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A representation of unital completely positive maps

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Abstract

Let $M_n(\mathbb{C})$ be the algebra of $n\times n$ complex mairices, and let Φ be a unital completely positive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. With the notion of the von Neumann entropy for a state in mind, we give a model of r-tupple $\{v_j\}_{j=1}^{r}$ so that $\Phi(x) = v_1^*xv_1 + \cdots + v_r^*xv_r, (x \in M_n(\mathbb{C}))$. The r is uniquely determined for Φ and the r-tupple is also unique up to a $r \times r$ unitary matrix.

1 Introduction

In the framework of the theory of operator algebras, the notion of entropy for automorphisms was introduced by Connes-Stømer in [8], Connes-Narnhofer-Thirring in [9] and Voiculescu in [14] (which is extended by Brown [4]). The Connes-Stømer entropy $H(\theta)$ is defined for a *-automorphism θ of finite von Neumann algebra M with $\tau = \tau\circ\theta$, where τ is a fixed given finite trace of M. After then, the Connes-Narnhofer-Thirring entropy $h_\phi(\theta)$ is given as an extended version of $H(\theta)$ for a *-automorphism θ of a C^*-algebra A by replacing the trace τ to a state ϕ of A, and if A is a finite von Neumann algebra then $h_\tau(\theta) = H(\theta)$. Voiculescu's topological entropy $ht(\theta)$ is defined as an independent version of any state of A.

We studied these entropies in [6] and [7] for not only *-automorphisms but also *-endomorphisms like so called canonical shifts. As one of interesting such *-endomorphisms, we picked up the Cuntz canonical endomorphism Φ_n on the Cuntz algebra O_n which has a strong connection to Longo's canonical shift (cf. [6]). The O_n is the C^*-algebra generated by isometries $\{S_1, \cdots, S_n\}$ such that $S_1S_1^* + \cdots + S_nS_n^* = 1$, and the Φ_n is defined as

$$\Phi_n(x) = S_1xS_1^* + \cdots + S_nxS_n^*, \quad (x \in O_n). \tag{1.1}$$

Such maps given by the form as the right hand side of (1.1) are unital completely positive maps, and the above notions $H(\cdot)$, $h_\phi(\cdot)$ and $ht(\cdot)$ are available for unital completely positive maps too.

Conditional expectations are the most typical examples of unital completely positive maps, ans states of the matrix algebras $M_n(\mathbb{C})$ are considered as the most elementary example of conditional expectations. However, for a conditional expectation E, by their definitions it holds always that $H(E) = h_\phi(E) = ht(E) = 0$. On the other hand, in the case of the von Neumann entropy $S(\phi)$ for a state ϕ of $M_n(\mathbb{C})$, it is possible that $S(\phi) \neq 0$.

In order to define "entropy", we need the notion of "finite partition of unity" (see for example, [11]). The most generalized one of "finite partition of unity" was introduced by Lindblad ([10]), and it is called the "finite operational partition of unity".

With these facts in mind, here we give a method to induce the finite operational partition of unity for a given unital completely positive map. That is, let A and B be unital C^*-algebras and let Φ be a unital completely positive map of A to B. We give a method to get a model of r-tupple $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ such that

$$\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r, \quad (x \in A). \tag{1.2}$$

When A and B are matrix algebras, such a representation is called Kraus representation (cf. Appendix in [13]), or obtained as a straightforward application of Stinespring's theorem (see for example, [1, 3]). We note that this representation is not unique

Our main purpose in this note is to show, for a given completely positive map Φ, a unique r-tupple $v(\Phi) = \{v_1, \cdots, v_r\}$ which is suitable to extend the notion of von Neumann entropy $S(\phi)$ for a state ϕ of matrix algebras to the entropy $S(\Phi)$ for a unital completely positive map Φ.

First, for a given completely positive map Φ from $B(H)$ to $B(K)$ of finite dimensional Hilbert spaces H, K, we construct the Hilbert spaces $H \otimes_\Phi K$. Let $r = \dim(H \otimes_\Phi K)$. Next, we give a r-tupple $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ which satisfy that $\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r$. The r-tupple is unique up to unitaties and induces $S(\Phi)$. After then, we apply these to the non-commutative Bernoulli shift β and we define the entropy $S_\varphi(\Phi)$ with respect to a state φ with $\varphi = \varphi \cdot \beta$. These results in this note are in [5].

