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Entropy via partitions of unity

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1 Introduction

In this paper, we take up the notions of *"finite partion of unity"*. The first is a finite partion of 1 in real numbers. The second is it in the ergodic theory. The third is those in operator algebras, two kinds of "finite partion of unity". One is finite partions of unity *via positive operator*. The other is *finite operationsl partion* of unity.

We state relations between "mutual orthogonality for two subalgebras A and B" of the $n \times n$ complex matrix algebra and the entropy for "finite partion of unity" induced from unitary operators which are related to the pair $\{A, B\}$. Generally speaking, isomorphic two subalgebras are mutually orthogonal if and only if the related entropy takes the maximum value, and the value is the logarithm of the dimension of the subalgebra.

This relation reminde us that the area of a rhombus takes the maximal value if and only if two sides intersect at right angles, and the maximal value is the square of the length of the side.

Furthermore, by applying the notion of *finite operationsl partion* of unity to unital completely positive (called the UCP for short) maps, we extend the notion of von Neumann entropy for states to that for UCP maps, and show computations of entropy for UCP maps (in special Cuntz's canonical endomorphisms).

In this paper we denote by $M_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices, and by Tr_n the standard trace, that is, the sum of all diagonal components. A matrix $D \in M_n(\mathbb{C})$ is called a *density matrix* if D is a positive operator with $\mathrm{Tr}_n(D) = 1$(cf. [11] [12]).

The notation η is called the entropy function in usual, and it is the function defined by

$$\eta(t) = \begin{cases} -t \log t, & (0 < t \leq 1) \\ 0, & t = 0 \end{cases}$$

2 Several kinds of "finite partitions of unity"

Here, we discuss on several kinds of notions which we may call a finite partition of unity, and we apply the entropy function η to those finite partition of unity.

2.1 Entropy for finite partitions of 1

The first one is discuused in the real numbers $\mathbb{R}$. Let

$$\lambda = \{\lambda_1, \cdots, \lambda_n\}$$

be the set of real numbers $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$. We say that the set λ is a *finite partition of* 1. Let

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n), \tag{2.1}$$

and we say that $H(\lambda)$ is the *entropy* for the finite partiton λ of 1.

2.2 Entropy for finite measurable partitions

Let us remember the definition of the entropy for finite measurable partitions in the ergodic theory. The notation $H(\lambda)$ is discussed under the following setting: Let (X, μ) be a Lebesque space. Let

$$\xi = \{X_1, \cdots, X_n\}$$

be a finite measurable partition of X. Then ξ induces the finite partition of 1 as follows:

$$\lambda_\mu(\xi) = \{\mu(X_1), \cdots, \mu(X_n)\}. \tag{2.2}$$

The *entropy* $H(\xi)$ for the finite partition ξ of X is nothing else but $H(\lambda_\mu(\xi))$, that is,

$$H(\xi) = \eta(\mu(X_1)) + \cdots + \eta(\mu(X_n)).$$

2.3 Finite partitions via projections

We can discuss the notion of finite measurable partitions in the abelian algebra $L^\infty(X,\mu)$. Let χ_i be the characteristic function of X_i. Then $\chi = \{\chi_1, \cdots, \chi_n\}$ is a partition of unity in $L^\infty(X,\mu)$:

$$\chi_1 + \cdots + \chi_n = 1_{L^\infty(X,\mu)}$$

Let π be the multiplicative representation of $L^\infty(X,\mu)$ on the Hilbert space $H = L^2(X,\mu)$. We denote by M the von Neumann algebra $\pi(L^\infty(X,\mu))$, and by p_i the projection $\pi(\chi_i)$ for $i = 1, \cdots, n$. Then $p_1 + \cdots + p_n = 1_M$. Let

$$p = \{p_1, \cdots, p_n\}.$$

Thus the set p is a *finite partition via projections* of unity 1 in M, and its *entropy* $H_\mu(p)$ with respect to the measure μ is considered as

$$H_\mu(p) = \eta(\mu(p_1)) + \cdots + \eta(\mu(p_n)). \tag{2.3}$$

2.4 Finite partitions via positive operators

Connes-Stømer([8]) extended the notion of "finite partition via projections of unity" in the abelian von Neumann algebra $\pi(L^\infty(X,\mu))$ to the case of a finite von Neumann algebra M.

