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The q-Meixner self-adjoint operators on the q-deformed Fock space

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1. Introduction

The Schrödinger algebra plays an important role in mathematical physics and its applications. It has been introduced and studied as the algebra of symmetries of the Schrödinger equation (see, for instance, [1], [2]). Although, in general, the Schrödinger algebra can be considered in $(n+1)$-dimensional (space-time), in this paper, we treat the centrally extended Schrödinger algebra of $n=1$.

It was noticed that the $(1+1)$ Schrödinger algebra with central extension $\mathcal{S}_1$ can be embedded into the two-photon algebra [3] (a low-dimensional Wick algebra), which gives a sort of boson Fock realization of $\mathcal{S}_1$ (the two-photon realization) and also helps us to understand the structure of $\mathcal{S}_1$ such as semidirect product of the Heisenberg algebra and $sl(2)$.

In the paper [4], the structure of Schrödinger algebra $\mathcal{S}_1$ was investigated related to the representation theory. Especially, they constructed the canonical Appell system and found a family of the probability distributions associated to the Lie algebraic structure of the Schrödinger algebra $\mathcal{S}_1$. Concretely, they construct the Hilbert space on which certain two commuting operators act as self-adjoint operators with an adjustment of the inner product. Such a self-adjointization is important because it yields a probabilistic interpretation of these operators as random variables in non-commutative probability space.

2. The Schrödinger algebra $\mathcal{S}_1$ and its boson realization

The centrally extended $(1+1)$ Schrödinger algebra $\mathcal{S}_1$ is a six-dimensional Lie algebra generated by the operators, K, G, P_x, D, P_t, M with the following non-trivial commutation relations (see [5]):

$$\left.\begin{array}{llll} [P_t,G]=P_x, & [P_x,K]=G, & [D,G]=G, & [P_x,D]=P_x, \\ [P_t,D]=2P_t, & [D,K]=2K, & [P_t,K]=D, & [P_x,G]=M, \end{array}\right\} \quad (2.1)$$

where the last commutation relation corresponds to the property of the central extension. For simplicity, if we say Schrödinger algebra in this paper, then it means to be centrally extended.

It is also known that the Schrödinger algebra $\mathcal{S}_1$ has the following vector fields realization with the multiplication and the partial differentiation by x and t:

$$
\left.\begin{array}{lll}
K = t^2\,\partial_t + t\,x\,\partial_x + \frac{m}{2}\,x^2 - d\,t & \quad & \text{conformal transformation,} \\
G = t\,\partial_x + m\,x & & \text{Galilei boost,} \\
P_x = \partial_x & & \text{spatial translation,} \\
D = 2\,t\,\partial_t + x\,\partial_x - d & & \text{dilation,} \\
P_t = \partial_t & & \text{time translation,} \\
M = m\,\mathbf{1} & & \text{mass,}
\end{array}\right\} \tag{2.2}
$$

where m and d are given parameters, and $\mathbf{1}$ is the identity operator.

The operators $\{M, G, P_x\}$ span a Heisenberg-Weyl subalgebra, and $\{K, D, P_t\}$ span an $sl(2)$ subalgebra. Indeed the Schrödinger algebra can be decomposed into the semidirect product as $\mathcal{S}_1 \cong \mathcal{H} \oplus_s sl(2)$.

Let $a^\dagger$ and a be the boson creation and the boson annihilation operators on the symmetric (boson) Fock space, respectively, that is, these two operators satisfy the canonical commutation relation $[a, a^\dagger] = \mathbf{1}$. The two-photon algebra h_6 (see, for instance, [3]) is generated by

$$
\begin{array}{lll}
N = a^\dagger a, & A_+ = a^\dagger, & A_- = a, \\
M = \mathbf{1}, & B_+ = (a^\dagger)^2, & B_- = a^2.
\end{array} \tag{2.3}
$$

