end{array}$$

Definition 2.10. [Functors] Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of two functions,
$F_{\mathcal{O}} : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$ and $F_{\mathcal{M}} : \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{D}}$ satisfying

1. $f : A \to B \implies F(f) : F(A) \to F(B)$,
2. $F(g \circ f) = F(g) \circ F(f)$,
3. $F(1_A) = 1_{F(A)}$.

Definition 2.11. [Contravariant functors] A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is called a *contravariant functor*. if two conditions 1 and 2 in Definition 2.10 are replaced by

1. $f : A \to B \implies F(f) : F(B) \to F(A)$,
2. $F(g \circ f) = F(f) \circ F(g)$.

Example 2.12. [Contravariant Functor]

$$\mathcal{C}^{op} \xrightarrow{Hom_{\mathcal{C}}(-,C)} \mathbf{Set}$$

$$\begin{array}{ccccc} A & \quad & \mathrm{Hom}_{\mathcal{C}}(A, C) & \ni & g \circ f \\ f \big\downarrow & & \big\uparrow \mathrm{Hom}_{\mathcal{C}}(f,C) & & \big\uparrow \\ B & & \mathrm{Hom}_{\mathcal{C}}(B, C) & \ni & g \end{array}$$

Definition 2.13. [Natural Transformations] Let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be two functors. A *natural transformation* $\alpha : F \dot{\to} G$ consists of a family of arrows $\langle \alpha_C | C \in \mathcal{O}_{\mathcal{C}} \rangle$ making the following diagram commute:

$$\begin{array}{ccccc} C_1 & \quad & F(C_1) & \xrightarrow{\alpha_{C_1}} & G(C_1) \\ f \big\downarrow & & F(f) \big\downarrow & & \big\downarrow G(f) \\ C_2 & & F(C_2) & \xrightarrow[\alpha_{C_2}]{} & G(C_2) \end{array}$$

Definition 2.14. [Functor Categories] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category such that

- $\mathcal{O}_{\mathcal{D}^{\mathcal{C}}}$:= collection of all functors from $\mathcal{C}$ to $\mathcal{D}$,
- $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$:= collection of all natural transformations from F to G.

3 Monetary Value Measures

Now we start defining monetary value measures with the language of category theory. First, we introduce a simple category that is actually a partially ordered set derived by the σ-field $\mathcal{F}$.

Definition 3.1. [Category χ]

1. Let $\chi := \chi(\mathcal{F})$ be the set of all sub-σ-fields of $\mathcal{F}$. Then, it becomes a poset with the set-inclusion relation $\subset$. Moreover, as shown in Example 2.9, χ becomes a category whose hom set $\mathrm{Hom}_\chi(\mathcal{V},\mathcal{U})$ for $\mathcal{U},\mathcal{V} \in \chi$ is defined by

$$\mathrm{Hom}_\chi(\mathcal{V},\mathcal{U}) := \begin{cases} \{i^{\mathcal{V}}_{\mathcal{U}}\} & \text{if } \mathcal{V} \subset \mathcal{U}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.1}$$

   The arrow $i^{\mathcal{V}}_{\mathcal{U}}$ is called an *inclusion map*.

2. $\bot := \{\Omega, \emptyset\}$, which is the least element of χ.

All the discussions below depend on this particular category χ. But, you may notice in many cases that we can replace χ with more restricted collections of σ-fields such as the full subcategory χ_c of χ whose objects are $\mathbb{P}$-complete, or a totally ordered subset of χ which is considered as a (strictly increasing) filtration.

We restrict the space of random variables bounded in the norm $\|\cdot\|_\infty$, which is actually necessary when showing the local property in Proposition 3.5. But, in many places, you can relax it to $\|\cdot\|_p$ with an arbitrary $p (p \geq 1)$ instead of ∞.

Definition 3.2. [Monetary Value Measures] A *monetary value measure* is a contravariant functor

$$\varphi : \chi^{op} \to \mathbf{Set}$$

satisfying the following two conditions:

1. for $\mathcal{U} \in \chi$, $\varphi(\mathcal{U}) := L(\mathcal{U})$,
2. for $\mathcal{U},\mathcal{V} \in \chi$ such that $\mathcal{V} \subset \mathcal{U}$, the map $\varphi^{\mathcal{V}}_{\mathcal{U}} := \varphi(i^{\mathcal{V}}_{\mathcal{U}}) : L(\mathcal{U}) \to L(\mathcal{V})$ satisfies
   - *Cash invariance:* $(\forall X \in L(\mathcal{U}))(\forall Z \in L(\mathcal{V}))\ \varphi^{\mathcal{V}}_{\mathcal{U}}(X + Z) = \varphi^{\mathcal{V}}_{\mathcal{U}}(X) + Z$,
   - *Monotonicity:* $(\forall X \in L(\mathcal{U}))(\forall Y \in L(\mathcal{U}))\ X \leq Y \Rightarrow \varphi^{\mathcal{V}}_{\mathcal{U}}(X) \leq \varphi^{\mathcal{V}}_{\mathcal{U}}(Y)$,
   - *Normalization:* $\varphi^{\mathcal{V}}_{\mathcal{U}}(0) = 0$.