2 Preliminaries

Here, we denote some notaions and terminologies which we use later.

We denote by $M_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices, and by Tr_n the standard trace, that is, the sum of all diagonal components. A matrix $D \in M_n(\mathbb{C})$ is called a *density matrix* if D is a positive operator with $\mathrm{Tr}_n(D) = 1$(cf. [11] [12]).

The notation η is called the entropy function in usual, and it is the function defined by

$$\eta(t) = \begin{cases} -t \log t, & (0 < t \leq 1) \\ 0, & t = 0 \end{cases}$$

2.1 Finite partitions

The notion of "a finite partition of unity" is the starting point of our study.

2.1.1 Finite partitions of 1

The first one is discuused in the real numbers $\mathbb{R}$. Let

$$\lambda = \{\lambda_1, \cdots, \lambda_n\}$$

be the set of real numbers $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$. We say that the n-tupple $\lambda \subset \mathbb{R}$ is a *finite partition of* 1.

2.1.2 Finite operational partition of unity

The terminology, a *finite operational partition of unity*, was first given by Lindblad ([10]) and after then it is used by Alicki-Fannes([2]).

Let A be a unital C^*-algebra. Let $x = \{x_1, ..., x_k\} \subset A$. Then x is said to be a *finite operational partition of unity* of size k if

$$\sum_i^k x_i^* x_i = 1_A. \tag{2.1}$$

Such a finite operational partition of unity $x = \{x_1, ..., x_k\}$ in A induces an A-coefficient in $M_k(A)$, whose (i,j) coefficient $x(j,i)$ is given by the following:

$$x(j,i) = x_i^* x_j, \quad (1 \leq i,j \leq k). \tag{2.2}$$

We denote this matrix by $[x]$. Then $[x]$ is an A-coefficient density matrix in $M_k(A)$, that is, $[x]$ is a positive operator with $Tr([x]) = \sum_{i=1}^k x(i,i) = 1_A$.

2.2 Entropy for finite partitions of unity

2.2.1 Entropy for finite partitions of 1

Let a n-tupple $\lambda \subset \mathbb{R}$ be a a *finite partition of* 1. Let

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n). \tag{2.3}$$

Then $H(\lambda)$ is called the *entropy* for the finite partiton λ of 1.

2.2.2 Entropy for Finite operational partition of unity

Let $x = \{x_1, ..., x_k\}$ be an operational partition of unity in a unital C^*-algebra A, and let φ be a state of A. The $\rho_\varphi[x]$ is the $k \times k$ matrix whose $\{i, j\}$-component is defined by

$$\rho_\varphi[x](i,j) = \varphi(x_j^* x_i), \quad (i, j = 1, \cdots, k). \tag{2.4}$$

Then $\rho_\varphi[x]$ is a density matrix. We call $\rho_\varphi[x]$ the *density matrix associate with x and φ*. If φ is a unique tracial state, we denote $\rho_\varphi[x]$ by $\rho[x]$ simply.

Let $\lambda(\rho_\varphi[x]) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ be the eigenvalues of the matrix $\rho_\varphi[x]$. Then $\lambda(\rho_\varphi[x])$ is a finite partition of 1 because $\rho_\varphi[x]$ is a density matrix. Hence we have the entropy $H(\lambda(\rho_\varphi[x]))$.