Namely, B_+, B_-, and N are the double creation, the double annihilation, and the number operators, respectively. The non-trivial commutation relations among these generators are

$$
\left.\begin{array}{ll}
[A_-,\, A_+] = M, & [B_-,\, B_+] = 4N + 2M, \\
[N,\, A_+] = A_+, & [N,\, A_-] = -A_-, \\
[N,\, B_+] = 2B_+, & [N,\, B_-] = -2B_-, \\
[A_+,\, B_-] = -2A_-, & [A_-,\, B_+] = 2A_+,
\end{array}\right\} \tag{2.4}
$$

from which we can have an embedding of the Schrödinger algebra $\mathcal{S}_1$ into the two photon algebra h_6 explicitly as follows:

$$
\begin{array}{lll}
K = \frac{1}{2}B_+, & G = A_+, & P_x = A_-, \\
D = N + \frac{1}{2}M, & P_t = \frac{1}{2}B_-, & M = M.
\end{array} \tag{2.5}
$$

Combining with (2.3), this embedding gives the two-photon realization of the Schrödinger algebra $\mathcal{S}_1$.

Here we shall remind the commuting operators of our interest. In [4] they investigated the following two commuting operators $\mathbf{x}_1$ and $\mathbf{x}_2$ obtained by the adjoint action of the exponentiating P_t: For $\beta > 0$

$$\begin{aligned} \mathbf{x}_1 &= e^{\beta P_t} K e^{-\beta P_t} = K + \beta D + \beta^2 P_t, \\ \mathbf{x}_2 &= e^{\beta P_t} G e^{-\beta P_t} = G + \beta P_x. \end{aligned} \tag{2.6}$$

They made adjustment an inner product so that $K^* = \beta^2 P_t$ and $G^* = \beta P_x$, that is, $\mathbf{x}_1$ and $\mathbf{x}_2$ are self-adjoint, and gave the probabilistic observations. Especially, the joint distribution of $\mathbf{x}_1$ and $\mathbf{x}_2$ was determined explicitly based on the Appell system.

The inner product that they assumed was enough for their manipulation but a little implicit for making an extension to a deformed case. In the next section, we shall introduce an inner product which makes $\mathbf{x}_i$ be self-adjoint more explicitly, by using a deformation of the symmetric Fock space. Then we shall find the probability distribution of $\mathbf{x}_i$, in which the boson Fock realization of the Schrödinger algebra $\mathcal{S}_1$ will be crucial.

3. The β-symmetric Fock space

We shall slightly change the inner product on the symmetric Fock space, and construct the β-symmetric Fock space, on which $\mathbf{x}_1$ and $\mathbf{x}_2$ can be regarded to be self-adjoint.

Let $\mathscr{H}$ be a real Hilbert space equipped with the inner product $\langle \cdot | \cdot \rangle$, and Ω be a distinguished unit vector, called vacuum. We denote by $\mathcal{F}^{fin}(\mathscr{H})$ the set of all the finite linear combinations of the elementary vectors $\xi_1 \otimes \cdots \otimes \xi_n \in \mathscr{H}^{\otimes n}$ $(n = 1, 2, \ldots)$

.

We introduce the inner product $(\cdot \mid \cdot)_\beta$ on $\mathcal{F}^{fin}(\mathscr{H})$ by

$$(\xi_1 \otimes \cdots \otimes \xi_n \mid \eta_1 \otimes \cdots \otimes \eta_m)_\beta = \delta_{m,n} \, \beta^n \sum_{\sigma \in \mathfrak{S}_n} \langle \xi_1 \mid \eta_{\sigma(1)} \rangle \cdots \langle \xi_n \mid \eta_{\sigma(n)} \rangle,$$

where $\mathfrak{S}_n$ is the nth symmetric group of permutations.