Let τ be a tracial state of M. A set $x = \{x_1, \cdots, x_n\} \subset M$ is said to be a *finite partition of unity in M* if each $x_i \in M$ is a positive operator for $i = 1, \cdots, n$ and $\sum_{i-1}^n x_i = 1_M$. The entropy for x (finite partition of unity in M) with respect to τ is given by

$$H_\tau(x) = \eta(\tau(x_1)) + \cdots + \eta(\tau(x_n)). \tag{2.4}$$

2.5 Finite operational partition of unity

The terminology, a *finite operational partition of unity,* was first given by Lindblad ([10]) and after then it is used by Alicki-Fannes([1]).

Let A be a unital C^*-algebra. Let $x = \{x_1, ..., x_k\} \subset A$. Then x is said to be a *finite operational partition of unity* of size k if

$$\sum_i^k x_i^* x_i = 1_A.$$

Let φ be a state of A, and let $x = \{x_1, ..., x_k\}$ be an operational partition of unity in A. We denote by $\rho_\varphi[x]$ the $k \times k$ matrix whose $\{i, j\}$ component $\rho_\varphi[x](i, j)$ is given by

$$\rho_\varphi[x](i,j) = \varphi(x_j^* x_i), \quad (i, j = 1, \cdots, k).$$

Then $\rho_\varphi[x]$ is a positive operator in $M_k(\mathbb{C})$ and $\mathrm{Tr}_k(\rho_\varphi[x]) = 1$. We call $\rho_\varphi[x]$ the *density matrix associate with x and φ*.

Let $\lambda(\rho_\varphi[x]) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ be the eigenvalues of the density matrix $\rho_\varphi[x]$. Then $\lambda(\rho_\varphi[x])$ is a finite partition of 1 because $\rho_\varphi[x]$ is a positive operator in $M_k(\mathbb{C})$ and $\mathrm{Tr}_k(\rho_\varphi[x]) = 1$. Hence we have the entropy $H(\lambda(\rho_\varphi[x]))$ in (3.1).

Let $S(\rho_\varphi[x])$ be the von Neumann entropy ([11, 12]) for the density operator $\rho_\varphi[x]$. Then $S(\rho_\varphi[x])$ is nothing else but $H(\lambda(\rho_\varphi[x]))$, that is,

$$S(\rho_\varphi[x]) = \mathrm{Tr}_k(\eta(\rho[x])) = H(\lambda(\rho_\varphi[x])) = \sum_i \eta(\lambda_i). \tag{2.5}$$

2.6 Unitary matrix and patition of 1

Let $u = ((u(i,j))_{ij}$ be an $n \times n$ unitary matrix. Then

$$\sum_{i=1}^n |u(i,j)|^2 = 1 = \sum_{j=1}^n |u(i,j)|^2, \quad \text{for all } i, j.$$

Let

$$\lambda(u) = \{\frac{|u(1,1)|^2}{n}, \frac{|u(1,2)|^2}{n}, \cdots, \frac{|u(n,n)|^2}{n}\}.$$

Then $\lambda(u)$ is a finite partition of 1, and we call $\lambda(u)$ the *finite partition of* 1 *induced from u*. The entropy $H(\lambda(u))$ of $\lambda(u)$ satisfies the following equality by (3.1):

$$H(\lambda(u)) = \frac{1}{n} \sum_{i,j} \eta(|u(i,j)|^2) + \log n. \tag{2.6}$$

2.7 Entropy for unistochastic matrices

A matrix $b \in M_n(\mathbb{C})$ is said to be *bistochastic* if

$$b(i,j) \geq 0, \quad \text{and} \quad \sum_{i=1}^n b(i,j) = \sum_{j=1}^n b(i,j) = 1, \ (\forall i, j).$$

Życzkowski - Kuś - Słomczyński - Sommers defined in [17] the entropy $H(b)$ for a bistochastic matrix b by

$$H(b) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\eta(b(i,j)).$$

We pick up a bistochastic matrix which is called a *unistochastic* matrix. Let us consider a unitary matrix $u \in M_n(\mathbb{C})$, and let

$$(b(u))(i,j) = |u(i,j)|^2 \quad \forall i,j.$$

The $b(u)$ is a bistochastic matrix which is called unistochastic matrix induced from u.

We have the following relation between two kinds of entropy $H(\lambda(u))$ and $H(b(u))$:

$$H(\lambda(u)) = H(b(u)) + \log n. \tag{2.7}$$

where $H(\lambda(u))$ is the entropy for the finite partition $\lambda(u)$ of 1 induced from u and $H(b(u))$ is the entropy for the unistochastic matrix induced from u.