Let $S(\rho_\varphi[x])$ be the von Neumann entropy (cf. [11, 12]) for the density matrix $\rho_\varphi[x]$. Then $S(\rho_\varphi[x])$ is nothing else but $H(\lambda(\rho_\varphi[x]))$, that is,

$$S(\rho_\varphi[x]) = \mathrm{Tr}_k(\eta(\rho[x])) = H(\lambda(\rho_\varphi[x])) = \sum_i \eta(\lambda_i). \tag{2.5}$$

3 Representation of completely posive maps

Let Φ be a completely posive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. Put $A = M_n(\mathbb{C})$. We give a method to get a "finite" family $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ for Φ which satisfies that $\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r$ for all $x \in A$. We remark that if Φ is unital, then $v(\Phi)$ is a finite operational partition:

$$\sum_{j=1}^{r} v_j^* v_j = \Phi(1_A) = 1_{B(K)} \tag{3.1}$$

3.1 Hilbert space $H \otimes_\Phi K$

Let H be an n-dimensional Hilbert space, and let $\Phi : A \to B(K)$ be a completely positive linear map. Let $\{e_1, \cdots, e_m\}$ be the set of mutually orthogonal minimal projections in $B(H)$ with $\Phi(e_i) \neq 0$ for all i. Let $\xi_i \in e_i(H)$ be a vector with $\|\xi_i\| = 1$ for $i = 1, \cdots, m$ and we extend $\{\xi_1, \cdots, \xi_m\}$ to an orthonormal basis of H as $\{\xi_1, \cdots, \xi_n\}$. Let $\{e_{ij}; i, j = 1, \cdots, n\}$ be a matrix units of A with $e_{ij}\xi_j = \xi_i$ so that $e_{ii} = e_i$ for $i = 1, \cdots, m$. Then each $\zeta \in H \odot K$ (the algebraic tensor product $H \odot K$ of H and K) is written by

$$\zeta = \sum_{i=1}^{n} \xi_i \otimes \mu_i, \quad \text{for some} \quad \mu_i \in K. \tag{3.2}$$

Definition 3.1.1. We define a sesquilinear form $< \cdot, \cdot >_\Phi$ on the space $H \odot K$ by

$$< \sum_{i=1}^{n} \xi_i \otimes \mu_i, \sum_{j=1}^{n} \xi_j \otimes \nu_j >_\Phi = \sum_{i,j} < \Phi(e_{ji})\mu_i, \nu_j >_K, \tag{3.3}$$

where $< \cdot, \cdot >_K$ means the inner product of the Hilbert space K.

Since Φ is completely posive, this form $< \cdot, \cdot >_\Phi$ turns out positive semidefinite. The value $< \cdot, \cdot >_\Phi$ depends on the choise of the orthonormal basis of H. However the kernel of this sesquilinear form $< \cdot, \cdot >_\Phi$ is unique up to unitaries on $H \otimes K$ as follows:

Proposition 3.1.2. *Let $\{\xi_1, \cdots, \xi_n\}$ (resp., $\{\xi'_1, \cdots, \xi'_n\}$) be an orthonormal basis of H, and let u be a unitary on $H \otimes K$ with $\xi'_i = u\xi_i$ for all $i = 1, \cdots, n$. Let $Ker(\Phi)$ (resp., $Ker'(\Phi)$) be the kernel of this form via $\{\xi_i\}_i$ (resp., $\{\xi'_i\}_i$) Then*

$$Ker'(\Phi) = (\bar{u}u^* \otimes 1)Ker(\Phi)$$

where $\bar{u}$ is the unitary matrix on H whose (i,j)-entry is the conjugate complex number of $u(i,j)$ for all $i, j = 1, \cdots, n$.

Definition 3.1.3. Now taking the quotient by the space $Ker(\Phi)$, we have a preHilbert space and complete to get the Hilbert space $H \otimes_\Phi K$.

We denote by $(\sum_{i=1}^{n} \xi_i \otimes \mu_i)_\Phi$ the element in $H \otimes_\Phi K$ corresponding to $\sum_{i=1}^{n} \xi_i \otimes \mu_i \in H \odot K$.

If K is finite dimensional, of course the $H \otimes_\Phi K$ is finite dimensional. We can extend this method to get the Hilbert space $H \otimes_\Phi K$ to (for an example) non-commutative Bernoulli shifts, and it induces finite dimensional $H \otimes_\Phi K$ even if H and K are infinite dimensional. By Proposition 3.1.2, we have the following:

Proposition 3.1.4. *The dimension of $H \otimes_\Phi K$ does not depend on the choice of orthonormal basis of H.*

Example 3.1.5.