The strict positivity of the inner product $(\cdot \mid \cdot)_\beta$ follows immediately from that of the inner product on the symmetric (boson) Fock space. Thus we can have the following definitions:

Definition 3.1. The β-symmetric Fock space $\mathcal{F}_\beta(\mathscr{H})$ is given by the completion of $\mathcal{F}^{fin}(\mathscr{H})$ by the inner product $(\cdot | \cdot)_\beta$. Given the vector $\xi \in \mathcal{H}$, the β-creation operator $a^\dagger_\beta(\xi)$ is defined by the canonical left creation that

$$\begin{aligned} &a^\dagger_\beta(\xi) \, \Omega = \xi, \\ &a^\dagger_\beta(\xi) \, \xi_1 \otimes \cdots \otimes \xi_n = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \quad n \geq 1, \end{aligned}$$

and the β-annihilation operator $a_\beta(\xi)$ is defined to be the adjoint operator of $a^\dagger_\beta(\xi)$ with respect to the inner product $(\cdot \mid \cdot)_\beta$, that is, $a_\beta(\xi) = (a^\dagger_\beta(\xi))^*$.

The followings are direct consequences of the definition:

Proposition 3.2.

(1) *The β-annihilation operator $a_\beta(\xi)$ acts on the elementary vectors as follows:*

$$a_\beta(\xi)\,\Omega = 0, \quad a_\beta(\xi)\,\xi_1 = \beta\,\langle\xi|\xi_1\rangle\Omega,$$
$$a_\beta(\xi)\,\xi_1\otimes\cdots\otimes\xi_n = \beta\sum_{k=1}^{n}\langle\xi|\xi_k\rangle\,\xi_1\otimes\cdots\otimes\overset{\vee}{\xi_k}\otimes\cdots\otimes\xi_n \quad n\geq 2,$$

where $\overset{\vee}{\xi_i}$ means that ξ_i should be deleted from tensor product.

(2) *The β-creation and the β-annihilation operators satisfy the β-scaled canonical commutation relation*

$$a_\beta(\xi)\,a_\beta^\dagger(\eta) - a_\beta^\dagger(\eta)\,a_\beta(\xi) = \beta\,\langle\xi\,|\,\eta\rangle\,\mathbf{1}$$

We shall work on the β-symmetric Fock space of one-mode, and employ the β-creation and the β-annihilation to construct the operators $A_\pm$, $B_\pm$, and N in (2.3) instead of the boson creation and the boson annihilation operators. Under the same embedding in (2.5), we can obtain the self-adjointization of $\mathbf{x}_i$ on the β-symmetric Fock space, namely,

$$\mathbf{x}_1^{(\beta)} = \frac{1}{2}(a_\beta^\dagger)^2 + \frac{1}{2}(a_\beta)^2 + a_\beta^\dagger a_\beta + \frac{1}{2}\mathbf{1},$$
$$\mathbf{x}_2^{(\beta)} = a_\beta^\dagger + a_\beta,$$

where, for the unit base vector ξ of the one-mode β-symmetric Fock space, we shall simply denote $a_\beta^\dagger(\xi)$ and $a_\beta(\xi)$ by $a_\beta^\dagger$ and a_β, respectively.

Now we shall find the probability distributions of the self-adjoint operators $\mathbf{x}_i^{(\beta)}$ with respect to the vacuum expectation on the β-symmetric Fock space. We will apply the theory of orthogonal polynomials to finding the probability distributions.

Proposition 3.3. *For $\beta > 0$, we define the sequence of polynomials $\{P_n^{(\beta)}(X)\}_{n\geq 0}$ by the recurrence formula*

$$P_0^{(\beta)}(X) = 1, \quad P_1^{(\beta)}(X) = X - \frac{1}{2},$$
$$P_{n+1}^{(\beta)}(X) = \Big(X - (\tfrac{1}{2} + 2\beta n)\Big)P_n^{(\beta)}(X) - \frac{\beta^2 2n(2n-1)}{4}P_{n-1}^{(\beta)}(X) \quad n\geq 1, \tag{3.1}$$

and ξ stands for the unit base vector for the one-mode β-symmetric Fock space.