We show in the next section the meaning of the value $H(b(u))$ from the view point of the theory of operator algebras.

3 Characterization of Orthogonality

In this section, we give a characterization for the mutual orthogonality of two subalgebras by the maximal value of related entropy to the pair of subalgebras.

3.1 Mutually orthogonal subalgebras

Let M be a finite von Neumann algebra and let τ be a fixed normal faithful tracial state. If $M = M_n(\mathbb{C})$, then $\tau(x) = \mathrm{Tr_n}(x)/n$. The inner product $< x, y >$ for $x, y \in M$ is given by $< x, y >= \tau(y^*x)$. Let A and B be von Neumann subalgebras of M such that $1_M \in A$ and $1_M \in B$. Then A and B are said to be *mutually orthogonal* by Popa([15]) if

$$\tau(a) = 0 = \tau(b) \Longrightarrow \tau(ab) = 0 \tag{3.1}$$

for all $a \in A$ and $b \in B$. This means that

$$(A \ominus \mathbb{C}1_A) \perp (B \ominus \mathbb{C}1_B)$$

with respect to the above inner product.

We remark that in some papers (for example [14]) mutual orthogonality for subalgebras is called *complementarity*.

3.2 Conditional relative entropy $h(A|B)$

Let M be a finite von Neumann algebra, and let A and B be two von Neumann subalgebras of M. Let τ be a tracial state of M. Then there exists the conditional expectations $E_A : M \to A$ and $E_B : M \to B$ conditiones by τ. We modify the relative entropy $H(A \mid B)$ of A and B due to Connes and Stømer([8]) as follows:

Definition 3.2. ([2]). Let

$$h(A \mid B) = \sup_{(x_i)} \sum_i (\tau\eta E_B(x_i) - \tau\eta x_i) \tag{3.2}$$

where (x_i) is a finite partition of unity in A, that is, (x_i) is a finite set of positive operators contained in A and $1_A = \sum_i x_i$.

We remark that in the definition of $H(A \mid B)$, Connes and Stømer([8]) take (x_i) as a finite partition of unity in M.

3.3 Mutual orthogonality is maximality of entropy

We have the following relation between the entroppy $h(D_n(\mathbb{C})|uD_n(\mathbb{C})u^*)$ and the entroppy $H(b(u))$:

Theorem 3.3. ([2]). *Let $D = D_n(\mathbb{C})$ be the algebra of the diagonal matrices in $M_n(\mathbb{C})$, and let $u \in M_n(\mathbb{C})$ be a unitary matrix. Then*

$$h(D \mid uDu^*) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \eta(|u(i,j)|^2) = H(b(u)). \tag{3.3}$$

so that

$$\{h(D \mid uDu^*) : u \in M_n(\mathbb{C}) \ \textit{unitary}\ \} = [0, \log n].$$

Remark 3.3. We remark that the above relations do not hold in the case of Connes and Størmer relative entropy $H(A \mid B)$. See for example [14].

Corollary 3.3. ([2]). *The following conditions are equivalent:*

1. *D and uDu^* are mutually orthogonal,*
2. *$h(D \mid uDu^*) = \log n$,*
3. *$|u(i,j)| = 1/\sqrt{n} \quad ,\forall i,j$.*

Let A and B be maximal abelian subalgebras (called MASA for short) in $M_n(\mathbb{C})$. Then there exists a unitary $u(A,B) \in M_n(\mathbb{C})$ and $D_n(\mathbb{C})$ is the typical MASA of $M_n(\mathbb{C})$ so that we have the following:

Conclusion 3.3. ([2]). Let A_0 and B_0 be maximal abelian subalgebras in $M_n(\mathbb{C})$. Then following conditions are equivalent:

1. A_0 and B_0 are mutually orthogonal,
2. $h(A_0|B_0) = \max\{h(A|B) : A, B \subset M_n(\mathbb{C}),\ \text{MASA}\}$,
3. $H(b(A_0, B_0)) = \max\{H(b(A,B)) : A, B \subset M_n(\mathbb{C}),\ \text{MASA}\}$
4. $h(A_0|B_0) = H(b(A_0, B_0)) = \log n$

Extended version to II_1 factor. In the paper [3], the above equivalent three relations $\{1, 2, 4\}$ were extended in a connection to Jones index theory, to some kinds of subfactors in II_1 factors, where the notion of mutual orthogonality is replaced by the "commuting square condition".