1. If ϕ is a state of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_\phi \mathbb{C}) = \text{rank of } \phi, \text{(i.e., the rank of the density matrx of } \phi).$$

2. If E is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra A of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_E \mathbb{C}^n) = n.$$

   Here, we ramark that A is isomorphic to the diagonal subalgebra $D_n(\mathbb{C})$.

3. Let B be a subfactor of $M_n(\mathbb{C})$. If E is the conditional expectation $M_n(\mathbb{C})$ to B, then

$$\dim(\mathbb{C}^n \otimes_E \mathbb{C}^m) = \frac{n}{m}$$

   Here, we ramark that B is isomorphic to $M_m(\mathbb{C})$ for some m by which n can be divided.

4. If α is an automorphism of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_\alpha \mathbb{C}^n) = 1.$$

The following shows that $\dim(H \otimes_\Phi K)$ can be finite, even if H and K are infinite dimensional.

Example 3.1.6. Let β be the non-commutative Bernoulli shift of $A = \otimes_{i=1}^\infty M_n(\mathbb{C})$. That is, let $A_i = M_n(\mathbb{C})$ for all $i = 1, 2, \cdots$, and for each $m \in \mathbb{N}$, let

$$A(m) = A_1 \otimes \cdots \otimes A_m \otimes 1 \otimes \cdots \subset A, \tag{3.4}$$

where 1 is the unit of $M_n(\mathbb{C})$.

The β is given as the shift as the followings:

$$\beta(x) = 1 \otimes x \otimes 1 \otimes \cdots, \quad \text{for all} \quad m, \ x \in A(m). \tag{3.5}$$

Let $H_i = \mathbb{C}^n$ for all $i = 1, 2, \cdots$, and for each $m \in \mathbb{N}$. Fix an vector $\Omega \in \mathbb{C}$ with $||\Omega|| = 1$, and let

$$H(m) = H_1 \otimes \cdots \otimes H_m \otimes \Omega \otimes \cdots \subset H = \otimes_{i=1}^{\infty} H_i, \tag{3.6}$$

The β is a unital completely positive map from $A \subset B(H)$ to $B(H)$, and the restriction $\beta|_{A(m)}$ of β to $A(m)$ is a unital completely positive map from $A(m) \subset B(H(m))$ to $B(H(m+1))$. Apply the above method to $\beta|_{A(m)}$, we have always that

$$\dim(H(m) \bigotimes_{\beta|_{A(m)}} H(m+1)) = n, \quad \text{for all} \quad m$$

As a result, we have that

$$\dim(H \bigotimes_{\beta} H) = n = \dim(H \bigotimes_{E} H),$$

where $E : A \to \beta(A)$ is the conditional expectation.

Now, we call the dimension of $H \otimes_\Phi K$ the *rank* of Φ.

A phenomenon As an example, we show a phenomenon of the above discussion in the case of a state of $M_2(\mathbb{C})$ which indicates how the dimension of $H \otimes_\phi K$ coincides with the rank of a state ϕ of the usual sense.

Example 3.1.7. Let $\{\xi_1, \xi_2\}$ be an orthonormal basis of $\mathbb{C}^2$ and let $\{e_{ij}; i, j = 1, 2\}$ be a matrix units of $\mathbb{C}^2$ with $e_{ij}\xi_j = \xi_i$. We give a vector representation for each $\xi \in \mathbb{C}^2$ relative to this $\{\xi_i; i = 1, 2\}$ and a matrix representation for each $x \in M_2(\mathbb{C})$ relative to this $\{e_{ij}; i, j = 1, 2\}$:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7}$$

and

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.8}$$

Assume that ϕ is a state given by

$$\phi(x) = \phi(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}) = \frac{1}{2}\sum_{i,j=1}^{2} x_{ij}. \tag{3.9}$$

Then $\phi(e_i) = 1/2$ for $i = 1, 2$ and the discussion with respect to $\{\xi_1, \xi_2\}$ and $\{e_{ij}; i, j = 1, 2\}$ is as follows:

Let Ω be a fixed unit vector in $\mathbb{C}$ and let

$$\zeta_i = \sqrt{2}\xi_i \otimes \Omega \in \mathbb{C}^2 \otimes_\phi \mathbb{C}. \tag{3.10}$$

Then

$$< \zeta_i, \zeta_j >_\phi = 2\phi(e_{ij}) = 1, \quad \text{for all} \quad i, j = 1, 2. \tag{3.11}$$

This implies that $\zeta_i \in \mathbb{C}^2 \otimes_\phi \mathbb{C}$ has norm 1 for $i = 1, 2$ and

$$\zeta_1 = \sqrt{2}\xi_1 \otimes \Omega = \sqrt{2}\xi_2 \otimes \Omega = \zeta_2. \tag{3.12}$$

Next we choose another family of minimal projections with $\phi(p) \neq 0$ and orthonormal basis of $\mathbb{C}^2$. Let

$$e'_{11} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{3.13}$$

then $\phi(e'_{11}) = 1$ so that the set containing e'_{11} of minimal projections with $\phi(\cdot) \neq 0$ is the one point set $\{e'_{11}\}$. The corresponding orthonormal basis of $\mathbb{C}^2$ and the corresponding matrix units are as follows:

$$\xi'_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi'_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.14}$$

and

$$e'_{11}, \; e'_{12} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \; e'_{21} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \; e'_{22} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.15}$$

Let

$$\zeta'_i = \frac{1}{\sqrt{2}}\xi'_i \otimes \Omega \in \mathbb{C}^2 \otimes_\phi \mathbb{C}, \quad (i = 1, 2). \tag{3.16}$$

Then

$$< \zeta'_1, \zeta'_1 >_\phi = \frac{1}{2}\phi(e'_{11}) = 1, \tag{3.17}$$

and

$$< \zeta_2', \zeta_2' >_\phi = \frac{1}{2}\phi(e_{22}') = 0 \tag{3.18}$$

This means that

$$\mathbb{C}^2 \otimes_\phi \mathbb{C} = \mathbb{C}\zeta_i' = \xi_i' \otimes \mathbb{C}.$$

We remark that $\{\xi_1, \xi_2\}$ and $\{\xi_1', \xi_2'\}$ are combined as $u\xi_i = \xi_i', (i = 1, 2)$ by the unitary u:

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.19}$$

Some relation to the Choi matrix. For a completely posive map Φ of $M_n(\mathbb{C})$, it is given the so called the Choi matrix C_Φ.

In the case of Φ is a state ϕ of $M_n(\mathbb{C})$, we have the following relation between the sesquilinear form $< \cdot, \cdot >_\phi$ and the coefficient of C_ϕ :

$$< \xi_i \otimes \Omega, \xi_j \otimes \Omega >_\phi = C_\phi(j, i) \tag{3.20}$$

where $\{\xi_1, \cdots \xi_n\}$ is a orthonormal basis of $\mathbb{C}^n$ and Ω is 1 considered as the vector in $\mathbb{C}$.

3.2 Operators $\{v_j; j = 1, \cdots, r\}$

As the above section, let $\Phi : B(H) \to B(K)$ be a completely positive linear map. Here, we assume that H and K be finite dimensional, and let $r = \dim(H \otimes_\Phi K)$.

Definition 3.2.1. Let $\{\xi_i; i = 1, \cdots, n\}$ be an orthonormal basis of H, and let $\{\zeta_j; j = 1, \cdots, r\}$ be an orthonormal basis of $H \otimes_\Phi K$. Define $v_j : K \to H$ by

$$v_j(\mu) = \sum_{i=1}^{n} < \xi_i \otimes \mu, \zeta_j >_\Phi \xi_i, \quad (\mu \in K) \tag{3.21}$$

Proposition 3.2.2. *Let $\{v_1, v_2, \cdots, v_r\}$ be the tupple obtained by (3.17). Then*

(i) *They satisfies the desired following property:*

$$\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r, \quad (x \in B(H)) \tag{3.22}$$

(ii) $||v_i|| \leq 1$, *for all* $i = 1, \cdots, r$;