Then we obtain

$$P_n^{(\beta)}(\mathbf{x}_1^{(\beta)})\Omega = \frac{1}{2^n}\xi^{\otimes 2n} \quad n\geq 0,$$

where Ω is the vacuum vector, and we use the convention that $\xi^{\otimes 0} = \Omega$.

This proposition can be proved by induction without much difficulties.

In order to determine the measure μ, we shall reformulate the three-steps recurrence relation in (3.1) as follows: Divide by β^{n+1}, then we have

$$\frac{P_{n+1}^{(\beta)}(X)}{\beta^{n+1}} = \Big(\frac{1}{\beta}(X - \frac{1}{2} + \frac{\beta}{2}) - (\frac{1}{2} + 2n)\Big)\frac{P_n^{(\beta)}(X)}{\beta^n} - n(n - \frac{1}{2})\frac{P_{n-1}^{(\beta)}(X)}{\beta^{n-1}},$$

and change the variable X by Y as

$$Q_n(Y) = \frac{P_n(\beta Y + (\frac{1}{2} - \frac{\beta}{2}))}{\beta^n}.$$

Consequently, we obtain the sequence of the monic polynomials $\{Q_n(Y)\}$ which satisfies the recurrence relation

$$Q_0(Y) = 1, \quad Q_1(Y) = Y - \frac{1}{2},$$
$$Q_{n+1}(Y) = \Big(Y - (\frac{1}{2} + 2n)\Big)Q_n(Y) - n(n - \frac{1}{2})Q_{n-1}(Y) \quad n \geq 1.$$

This recurrence relation is known as one for the Laguerre polynomials of the parameter $\alpha = -\frac{1}{2}$ (see, for instance, [7, Sec. 2.11]), and the corresponding orthogonalizing probability measure can be given by the density function

$$g(t) = \frac{1}{\Gamma(\frac{1}{2})}\frac{e^{-t}}{\sqrt{t}}I_{t\geq 0},$$

where $I_{t\geq 0}$ is the indicate function on $\{t \,|\, t \geq 0\}$.

Remark 3.4. By the form of the density function $g(t)$, we can find that it is in the type of gamma distributions. More precisely, it is given as $\frac{1}{2}\chi^2(1)$, that is, the $\frac{1}{2}$-dilation of the chi-square distribution with 1 degree of freedom, because the density of the distribution $\chi^2(1)$ is given by

$$f(t) = \frac{1}{\Gamma(\frac{1}{2})}\frac{e^{-t/2}}{\sqrt{2t}}I_{t\geq 0}.$$

Since the variable X in the sequence of polynomials $\{P_n^{(\beta)}(X)\}$ is related to Y in $\{Q_n(Y)\}$ by $X = \beta Y + (\frac{1}{2} - \frac{\beta}{2})$, the orthogonalizing probability measure for $\{P_n^{(\beta)}(X)\}$ is given by the β-dilation and the $(\frac{1}{2} - \frac{\beta}{2})$-right translation of one for $\{Q_n(Y)\}$.

Theorem 3.5. *The probability distribution of the operator* $\mathbf{x}_1^{(\beta)}$ *with respect to the vacuum expectation, namely, the orthogonalizing probability measure* μ *for the sequence of polynomials* $\{P_n^{(\beta)}(X)\}$ *defined in (3.1), is given by*

$$d\mu = \frac{2}{\beta}f\Big(\frac{2}{\beta}(x - (\frac{1}{2} - \frac{\beta}{2}))\Big)\, I_{t\geq\frac{1}{2}-\frac{\beta}{2}}dt$$

where f is the density function of $\chi^2(1)$, and dt *is the Lebesgue measure on* $\mathbb{R}$. *Namely, the distribution μ can be written symbolically by*