At this point, we do not require the monetary value measures to satisfy some familiar conditions such as concavity or law invariance. Instead of doing so, we want to see what kind of properties are deduced from this minimal setting.

One of the key points of Definition 3.2 is that φ is a contravariant functor. So, for any triple of σ-fields $\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}$ in χ, we have, as seeing in Diagram 3.1,

$$\varphi^{\mathcal{U}}_{\mathcal{U}} = 1_{L(\mathcal{U})} \text{ and } \varphi^{\mathcal{W}}_{\mathcal{V}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}} = \varphi^{\mathcal{W}}_{\mathcal{U}}. \tag{3.2}$$

Example 3.3. [Entropic Value Measure] Let λ be a non-zero real number. Then the functor $\varphi : \chi^{op} \to \mathbf{Set}$ defined by

$$\varphi^{\mathcal{V}}_{\mathcal{U}}(X) := \lambda^{-1} \log \mathbb{E}^{\mathbb{P}}[e^{\lambda X} \mid \mathcal{V}] \tag{3.3}$$

where $\mathcal{V} \subset \mathcal{U}$ in χ and $X \in L(\mathcal{U})$ is a monetary value measure.

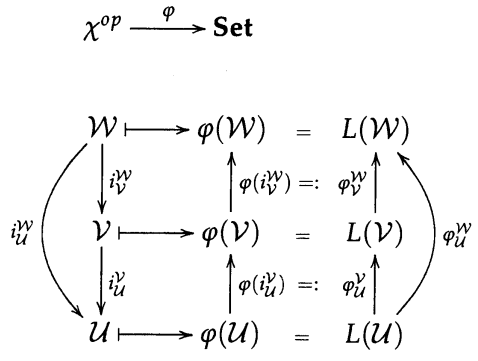


Diagram 3.1

Definition 3.4. [Concave Monetary Value Measure] A monetary value measure φ is said to be *concave* if for any $\mathcal{V} \subset \mathcal{U}$ in χ, $X, Y \in L(\mathcal{U})$ and $\lambda \in [0,1]$,

$$\varphi_{\mathcal{U}}^{\mathcal{V}}(\lambda X + (1-\lambda)Y) \geq \lambda \varphi_{\mathcal{U}}^{\mathcal{V}}(X) + (1-\lambda)\varphi_{\mathcal{U}}^{\mathcal{V}}(Y). \tag{3.4}$$

An entropic value measure is concave.

Here are some properties of monetary value measures.

Proposition 3.5. *Let $\varphi : \chi^{op} \to$ **Set** be a monetary value measure, and $\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}$ be σ-fields in χ.*

1. $(\forall X \in L(\mathcal{V}))\ \varphi_{\mathcal{U}}^{\mathcal{V}}(X) = X,$
2. *Idempotentness:* $(\forall X \in L(\mathcal{U}))\ \varphi_{\mathcal{U}}^{\mathcal{V}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) = \varphi_{\mathcal{U}}^{\mathcal{V}}(X),$
3. *Local property:* $(\forall X \in L(\mathcal{U}))(\forall Y \in L(\mathcal{U}))(\forall A \in \mathcal{V})\ \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) = \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) + \mathbb{1}_{A^c}\varphi_{\mathcal{U}}^{\mathcal{V}}(Y),$
4. *Dynamic programming principle:* $(\forall X \in L(\mathcal{U}))\ \varphi_{\mathcal{U}}^{\mathcal{W}}(X) = \varphi_{\mathcal{U}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)),$
5. *Time consistency:* $(\forall X \in L(\mathcal{U}))(\forall Y \in L(\mathcal{U}))\ \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{V}}(Y) \Rightarrow \varphi_{\mathcal{U}}^{\mathcal{W}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{W}}(Y).$

Proof. 1. By cash invariance and normalization, $\varphi_{\mathcal{U}}^{\mathcal{V}}(X) = \varphi_{\mathcal{U}}^{\mathcal{V}}(0+X) = \varphi_{\mathcal{U}}^{\mathcal{V}}(0) + X = X.$

2. Since $\varphi_{\mathcal{U}}^{\mathcal{V}}(X) \in L(\mathcal{V})$, it is obvious by 1.

3. First, we show that for any $A \in \mathcal{V}$,

$$\mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) = \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X). \tag{3.5}$$

Since $X \in L^\infty(\Omega, \mathcal{U}, \mathbb{P})$, we have $|X| \leq \|X\|_\infty$. Therefore,

$$\mathbb{1}_A X - \mathbb{1}_{A^c}\|X\|_\infty \leq \mathbb{1}_A X + \mathbb{1}_{A^c} X \leq \mathbb{1}_A X + \mathbb{1}_{A^c}\|X\|_\infty.$$

Then, by cash invariance and monotonicity,

$$\begin{aligned}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) - \mathbb{1}_{A^c}\|X\|_\infty &= \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X - \mathbb{1}_{A^c}\|X\|_\infty) \\ &\leq \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \\ &\leq \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c}\|X\|_\infty) = \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c}\|X\|_\infty.\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) &= \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) - \mathbb{1}_{A^c}\|X\|_\infty) \\ &\leq \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \\ &\leq \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c}\|X\|_\infty) = \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X).\end{aligned}$$

Therefore, we get (3.5).