(iii) $\{v_1, v_2, \cdots, v_r\}$ *are linearly independent.*

Proposition 3.2.3. *The tupple $\{v_1, v_2, \cdots, v_r\}$ for unital completely positive map Φ satisfy the following convenient properties to compute the von Neumann type entropy $S(\Phi)$.*

1. *If E is a conditional expectation of $M_n(\mathbb{C})$ onto $D_n(\mathbb{C})$ then $\{v_j : 1 \leq j \leq r\}$ are mutually orthogonal minimal projections.*
2. *If Φ is a *-homomorphism, then $\{v_j : 1 \leq j \leq r\}$ are isometries with*
$$v_i v_j^* = \delta_{ij} 1. \tag{3.23}$$

We remark that the following results are well known (see for example [1, 13]) so that our tupple $\{v_1, v_2, \cdots, v_r\}$ for unital completely positive map Φ is unique up to a unitary matrix:

Proposition 3.2.4. *Let H and K be finite dimensional Hilbert spaces. Assume that $v = \{v_1, v_2, \cdots, v_r\}$ and $w = \{w_1, \cdots, w_r\}$ are two families of operators in $B(K, H)$. Then*

$$v_1^* x v_1 + \cdots + v_r^* x v_r = w_1^* x w_1 + \cdots + w_r^* x w_r, \quad \textit{for all} \quad x \in B(H)$$

if and only if there is a unitary matrix $[u(i,j)] \in M_r(\mathbb{C})$ such that

$$v_i = \sum_{j=1}^{r} u(i,j) w_j, \quad i = 1, \cdots, r.$$

Example 3.2.5. Let β be the non-commutative Bernoulli shift on $A = \otimes_{i=1}^{\infty} M_n(\mathbb{C})$. The n-tupple $\{v_j\}_j$ of β are as follows. We use the same notation as Example 3.1.6. Let

$$W_m = \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m), \quad \alpha_j \in \{1, 2, \cdots, n\}\}. \tag{3.24}$$

and let

$$\xi_\alpha = \xi_{\alpha_1} \otimes \xi_{\alpha_2} \otimes \cdots \otimes \xi_{\alpha_m} \otimes \Omega \otimes \cdots \in H(m) \tag{3.25}$$

Then $\{\xi_\alpha; \alpha \in W_m\}$ is an orthonormal basis of $H(m)$, and

$$\{\xi_\alpha \otimes_\beta (\xi_i \otimes \xi_\alpha); i = 1, 2, \cdots, n\} \tag{3.26}$$

is an orthonormal basis of $H(m) \otimes_\beta H(m+1)$. Then our tupple for β is given by

$$v_j(\xi_i \otimes \xi_\beta) = \delta_{ij}\, \xi_\beta, \quad \text{for all} m, \beta \in W_m. \tag{3.27}$$

We remark that $\{v_j^*\}_j$ are isometries satisfying the Cuntz relation.

3.3 Relation to Stinespring's theorem

Let $\Phi : A \subset B(H) \to B(K)$ be a copletely positive map, where H and K are finite dimensional. Let $\{v_j : K \to H\}_{j=1}^{r}$ be the tupple by (3.17). We denote by L the Hilbert space of r-direct sum of H, i.e., $H \oplus \cdots \oplus H$. Let

$$V(\xi) = (v_1\xi, \cdots, v_r\xi) \in L, \quad \xi \in K \tag{3.28}$$

and let

$$\pi(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ \vdots & x & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix}, \quad x \in A. \tag{3.29}$$

Then we have the followings:

1. The π is a representation of the C^*-algebra A on the Hilbert space L.
2. The property that $\sum_{i=1}^{r} v_i^* v_i = 1_K$ means that the operator V defined by (3.20) is an isometry from K to L.
3. The property $\Phi(x) = \sum_{j=1}^{r} v_j^* x v_j$ is written as

$$\Phi(x) = (v_1^*, \cdots, v_r^*) \begin{bmatrix} x & 0 & \cdots & 0 \\ \vdots & x & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ v_r \end{pmatrix} = V^* \pi(x) V \tag{3.30}$$

4. These imply that (π, V) can be considerefd as the pair obtained by the Stinespring's representation.
5. The property that $\{v_j\}_j$ are linearly independent satisfies that (π, V) is the minimal pair in the sense of Arveson [1].