$$\frac{\beta}{2}\chi^2(1)+(\frac{1}{2}-\frac{\beta}{2}).$$

Remark 3.6. In [4], they investigated the joint distribution of $\mathbf{x}_1^{(\beta)}$ and $\mathbf{x}_2^{(\beta)}$ with respect to the vacuum expectation. We shall, here, pay our attention upon the algebraic relation between the operators $\mathbf{x}_1^{(\beta)}$ and $\mathbf{x}_2^{(\beta)}$, namely,

$$\begin{aligned}\frac{1}{2}\,(\mathbf{x}_2^{(\beta)})^2 &= \frac{1}{2}\,(a_\beta^\dagger + a_\beta)^2 \\ &= \frac{1}{2}\,(a_\beta^\dagger)^2 + \frac{1}{2}\,(a_\beta)^2 + \frac{1}{2}\,(a_\beta^\dagger a_\beta + a_\beta a_\beta^\dagger) \\ &= \frac{1}{2}\,(a_\beta^\dagger)^2 + \frac{1}{2}\,(a_\beta)^2 + a_\beta^\dagger a_\beta + \frac{\beta}{2}\,\mathbf{1},\end{aligned} \tag{3.2}$$

where, in the last equality, we have used the commutation relation $[a_\beta, a_\beta^\dagger] = \beta\,\mathbf{1}$. Hence, we have the algebraic relation

$$\mathbf{x}_1^{(\beta)} = \frac{1}{2}\,(\mathbf{x}_2^{(\beta)})^2 + (\frac{1}{2}-\frac{\beta}{2})\,\mathbf{1}.$$

Although we have derived the distribution of the operator $\mathbf{x}_1^{(\beta)}$ via orthogonal polynomials, we can obtain it directly by using the above algebraic relation. Because it is known that the distribution of the field operator $\mathbf{x}_2^{\beta} = a_\beta^\dagger + a_\beta$ with respect to the vacuum expectation on the β-symmetric Fock space is given by $\mathcal{N}(0,\beta)$, the centered Gaussian of variance $((\mathbf{x}_2^{(\beta)})^2\,\Omega\,|\,\Omega)_\beta = \beta$, the operator $\frac{1}{\beta}\,(\mathbf{x}_2^{(\beta)})^2$ is distributed according to $\chi^2(1)$. Therefore we can find that $\mathbf{x}_2^{(\beta)}$ has the distribution $\frac{\beta}{2}\,\chi^2(1)+(\frac{1}{2}-\frac{\beta}{2})$ by the algebraic relation (3.2).

4. The case of the q-deformed Fock space

The β-symmetric Fock space $\mathcal{F}_\beta(\mathscr{H})$ that we have considered in the previous section, is more of a scaling than a deformation. Here we shall use the q-deformation of symmetric Fock space $\mathcal{F}_q(\mathscr{H})$ instead of $\mathcal{F}_\beta(\mathscr{H})$. The q-deformed symmetric Fock space was introduced in [8], which gives an interpolation between the symmetric (boson) and the anti-symmetric (fermion) Fock spaces and, especially, the case $q = 0$ of which yields the canonical model in the free probability theory (see, for instance, [9]).

We shall assume q to be non-negative, that is, we restrict to $0 \le q < 1$. We consider the one-mode case and simply denote the q-creation operator $a_q^\dagger(\xi)$ and the q-annihilation operator $a_q(\xi)$ for the unit base vector ξ of one-mode by $a_q^\dagger$ and a_q, respectively.