Next by using (3.5) twice, we have

$$\begin{aligned}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) &= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) \\ &= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A(\mathbb{1}_A X + \mathbb{1}_{A^c} Y)) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y)) \\ &= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c} Y) \\ &= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(Y).\end{aligned}$$

4. By 2 and (3.2), we have

$$\varphi_{\mathcal{U}}^{\mathcal{W}}(X) = \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) = \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X))) = (\varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}})(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) = \varphi_{\mathcal{U}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)).$$

5. Assume $\varphi_{\mathcal{U}}^{\mathcal{V}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{V}}(Y)$. Then, by monotonicity and (3.2),

$$\varphi_{\mathcal{U}}^{\mathcal{W}}(X) = \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) \leq \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(Y)) = \varphi_{\mathcal{U}}^{\mathcal{W}}(Y).$$

□

In Proposition 3.5, two properties, dynamic programming principle and time consistency are usually introduced as axioms ([DS06]). But, we derive them naturally here from the fact that the monetary value measure is a contravariant functor. This may be seen as another evidence that the two *axioms* are quite natural.

4 Monetary Value Measures as Sheaves

In general, a contravariant functor $\rho : \mathcal{C}^{op} \to \mathbf{Set}$ is called a *presheaf* for a category $\mathcal{C}$. By definition, a monetary value measure is a presheaf. The name *presheaf* suggests that it is related to another concept *sheaves*, which is a quite important concept in some classical branches in mathematics such as algebraic topology. [MM92]. So, what makes a presheaf be a sheaf?

For a given set, a topology defined on it provides a criteria to distinguish good (= continuous) functions from given functions on the set. In a similar way, there is a concept called

set	category	χ
topology	Grothendieck topology	
function	presheaf	value measure
continuous function	sheaf	value measure *satisfying axioms*
weakest topology	largest Grothendieck topology	

Figure 4.1: topology-as-axioms paradigm

a *Grothendieck topology* defined on a given category that gives a criteria to distinguish good presheaves (= sheaves) from given presheaves on the category. In both cases, a (Grothendieck) topology can be seen as a vehicle to identify good functions (presheaves) among general functions (presheaves).

On the other hand, if we have a set of functions that we want to make *good* (= continuous), we can find the weakest topology that makes the functions continuous. In a similar way, if we have a set of presheaves that we want to make *good*, it is known that we can pick a Grothendieck topology with which the presheaves become sheaves. See Figure 4.1 for the analogy.

Since a monetary value measure is a presheaf, if we have a set of *good* monetary value measures (= the monetary value measures that satisfy a given set of axioms), we may find a Grothendieck topology with which the monetary value measures become sheaves. We will see a concrete shape of the Grothendieck topology in Section 4.1.

Now suppose we have a weak topology that makes given functions continuous. This, however, does not imply the fact that any continuous function w.r.t. the topology is contained in the originally given functions. Similarly, Suppose that we have a Grothendieck topology that makes all monetary value measures satisfying a given set of axioms sheaves. It, however, does not mean that any sheaf w.r.t. the Grothendieck topology satisfies the given set of axioms. We will investigate this situation in Section 4.2.

4.1 A Grothendieck Topology as Axioms

In this subsection, we see a concrete shape of the Grothendieck topology with which all monetary value measures satisfying a given set of axioms become sheaves.

First, we review two concepts of Grothendieck typologies and sheaves.

Definition 4.1. Let $\mathcal{U} \in \chi$.

1. $\downarrow\mathcal{U} := \{\mathcal{V} \in \chi \mid \mathcal{V} \subset \mathcal{U}\}$.
2. A *sieve* on $\mathcal{U}$ is a set $I \subset\downarrow\mathcal{U}$ such that $(\forall\mathcal{V} \in\downarrow\mathcal{U})(\forall\mathcal{W} \in\downarrow\mathcal{U})\left[\mathcal{W} \subset \mathcal{V} \in I \Rightarrow \mathcal{W} \in I\right]$.
3. For a sieve I on $\mathcal{U}$ and $\mathcal{V} \subset \mathcal{U}$ in χ, $I \downarrow \mathcal{V} := I \cap \downarrow \mathcal{V}$.
4. A *family* of I is an element $X \in \prod_{\mathcal{V}\in I} L(\mathcal{V})$. We write $X = (X_{\mathcal{V}})_{\mathcal{V}\in I}$.

5. A family $X = (X_\mathcal{V})_{\mathcal{V}\in I}$ is called a $\mathbb{P}$-*martingale* if $(\forall \mathcal{V} \in I)(\forall \mathcal{W} \in I)\big[\mathcal{W} \subset \mathcal{V} \Rightarrow \mathbb{E}^{\mathbb{P}}[X_\mathcal{V} \mid \mathcal{W}] = X_\mathcal{W}\big]$.

A sieve on $\mathcal{U}$ is considered as a kind of a *time domain*. We sometimes call a family a *subprocess*.

Note that $I \downarrow \mathcal{V}$ is a sieve on $\mathcal{V}$.