4 Entropy for unital completely positive maps

In this section, we denote an application to the notion of entropy. Let Φ be a unital completely positive map of a C^*-algebra $A \subset B(H)$ to $B \subset B(K)$ and let $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ be the tupple for Φ obtained by (3.21). Then the tupple $v(\Phi)$ is a finite operational partition of unity in $B(K)$. Hence we can apply the discussion in the section 2.2.2 to unital completely positive map.

4.1 Case of a state ϕ of $M_n(\mathbb{C})$

First, we consider the cae of a state ϕ of $M_n(\mathbb{C})$. Let ϕ be a state of $M_n(\mathbb{C})$, and let $v(\phi) = \{v_j : 1 \leq j \leq r\}$ be the tuple associated with ϕ defined by (3.20).

Let τ be the unique tracial state of $M_n(\mathbb{C})$, that is $\tau(x) = Tr(x)/n$ for all $x \in M_n(\mathbb{C})$. The density matrix $\tau[v(\phi)]$ is given by (2.4). Let $\{e_i\}_{i=1}^m$ be the mutually orthogonal minimal projections in $M_n(\mathbb{C})$ such that $\phi(e_i) \neq 0$. Then we see that

$$\tau[v(\phi)](i,j) = \delta_{ij}\sqrt{\phi(e_i)\phi(e_j)}. \tag{4.1}$$

It is clear that $\tau[v(\phi)]$ is a diagonal matrix, and the entropy $S(\tau[v(\phi)])$ in the section 2.2.2 is nothing else but the von neumann entropy $S(\phi)$ of ϕ:

$$S(\tau[v(\phi)]) = \sum_{j=1}^{r} \eta(\phi(e_j)) = -\sum_{j=1}^{r} \phi(e_j)\log\phi(e_j) = S(\phi). \tag{4.2}$$

4.2 Entropy for unital completely positive maps

On the basis of the fact in the above section 4.1, we denote the $S(\rho[v(\Phi)])$ for a unital completely positive map $\Phi : A \to B$ by $S(\rho(\Phi))$, and in the case of the tracial state ρ we use the same notation $S(\Phi)$ simply.

Here, we show the case of A which has a unique taracial state τ and we number the values of typical examples of von Neumann type entropy $S(\Phi)$ for unital completely positive maps Φ.

1. If ϕ is a state of $M_n(\mathbb{C})$, then

$$S(\phi) = \sum_{j=1}^{r} \eta(\lambda_j) \tag{4.3}$$

   where $\{\lambda_j\}$ are eigenvalues of ϕ.

2. If E is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra B then

$$S(E) = \log n. \tag{4.4}$$

   Compare this fact to that $H(E) = ht(E) = 0$.

3. If E is the conditional expectation of $M_n(\mathbb{C})$ to a subfactor B then

$$S(E) = \log \frac{n}{k}. \tag{4.5}$$

   Here we remark that a subfactor B of $M_n(\mathbb{C})$ is isomorphic to $M_k(\mathbb{C})$ some k, and that n is divisible by k. Compare this fact to that $H(E) = ht(E) = 0$.

4. If α is an automorphism of $M_n(\mathbb{C})$), then

$$S(\alpha) = 0 \tag{4.6}$$

   and this coinsides with the fact that $H(\alpha) = ht(\alpha) = 0$.

5. If β is the non-commutative Bernoulli shift on $\otimes_{i=1}^{\infty} M_n(\mathbb{C})$, then

$$S(\beta) = \log n \tag{4.7}$$

   and this coinsides with the fact that $H(\beta) = ht(\beta) = \log n$.

6. If Φ_n is the Cuntz's canonical shift on O_n, then

$$S_\Psi(\Phi_n) = \log n \tag{4.8}$$

   and this coinsides with the fact that $h_\Psi(\Phi_n) = ht(\Phi_n) = \log n$. Here Ψ is the state of O_n which is given by the left inverse of Φ.

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