Then we shall give the q-deformation of $A_\pm$, $B_\pm$ and N in (2.3) by using $a_q^\dagger$ and a_q as follows:

$$A_+^{(q)} = a_q^\dagger, \qquad A_-^{(q)} = a_q, \qquad B_+^{(q)} = (a_q^\dagger)^2, \qquad B_-^{(q)} = (a_q)^2,$$
$$N^{(q)} = a_q^\dagger a_q, \qquad M = \mathbf{1},$$

where M remains undeformed as the scalar operator. The above operators satisfy the following non-trivial commutation relations, cf. (2.4):

Lemma 4.1.

$$\begin{aligned}
A_-^{(q)}A_+^{(q)} - &\ q\,A_+^{(q)}A_-^{(q)} = M,\\
N^{(q)}A_+^{(q)} - &\ q\,A_+^{(q)}N^{(q)} = A_+^{(q)},\\
N^{(q)}A_-^{(q)} - &\ q^{-1}\,A_-^{(q)}N^{(q)} = -q^{-1}A_-^{(q)},\\
B_-^{(q)}B_+^{(q)} - &\ q^4\,B_+^{(q)}B_-^{(q)} = q(1+q)^2N^{(q)} + (1+q)M,\\
N^{(q)}B_+^{(q)} - &\ q^2\,B_+^{(q)}N^{(q)} = (1+q)B_+^{(q)},\\
N^{(q)}B_-^{(q)} - &\ q^{-2}\,B_-^{(q)}N^{(q)} = -q^{-2}(1+q)B_-^{(q)},\\
A_+^{(q)}B_-^{(q)} - &\ q^{-2}\,B_-^{(q)}A_+^{(q)} = -q^{-2}(1+q)A_-^{(q)},\\
A_-^{(q)}B_+^{(q)} - &\ q^2\,B_+^{(q)}A_-^{(q)} = (1+q)A_+^{(q)}.
\end{aligned}$$

We can see Lemma 4.1 by direct calculation, similar manipulations concerning with the above commutation relations can be also found in [10], [11], [12], in which square or higher powers of q-white noise analysis are investigated.

As we mentioned in Section 1, the classical Schrödinger algebra is generated by the operators, K, G, P_x, D, P_t, M, which have the boson Fock realization as in (2.5). Taking in account the commutation relations in Lemma 4.1, we shall give the q-deformation of K, G, P_x, D, P_t by using the q-Fock space instead of the symmetric one. Namely, we put

$$G^{(q)} = A_+^{(q)}, \quad P_x^{(q)} = A_-^{(q)}, \quad K^{(q)} = \frac{1}{1+q}B_+^{(q)}, \quad P_t^{(q)} = \frac{1}{1+q}B_-^{(q)}$$
$$D^{(q)} = qN^{(q)} + \frac{1}{1+q}M.$$

Now we shall consider the q-deformation of the operators $\mathbf{x}_1$ and $\mathbf{x}_2$ and find their probability distributions. We give the q-deformation by replacing K, D, P_t, G, P_x in (2.6) to the q-deformed ones, where no scaling parameter is imposed, that is, $\beta = 1$. Hence we have

$$\mathbf{x}_1^{(q)} = K^{(q)} + D^{(q)} + P_t^{(q)},$$
$$\mathbf{x}_2^{(q)} = G^{(q)} + P_x^{(q)}.$$

The operator $\mathbf{x}_2^{(q)}$ is the field operator $a_q^\dagger + a_q$ on the q-Fock space and its probability distribution with respect to the vacuum expectation is rather well-known as the q-Gaussian and investigated by many authors, for instance, [8], [14], [15].

Therefore we shall pay our attention upon the operator

$$\mathbf{x}_1^{(q)} = \frac{1}{1+q}B_+^{(q)} + \frac{1}{1+q}B_-^{(q)} + qN^{(q)} + \frac{1}{1+q}M,$$

and determine its probability distribution with respect to the vacuum expectation on the q-Fock space.

Similar to the β-symmetric case, we will seek the sequence of polynomials which are orthogonal with respect to the distribution of $\mathbf{x}_1^{(q)}$.