3.4 Unitary and finite operational partition

Let L be a finite von Neumann algebra (the most simple case $L = M_k(\mathbb{C})$ for some integer k), and let τ_L be a fixed normal faithful tracial state of L. Let M be the tensor product $M_n(\mathbb{C}) \otimes L$, and let τ_M be the tracial state of M which is given as the tensor product $\mathrm{Tr_n}/n \otimes \tau_L$. Let $\{e_{ij}; i,j = 1, \cdots, n\}$

be a system of a matrix units of $M_n(\mathbb{C})$. Then each $u \in M$ is written as the unique form

$$u = \sum_{i,j=1}^{n} e_{ij} \otimes u_{ij}, \quad (u_{ij} \in L),$$

and the u is unitary if and only if

$$\sum_{j=1}^{n} u_{ij} u_{kj}^* = \delta_{ik} 1_L, \quad \sum_{i=1}^{n} u_{ij}^* u_{ik} = \delta_{jk} 1_L.$$

Assume that $u \in M_n(\mathbb{C}) \otimes L$ is unitary. Let

$$U = \{\frac{1}{\sqrt{n}} u_{11}, \frac{1}{\sqrt{n}} u_{12}, \cdots, \frac{1}{\sqrt{n}} u_{nn}\}.$$

Then U is a finite operational partition of unity in the von Neumann algebra L of size n^2.

Definition 3.4. ([4]). Let u be a uunitary in $M_n(\mathbb{C}) \otimes L$. By putting

$$\rho[U](ij, kl) = \tau(u_{kl}^* u_{ij}), \quad i, j, k, l = 1, \cdots, n^2, \tag{3.4}$$

we define a $n^2 \times n^2$ matrix $\rho[U]$ which associates with the unitary u. Here, $\rho[U](ij, kl)$ means the double indexed (ij, kl) component of the matrix $\rho[U]$.

It is obvious that $\rho[U]$ is a positive operator in $M_{n^2}(\mathbb{C})$ which satisfies that $\mathrm{Tr}_{n^2}(\rho[U]) = 1$, so that $\rho[U]$ is a density matrix. Hence we have the von Neumann entropy $S(\rho[U])$ of the density operator $\rho[U]$..

Remark 3.4. We remark that the following holds in general:

$$0 \leq S(\rho[U]) \leq 2 \log n.$$

Theorem 3.4. ([4]). *Let L be a finite von Neumann algebra and let τ_L be a normalized trace of L. We let $M = M_n(\mathbb{C}) \otimes L$ and $\tau = \mathrm{Tr}/n \otimes \tau_L$. Assume that $N = M_n(\mathbb{C}) \otimes 1_L$ and that u is a unitary operator in M.*

Then the following conditions are equivalent:

1. *N and uNu^* are mutually orthogonal;*

2. *$n^2 \rho[U]$ is the $n^2 \times n^2$ identity matrix;*

3. $S(\rho[U]) = 2\log n = \log\dim N$.

Here U is the finite operational partition of unity induced by u.

Below, we show some operator algebraic role of $S(\rho[U])$.

3.5 Subfactors of matrix algebras

Let A and B be subalgebras of $M_k(\mathbb{C})$ and assume that both subalgebras A and B are isomorphic to $M_n(\mathbb{C})$. Then $k = nm$, and we can assume that $M_k(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and $A = M_n(\mathbb{C}) \otimes \mathbb{C}1$. By the assumption, there exists a unitary matrix $u \in M_k(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ such that $B = uAu^*$. We denote by $u(A, B)$ this unitary. Now, by letting L be $M_m(\mathbb{C})$, we apply the discussion in the section 4.4 to $u(A, B)$. Then we have the density matrix $\rho[U(A, B)]$ induced by $u(A, B)$. We denote by $\mathfrak{F}_{k,n}$ the set of subalgebras of $M_k(\mathbb{C})$ which are isomorphic to $M_n(\mathbb{C})$.

Conclusion 3.5. ([5]). Let A_0 and B_0 be subalgebras which are contained in $\mathfrak{F}_{k,n}$. Then following conditions are equivalent:

1. A_0 and B_0 are mutually orthogonal,
2. $S(\rho[U(A_0, B_0)]) = \max\{S(\rho[U(A, B)]) : A, B \in \mathfrak{F}_{k,n}\}$,

The maximal value is $2\log n$ which is the logarithm of the dimension of the subalgebra A_0.