Definition 4.2. 1. $\Xi : \chi^{op} \to \mathbf{Set}$ is a contravariant functor such that for $i^{\mathcal{V}}_{\mathcal{U}} : \mathcal{V} \to \mathcal{U}$ in χ, $\Xi(\mathcal{U})$ is the set of all sieves on $\mathcal{U}$, and that $\Xi(i^{\mathcal{V}}_{\mathcal{U}})(I) = I \downarrow \mathcal{V}$ for $I \in \Xi(\mathcal{U})$.

2. A *Grothendieck topology* on χ is a subfunctor $J \rightarrowtail \Xi$ satisfying the following conditions:

   (a) $(\forall \mathcal{U} \in \chi) \downarrow \mathcal{U} \in J(\mathcal{U})$,

   (b) $(\forall \mathcal{U} \in \chi)(\forall I \in J(\mathcal{U}))(\forall K \in \Xi(\mathcal{U}))\big[(\forall \mathcal{V} \in I) K \downarrow \mathcal{V} \in J(\mathcal{V}) \Rightarrow K \in J(\mathcal{U})\big]$.

   We say a sieve I J-*covers* $\mathcal{U}$ if $I \in J(\mathcal{U})$.

$\mathcal{U}$ is considered as a time *horizon* of a time domain I if it is covered by I.

Here is a well-known property of Grothendieck topologies.

Theorem 4.3. *Let $\{J_a \mid a \in A\}$ be a collection of Grothendieck topologies on χ. Then the subfunctor $J \rightarrowtail \Xi$ defined by $J(\mathcal{U}) := \bigcap_{a\in A} J_a(\mathcal{U})$ is a Grothendieck topology. We write this J by $\bigcap_{a\in A} J_a$.*

Next we introduce concepts of families depending on a monetary value measure.

Definition 4.4. Let $\varphi \in \mathbf{Set}^{\chi^{op}}$ be a monetary value measure, and I be a sieve on $\mathcal{U} \in \chi$.

1. A family $X = (X_\mathcal{V})_{\mathcal{V}\in I}$ is called φ-*matching* if $(\forall \mathcal{V} \in I)(\forall \mathcal{W} \in I)\ \varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{V}}(X_\mathcal{V}) = \varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{W}}(X_\mathcal{W})$.

2. A random variable $\bar{X} \in L(\mathcal{U})$ is called a φ-*amalgamation* for a family $X = (X_\mathcal{V})_{\mathcal{V}\in I}$ if $(\forall \mathcal{V} \in I)\varphi^{\mathcal{V}}_{\mathcal{U}}(\bar{X}) = X_\mathcal{V}$.

The next two propositions give us some intuition about the relation between two concepts just introduced, φ-matching and φ-amalgamation.

Proposition 4.5. *Let $\varphi \in \mathbf{Set}^{\chi^{op}}$ be a monetary value measure, I be a sieve on $\mathcal{U} \in \chi$ and $X = (X_\mathcal{V})_{\mathcal{V}\in I}$ be a family that has a φ-amalgamation. Then, X is φ-matching.*

Proof. Let $\bar{X} \in L(\mathcal{U})$ be a φ-amalgamation. Then, for any $\mathcal{V} \in I$, $X_\mathcal{V} = \varphi^{\mathcal{V}}_{\mathcal{U}}(\bar{X})$. Therefore, for any $\mathcal{V}, \mathcal{W} \in I$, $\varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{V}}(X_\mathcal{V}) = X_{\mathcal{V}\wedge\mathcal{W}} = \varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{W}}(X_\mathcal{W})$.

□

Proposition 4.6. *Let $\varphi \in \mathbf{Set}^{\chi^{op}}$ be a monetary value measure, I be a sieve on $\mathcal{U} \in \chi$ and $X = (X_\mathcal{V})_{\mathcal{V}\in I}$ be a φ-matching family.*

1. *For $\mathcal{W}, \mathcal{V} \in I$, if $\mathcal{W} \subset \mathcal{V}$, we have $\varphi^{\mathcal{W}}_{\mathcal{V}}(X_\mathcal{V}) = X_\mathcal{W}$.*

2. *If $\mathcal{U} \in I$, $X_\mathcal{U}$ is the unique φ-amalgamation for X.*

Proof. 1. $\varphi^{\mathcal{W}}_{\mathcal{V}}(X_\mathcal{V}) = \varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{V}}(X_\mathcal{V}) = \varphi^{\mathcal{V}\wedge\mathcal{W}}_{\mathcal{W}}(X_\mathcal{W}) = \varphi^{\mathcal{W}}_{\mathcal{W}}(X_\mathcal{W}) = X_\mathcal{W}$.

2. By 1, $X_{\mathcal{U}}$ is a φ-amalgamation for X.

   Now let $\bar{X} \in L(U)$ be another φ-amalgamation for X. Then for every $\mathcal{V} \in I$, $X_{\mathcal{V}} = \varphi^{\mathcal{V}}_{\mathcal{U}}(\bar{X})$. Put $\mathcal{V} := \mathcal{U}$. Then, we have $X_{\mathcal{U}} = \varphi^{\mathcal{U}}_{\mathcal{U}}(\bar{X}) = 1_{\mathcal{U}}(\bar{X}) = \bar{X}$.