Proposition 4.2. *We define the sequence of polynomials $\{P_n^{(q)}(X)\}_{n\geq 0}$ by the recurrence formula*

$$P_0^{(q)}(X) = 1, \quad P_1^{(q)}(X) = X - \frac{1}{1+q},$$
$$P_{n+1}^{(q)}(X) = \Big(X - \big(\frac{1}{1+q} + q\,[2n]_q\big)\Big)P_n^{(q)}(X) - \frac{[2n]_q[2n-1]_q}{(1+q)^2}P_{n-1}^{(q)}(X) \quad n \geq 1.$$

and let ξ be the unit base vector for the one-mode q-Fock space. Then we obtain

$$P_n^{(q)}(\mathbf{x}_1^{(q)})\,\Omega = \frac{1}{(1+q)^n}\xi^{\otimes 2n} \quad n \geq 0,$$

where Ω is the vacuum vector, and we use the convention that $\xi^{\otimes 0} = \Omega$.

In order to describe the probability distribution of the operator $\mathbf{x}_1^{(q)}$, we will recall the q-deformed Meixner polynomials and fix the notations of the corresponding probability measures.

The classical infinitely divisible distributions of the Meixner family, that is, Gaussian, Poisson, gamma, Pascal, and (pure) Meixner types, can be determined as the orthogonalizing probability measures for the sequence of polynomials given by the following recurrence relations with 4 parameters, κ_1, κ_2, γ, and δ ($\kappa_2 > 0$, $\delta \geq 0$):

$$\begin{aligned} &P_0(X) = 1, \quad P_1(X) = X - \kappa_1, \\ &P_{n+1}(X) = (X-(\kappa_1+\gamma n))P_n(X) - (\kappa_2+\delta(n-1))\,n\,P_{n-1}(X) \quad n \geq 1. \end{aligned} \tag{4.1}$$

The parameters κ_1 and κ_2 correspond to the mean and the variance of the distribution, respectively. The sequence of polynomials $\{P_n(X)\}$ given by (4.1) is called the (classical) Meixner polynomials.

One of q-deformations of the Meixner polynomials is given as replacing the integers in the Szegö-Jacobi parameters in the recurrence relation (4.1) to the q-integers, and we will refer the corresponding orthogonalizing probability measures as the q-deformed Meixner distributions.

Definition 4.3. Let $q, \kappa_1, \kappa_2, \gamma, \delta$ be given constants with $0 \leq q < 1$, $\kappa_2 > 0$, $\delta \geq 0$. Then the q-deformed Meixner distribution on $\mathbb{R}$ of parameters κ_1, κ_2, γ, δ is defined to be the unique probability measure $\mu(q; \kappa_1, \kappa_2, \gamma, \delta)$ on $\mathbb{R}$ for which the sequence of polynomials $\{P_n\}$ given by the following recurrence relation are orthogonal:

$$\begin{aligned} &P_0(X) = 1, \quad P_1^{(q)}(X) = X - \kappa_1, \\ &P_{n+1}(X) = (X - (\kappa_1 + \gamma[n]_q))P_n(X) - (\kappa_2 + \delta[n-1]_q)[n]_q P_{n-1}(X), \quad n \geq 1. \end{aligned} \tag{4.2}$$

Remark 4.4. Although the polynomials defined in (4.2) are affine transformation of Al-Salam-Chihara polynomials [16], we shall adopt the above parameterization for emphasizing the relation to the five types of infinitely divisible distributions.

By comparing the Szegö-Jacobi parameters for the orthogonal polynomials in Proposition 4.2 with those in (4.2), we can derive the following theorem, for more details see [20]:

Theorem 4.5. *The probability distribution of the operator* $\mathrm{x}_1^{(q)}$ *with respect to the vacuum expectation on the q-Fock space is given as the deformed Meixner distribution*

$$\mu\left(q^2; \frac{1}{1+q}, \frac{1}{1+q}, q(1+q), q\right),$$

where we should note that the deformation parameter q for the q-deformed Meixner distributions is replaced to q^2.

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