4 Entropy for UCP maps

Let A and B be unital C^*-algebras and let $\Phi : A \to B$ be a linear map. If $\Phi \otimes \mathrm{id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$ is positive for all positive integer n, then Φ is said to be completely positive. If $\Phi(1_A) = 1_B$, then Φ is said to be unital. We call a unital completely positive map UCP map, and develope our entropy for UCP maps via finite operational partition of unity. ([5])

4.1 Finite operational partition and UCP map

Let A be a unital C^*-algebra, and let $v = \{v_1, \cdots, v_n\} \subset A$ be a finite operational partitions of unity. Let

$$\Phi(x) = \sum_{i=1}^{n} v_i^* x v_i, \quad x \in A \tag{4.1}$$

Since v is a finite operational partitions of unity (i.e. $\sum_{i=1}^{n} v_i^* v_i = 1$), it follows that Φ is a UCP map of A. We call this Φ the *UCP map associated with the finite operational partitions of unity* v. In the theory of C^*-algebras, we have an interesting example of finite operational partitions of unity.

4.2 Cuntz relation and finite operational partitions

Let $\{S_1, S_2, \cdots, S_n\}$ be n isometries on some Hilbert space such that:

$$S_1 {S_1}^* + S_2 {S_2}^* + \cdots S_n {S_n}^* = 1. \tag{4.2}$$

This (5.2) is called the *Cuntz relation.* Let $\mathsf{s} = \{S_1^*, S_2^*, \cdots, S_n^*\}$. Then the Cuntz relation implies that the set s is a finite operational partitions of unity. Let O_n be the universal C^*-algebra generated by the set $\{S_1, S_2, \cdots, S_n\}$. Then $O_n, (n \geq 2)$ is a unital, simple, purely infinite C^*-algebras called the Cuntz algebra.

4.3 Entropy for UCP via finite operational partition

Let A be a unital C^*-algebra, and let $v = \{v_1, \cdots, v_n\} \subset A$ be a finite operational partitions of unity. If φ is a state of A, then we have the density matrix $\rho_\varphi[v]$ and also we have the von Neumann entropy $S(\rho_\varphi[v])$ for $\rho_\varphi[v]$ as in the section 3.5.

Definition 4.3. ([5]). Let Φ be the UCP map associated with the finite operational partitions of unity v, and let

$$S(\Phi) = S(\rho_\varphi[v]) \tag{4.3}$$

We call the $S(\Phi)$ the *entropy for* Φ *associated with* v.

Remark 4.3. ([5]). The following two facts give us the ground that we consider $S(\Phi)$ the entropy for Φ.

1. Let $\Phi : B(K) \to B(H)$ be a UCP map, where H and K are finite dimensional. Then there exists a unique (up to unitary matrix) $v = \{v_1, \cdots, v_n\}$, which is a finite operational partition of unity associated with Φ. This implies that the value $S(\rho_\varphi[v])$ does not depend on the choice of v. Hence the value $S(\Phi)$ is determined by only Φ.

2. The most typical UCP map Φ is induced from a state ϕ by the following way: Let φ be a state of $M_n(\mathbb{C})$, and let $\Phi(x) = \varphi(x)1_{M_n(\mathbb{C})}$ for all $x \in M_n(\mathbb{C})$. In this case, $S(\rho_\varphi[v])$ is nothing else but the von Neumann entropy $S(\varphi)$ for φ.

4.4 Cuntz canonical shift Φ_n

The Cuntz canonical endomorphism Φ_n is defined by

$$\Phi_n(x) = \sum_{i=1}^{n} S_i x S_i^*, \quad x \in O_n.$$

That is, Φ_n is the UCP map associated with the s in the section 5.2. The standard left inverse $\hat{\Phi}_n$ of Φ_n is defined by

$$\hat{\Phi}_n(x) = \frac{1}{n}\sum_{i=1}^{n} S_i^* x S_i, \quad \text{for all} \quad x \in O_n,$$

and so-called canonical state φ of O_n is induced from the standard left inverse $\hat{\Phi}_n$. Then we have

Theorem 4.4. ([5]).

$$S(\Phi_n) = \log n. \tag{4.4}$$

Remark 4.4. We compare this value with those which we showed before:

1. The Voiclescu topological entropy $ht(\Phi_n)$ is $\log n$ ([6]).

2. The Connes-Narnhofer-Thirring entropy $h_\varphi(\Phi_n)$ with respect to φ is $\log n$ ([7]).

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