□

Now we are at the position where we can introduce the concept of sheaves.

Definition 4.7. Let J be a Grothendieck topology on χ. A monetary value measure $\varphi \in \mathbf{Set}^{\chi^{op}}$ is called a *sheaf* (for J) if for any $\mathcal{U} \in \chi$, any J-covering sieve $I \in J(\mathcal{U})$ and any φ-matching family $X = (X_{\mathcal{V}})_{\mathcal{V} \in I}$, X has a unique φ-amalgamation.

In the rest of this subsection, we will try to find a Grothendieck topology for which a given class of monetary value measures specified by a given set of (extra) axioms are sheaves.

Let us consider a sieve I on $\mathcal{U} \in \chi$ as a subfunctor $I \rightarrowtail \mathrm{Hom}_{\chi}(-,\mathcal{U})$, that is, a contravariant functor $I : \chi^{op} \to \mathbf{Set}$ defined by

$$I(\mathcal{V}) := \begin{cases} \{i^{\mathcal{V}}_{\mathcal{U}}\} & \text{if } \mathcal{V} \in I, \\ \emptyset & \text{if } \mathcal{V} \notin I. \end{cases} \tag{4.1}$$

for $\mathcal{V} \in \chi$.

Actually, by this convention, we can identify a φ-matching subprocess X on a sieve I with a natural transformation $X : I \dot{\to} \varphi$.

The following theorem assures the existence of a Grothendieck topology making a given monetary value measure a sheaf.

Proposition 4.8. *Let $\varphi \in \mathbf{Set}^{\chi^{op}}$ be a monetary value measure, and define a subfunctor $J_\varphi \rightarrowtail \Xi$ by*

$$J_\varphi(\mathcal{U}) := \left\{ I \in \Xi(\mathcal{U}) \mid (\forall \mathcal{V} \in \downarrow \mathcal{U}) \begin{array}{ccc} I \downarrow \mathcal{V} & \rightarrowtail & \downarrow \mathcal{V} \\ {\scriptstyle \forall X} \downarrow & \swarrow {\scriptstyle \exists ! Y} & \\ \varphi & & \end{array} \right\} \tag{4.2}$$

for $\mathcal{U} \in \chi$. Then, the subfunctor J_φ is the largest Grothendieck topology for which φ is a sheaf.

Proof. Refer Example 3.2.14c in [Bor94]. □

By combining Proposition 4.8 and Theorem 4.3, we have the following corollary.

Corollary 4.9. *Let $\mathcal{M} \subset \mathbf{Set}^{\chi^{op}}$ be the collection of all monetary value measures satisfying a given set of axioms. Then, there exists a Grothendieck topology for which all monetary value measures in $\mathcal{M}$ are sheaves, where the topology is largest among topologies representing the axioms. We write the topology by $J_{\mathcal{M}}$.*

Proof. Let $J_{\mathcal{M}} := \bigcap_{\varphi \in \mathcal{M}} J_\varphi$. Then, it is the largest Grothendieck topology for which every monetary value measure in $\mathcal{M}$ is a sheaf. □

4.2 Complete sets of Axioms

Let $\mathcal{A}$ be a *fixed* set of axioms. Then, for a given arbitrary monetary value measure φ, can we make a *good* alternative for it? In other words, can we find a monetary value measure that satisfies $\mathcal{A}$ and is the best approximation of the original φ? This is the theme of this subsection.

For a Grothendieck topology J on χ, define $Sh(\chi, J) \subset \mathbf{Set}^{\chi^{op}}$ to be a full subcategory whose objects are all sheaves for J. Then, it is well-known that there exists a *left adjoint* π_J in the following diagram.

$$\begin{array}{ccc} Sh(\chi, J) & \underset{\pi_J}{\overset{\rightarrowtail}{\longleftarrow}} & \mathbf{Set}^{\chi^{op}} \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \pi_J(\varphi) & \longleftarrow\!\!\shortmid & \varphi \end{array} \tag{4.3}$$

The functor π_J is well-known with the name *sheafification* functor, which comes with the following limit cone:

$$\begin{array}{ccccc} \cdots \longrightarrow Nat(I, \varphi) & & \xrightarrow{\quad Nat(i_I^K, \varphi) \quad} & & Nat(K, \varphi) \longrightarrow \cdots \\ & \searrow^{s_I^\varphi} & & {}^{s_K^\varphi}\swarrow & \\ \pi_J(\varphi)(\mathcal{U}) & := & \mathrm{colim}_{I \in J(\mathcal{U})}\, Nat(I, \varphi) & & \end{array} \tag{4.4}$$

for sieves I, K on $\mathcal{U}$. It also satisfies the following theorem.

Theorem 4.10. 1. *$\pi_J(\varphi)$ is a sheaf for J.*

2. *If φ is a sheaf for J, then for any $\mathcal{U} \in \chi$, $\pi_J(\varphi)(\mathcal{U}) \simeq L(\mathcal{U})$.*

Theorem 4.10 suggests that for an arbitrary monetary value measure, the sheafification functor provides one of its closest monetary value measures that *may* satisfy the given set of axioms. To make this certain, we need a following definition.

Definition 4.11. Let $\mathcal{A}$ be a set of axioms for monetary value measures.

1. $\mathcal{M}(\mathcal{A}) :=$ the collection of all monetary value measures satisfying $\mathcal{A}$.

2. $\mathcal{M}_0 :=$ the collection of all monetary value measures.

3. $\mathcal{A}$ is called *complete* if

$$\pi_{J_{\mathcal{M}(\mathcal{A})}}(\mathcal{M}_0) \subset \mathcal{M}(\mathcal{A}). \tag{4.5}$$

By Theorem 4.10, we have the following main result.

Theorem 4.12. *Let $\mathcal{A}$ be a complete set of axioms. Then, for a monetary value measure $\varphi \in \mathcal{M}_0$, $\pi_{J_{\mathcal{M}(\mathcal{A})}(\varphi)}$ is the monetary value measure that is the best approximation satisfying axioms $\mathcal{A}$.*

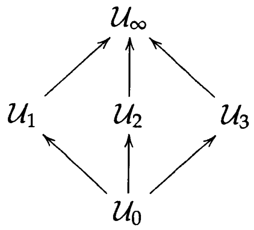
Now, we want to expect that some of the well-known sets of axioms such as those for concave monetary value measures are complete. However, we will see a counterexample in Section 5 in a quite simple case.

5 Completeness Condition on $\Omega = \{1,2,3\}$

In this section, we investigate if the set of axioms of concave monetary value measures is complete in the case $\Omega = \{1,2,3\}$ with a σ-field $\mathcal{F} := 2^{\Omega}$.

5.1 Some Consideration on the Shape of Monetary Value Measures on Ω

First, we enumerate all possible sub-σ-fields of Ω, that is, the shape of the category $\chi = \chi(\Omega)$ which is like following:



(5.1)

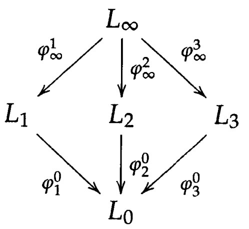
where

$$
\begin{aligned}
\mathcal{U}_\infty &:= \mathcal{F} := 2^{\Omega}, \\
\mathcal{U}_1 &:= \{\emptyset, \{1\}, \{2,3\}, \Omega\}, \\
\mathcal{U}_2 &:= \{\emptyset, \{2\}, \{1,3\}, \Omega\}, \\
\mathcal{U}_3 &:= \{\emptyset, \{3\}, \{1,2\}, \Omega\}, \\
\mathcal{U}_0 &:= \{\emptyset, \Omega\}.
\end{aligned} \tag{5.2}
$$

The Banach spaces derived by the elements of χ are:

$$
\begin{aligned}
L_\infty &:= L := L(\mathcal{U}_\infty) = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}, \\
L_1 &:= L(\mathcal{U}_1) = \{(a,b,b) \mid a,b \in \mathbb{R}\}, \\
L_2 &:= L(\mathcal{U}_2) = \{(a,b,a) \mid a,b \in \mathbb{R}\}, \\
L_3 &:= L(\mathcal{U}_3) = \{(a,a,c) \mid a,c \in \mathbb{R}\}, \\
L_0 &:= L(\mathcal{U}_0) = \{(a,a,a) \mid a \in \mathbb{R}\}.
\end{aligned} \tag{5.3}
$$

Then, a monetary value measure $\varphi : \chi^{op} \to \mathbf{Set}$ on χ is determined by the following six functions:



(5.4)

We will investigate its concrete shape one by one by considering axioms it satisfies.

For $\varphi_\infty^1 : L_\infty \to L_1$, we have by the cash invariance axiom,

$$\begin{aligned}\varphi_\infty^1(a,b,c) &= \varphi_\infty^1((0,b-c,0)+(a,c,c))\\ &= \varphi_\infty^1((0,b-c,0))+(a,c,c)\\ &= (f_{12}(b-c), f_{11}(b-c), f_{11}(b-c))+(a,c,c)\\ &= (f_{12}(b-c)+a, f_{11}(b-c)+c, f_{11}(b-c)+c)\end{aligned}$$

where $f_{11}, f_{12} : \mathbb{R} \to \mathbb{R}$ are defined by $(f_{12}(x), f_{11}(x), f_{11}(x)) = \varphi_\infty^1(0,x,0)$. Similarly, if we define nine functions $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32}, g_1, g_2, g_3 : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}(f_{12}(x), f_{11}(x), f_{11}(x)) &= \varphi_\infty^1(0,x,0),\\ (f_{21}(x), f_{22}(x), f_{21}(x)) &= \varphi_\infty^2(0,0,x),\\ (f_{31}(x), f_{31}(x), f_{32}(x)) &= \varphi_\infty^3(x,0,0),\\ (g_1(x), g_1(x), g_1(x)) &= \varphi_1^0(x,0,0),\\ (g_2(x), g_2(x), g_2(x)) &= \varphi_2^0(0,x,0),\\ (g_3(x), g_3(x), g_3(x)) &= \varphi_3^0(0,0,x).\end{aligned} \tag{5.5}$$

We can represent the original six functions in (5.4) by the nine functions defined in (5.5).

$$\begin{aligned}\varphi_\infty^1(a,b,c) &=(f_{12}(b-c)+a, f_{11}(b-c)+c, f_{11}(b-c)+c),\\ \varphi_\infty^2(a,b,c) &=(f_{21}(c-a)+a, f_{22}(c-a)+b, f_{21}(c-a)+a),\\ \varphi_\infty^3(a,b,c) &=(f_{31}(a-b)+b, f_{31}(a-b)+b, f_{32}(a-b)+c),\\ \varphi_1^0(a,b,b) &=(g_1(a-b)+b, g_1(a-b)+b, g_1(a-b)+b),\\ \varphi_2^0(a,b,a) &=(g_2(b-a)+a, g_2(b-a)+a, g_2(b-a)+a),\\ \varphi_3^0(a,a,c) &=(g_3(c-a)+a, g_3(c-a)+a, g_3(c-a)+a).\end{aligned} \tag{5.6}$$

Next by the normalization axiom, we have

$$f_{11}(0) = f_{12}(0) = f_{21}(0) = f_{22}(0) = f_{31}(0) = f_{32}(0) = g_1(0) = g_2(0) = g_3(0) = 0. \tag{5.7}$$

Now suppose that we can partially differentiate the function $\varphi_\infty^1(a,b,c)$ in all three arguments. Then, we have

$$\begin{aligned}\frac{\partial}{\partial a}\varphi_\infty^1(a,b,c) &= (1,0,0),\\ \frac{\partial}{\partial b}\varphi_\infty^1(a,b,c) &= (f_{12}'(b-c), f_{11}'(b-c), f_{11}'(b-c)),\\ \frac{\partial}{\partial c}\varphi_\infty^1(a,b,c) &= (-f_{12}'(b-c), 1-f_{11}'(b-c), 1-f_{11}'(b-c)).\end{aligned}$$

Therefore, by the monotonicity, we have $f_{12}'(x) = 0$ and $0 \le f_{11}'(x) \le 1$. Then by (5.7), we have for all $x \in \mathbb{R}$, $f_{12}(x) = 0$. Hence, for all $x \in \mathbb{R}$,

$$f_{12}(x) = f_{22}(x) = f_{32}(x) = 0. \tag{5.8}$$

With this knowledge, let us redefine the three functions $f_1, f_2, f_3, : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}(0, f_1(x), f_1(x)) &= \varphi^1_\infty(0, x, 0),\\ (f_2(x), 0, f_2(x)) &= \varphi^2_\infty(0, 0, x),\\ (f_3(x), f_3(x), 0) &= \varphi^3_\infty(x, 0, 0).\end{aligned} \tag{5.9}$$

Then, we have a new representation of the original six functions in (5.4):

$$\begin{aligned}\varphi^1_\infty(a,b,c) =&(a, f_1(b-c)+c, f_1(b-c)+c),\\ \varphi^2_\infty(a,b,c) =&(f_2(c-a)+a, b, f_2(c-a)+a),\\ \varphi^3_\infty(a,b,c) =&(f_3(a-b)+b, f_3(a-b)+b, c),\\ \varphi^0_1(a,b,b) =&(g_1(a-b)+b, g_1(a-b)+b, g_1(a-b)+b),\\ \varphi^0_2(a,b,a) =&(g_2(b-a)+a, g_2(b-a)+a, g_2(b-a)+a),\\ \varphi^0_3(a,a,c) =&(g_3(c-a)+a, g_3(c-a)+a, g_3(c-a)+a).\end{aligned} \tag{5.10}$$

Thinking about the composition rule $\varphi^0_\infty = \varphi^0_1 \circ \varphi^1_\infty = \varphi^0_2 \circ \varphi^2_\infty = \varphi^0_3 \circ \varphi^3_\infty$, we have

$$\begin{aligned}&g_1(a - f_1(b-c) - c) + f_1(b-c) + c\\ =&g_2(b - f_2(c-a) - a) + f_2(c-a) + a\\ =&g_3(c - f_3(a-b) - b) + f_3(a-b) + b.\end{aligned} \tag{5.11}$$

5.2 Grothendieck Topologies on χ

Any Grothendieck topology on χ we are discussing in the following has at least one sheaf for it. Therefore, we can assume any sieve I on $\mathcal{U}$ satisfies $\bigvee I = \mathcal{U}$.

Proposition 5.1. *Let J be a Grothendieck topology on χ. Then,*

$$J(\mathcal{U}_k) = \{\downarrow \mathcal{U}_k\} \tag{5.12}$$

for $k = 0, 1, 2$ or 3.

So, we only discuss about $J(\mathcal{U}_\infty)$ below.

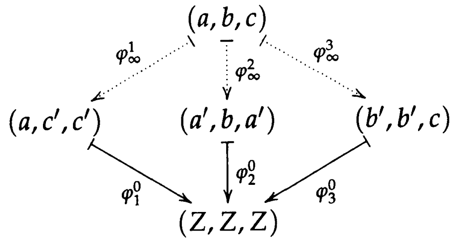
For $k = 0, 1, 2, 3, \infty$, define sieves I_k on $\mathcal{U}_k$ by $I_k :=\downarrow \mathcal{U}_k$. Followings are all possible sieves on $\mathcal{U}_\infty$. $I_{12} := I_1 \cup I_2,$, $I_{13} := I_1 \cup I_3,$, $I_{23} := I_2 \cup I_3$, and $I_{123} := I_1 \cup I_2 \cup I_3$..

Now, we define two Grothendieck Topologies J_0 and J_1.

- J_0 is defined by $J_0(\mathcal{U}_k) = \{I_k\}$ for $k = 0, 1, 2, 3$ or ∞.
- J_1 is defined by $J_1(\mathcal{U}_k) = \{I_k\}$ for $k = 0, 1, 2$ or 3 and $J_1(\mathcal{U}_\infty) = \{I_\infty, I_{123}\}$.

We can easily show that any Grothendieck topology on χ that has at least one sheaf on χ other than J_0 contains J_1. In other words, J_1 is the smallest Grothendieck topology on χ next to J_0.

The following diagram shows the unique extension from I_{123} to I_∞.



(5.13)

So, we have a necessary and sufficient condition for a monetary value measure to be a J_1-sheaf.

Proposition 5.2. *φ becomes a sheaf for J_1 iff for all $a, a', b, b', c, c' \in \mathbb{R}$,*

$$\begin{aligned} &g_1(a-c')+c' = g_2(b-a')+a' = g_3(c-b')+b' \\ &\Rightarrow\ (c' = f_1(b-c)+c)\ \wedge\ (a' = f_2(c-a)+a)\ \wedge\ (b' = f_3(a-b)+b). \end{aligned} \tag{5.14}$$

5.3 Entropic Value Measures on χ

Let $\mathbb{P}$ be a probability measure on Ω defined by $\mathbb{P} = (p_1, p_2, p_3)$ and φ be an entropic value measure defined by

$$\varphi^{\mathcal{V}}_{\mathcal{U}}(X) := \frac{1}{\lambda}\log \mathbb{E}^{\mathbb{P}}[e^{\lambda X} \mid \mathcal{V}]. \tag{5.15}$$

Then the function φ^1_∞ in (5.4) is

$$\begin{aligned} \varphi^1_\infty(a,b,c) &= \frac{1}{\lambda}\log \mathbb{E}^{\mathbb{P}}[(e^{\lambda a}, e^{\lambda b}, e^{\lambda c}) \mid \mathcal{U}_1] \\ &= \Big(a, \frac{1}{\lambda}\log\frac{p_2 e^{\lambda b}+p_3 e^{\lambda c}}{p_2+p_3}, \frac{1}{\lambda}\log\frac{p_2 e^{\lambda b}+p_3 e^{\lambda c}}{p_2+p_3}\Big). \end{aligned} \tag{5.16}$$

Therefore, the corresponding six functions defined in (5.5) and (5.9) are

$$\begin{aligned} f_1(x) &= \frac{1}{\lambda}\log\frac{p_2 e^{\lambda x}+p_3}{p_2+p_3}, \\ f_2(x) &= \frac{1}{\lambda}\log\frac{p_3 e^{\lambda x}+p_1}{p_3+p_1}, \\ f_3(x) &= \frac{1}{\lambda}\log\frac{p_1 e^{\lambda x}+p_2}{p_1+p_2}, \\ g_1(x) &= \frac{1}{\lambda}\log(p_1 e^{\lambda x}+p_2+p_3), \\ g_2(x) &= \frac{1}{\lambda}\log(p_1+p_2 e^{\lambda x}+p_3), \\ g_3(x) &= \frac{1}{\lambda}\log(p_1+p_2+p_3 e^{\lambda x}). \end{aligned}$$

So, the question is if the entropic value measure is a J_1-sheaf. By Proposition 5.2, its necessary and sufficient condition becomes like the following:

$$\begin{aligned} &p_1 e^{\lambda a}+(1-p_1)e^{\lambda c'} = p_2 e^{\lambda b}+(1-p_2)e^{\lambda a'} = p_3 e^{\lambda c}+(1-p_3)e^{\lambda b'} =: Z \\ &\Rightarrow\ Z = p_1 e^{\lambda a}+p_2 e^{\lambda b}+p_3 e^{\lambda c}. \end{aligned}$$

However, this does not hold in general.

Theorem 5.3. *φ is not a J_1-sheaf.*

Corollary 5.4. *Any set of axioms over $\Omega = \{1,2,3\}$ that accepts concave monetary value measures is not complete.*

6 Conclusion

We specified a concept of monetary value measures through the language of category theory. It is defined as an appropriate class of presheaves over a set of σ-fields as a poset. The resulting monetary value measures satisfy naturally so-called time consistency condition as well as dynamic programming principle.

Next, we showed a concrete shape of the largest Grothendieck topology for which monetary value measures satisfying given axioms become sheaves. By using sheafification functors, for any monetary value measure, we constructed its best approximation of the monetary value measure that satisfies given axioms in case the axioms are complete.

As a list of future's investigation, we will try to formulate a robust representation of concave monetary value measures in a category-theoretic language. We also seek the possibility to represent each individual axiom of monetary value measures as a specific Grothendieck topology which may give us an insight about different aspects of the axioms of monetary value measures. We investigate the *completeness* condition of sets of axioms for more realistic Ω in order to make sheafification functors work better.

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