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Free independence and its generalization

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1 Introduction

1.1 Algebraic probability space

The mathematical description of quantum theory can be seen as a probability theory on an non-commutative algebra. This viewpoint has led to the development of non-commutative probability theory (or quantum probability theory). Many probabilistic concepts can be extended to non-commutative algebras: for example, independence of random variables, moments of random variables and probability distributions of random variables. In particular, it turns out that independence of random variables, the basic concept of probability theory, is not uniquely determined. The most famous independence, which differs from the usual independence, is free independence [8]. On the other hand, monotone independence was introduced by Muraki [5] as another possible independence of random variables in a non-commutative algebra.

The main purpose of this article is to explain a key idea to unify free and monotone independences[1], with a new look at free independence. Let us start from basic concepts on non-commutative probability theory.

Let $\mathcal{A}$ be a unital $*$-algebra over $\mathbb{C}$, that is, a unital algebra with an involution $*$: $\mathcal{A}\to\mathcal{A}$ which is anti-linear. A **state** φ on $\mathcal{A}$ is a linear functional from $\mathcal{A}$ to $\mathbb{C}$, which is unital and positive: $\varphi(1_{\mathcal{A}})=1$ and $\varphi(a^{*}a)\geq 0$ for any $a\in \mathcal{A}$. A typical example of $\mathcal{A}$ is the set of all bounded linear operators $\mathbb{B}(H)$ on a Hilbert space H. If we denote by $\langle\cdot,\cdot\rangle$ the inner product of H[2], an involution $*$ is the usual adjoint operation: $\langle x, ay\rangle=\langle a^{*}x, y\rangle$ $(a\in \mathbb{B}(H), x, y\in H)$. A typical state is a vector state defined by $\varphi(a):=\langle v, av\rangle$, where v is a unit vector of H.

An **algebraic probability space** is a pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state on it. An element $X\in \mathcal{A}$ is called a **random variable**.

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[1]This article is based on a preprint [4].
[2]The inner product here is linear with respect to the right component.

From now on we assume that $\mathcal{A}$ is a von Neumann algebra, i.e., $\mathcal{A}$ is a $*$-subalgebra of $\mathbb{B}(H)$, the set of bounded linear operators on a Hilbert space H, and $\mathcal{A}$ is closed under the strong topology.

The **probability distribution** of a self-adjoint random variable $X \in \mathcal{A}$ is defined by

$$\mu_X(B) := \varphi(E_X(B)),$$

where $E_X(B)$ is the spectral projection associated to X and B is a Borel set of $\mathbb{R}$.

We can also consider some unbounded operators on H. A self-adjoint operator X is said to be affiliated to $\mathcal{A}$ if its spectral projections $E_X(B)$, B Borel sets, all belong to $\mathcal{A}$. Then we can define μ_X in the same way.

Example 1.1. The usual probability theory is recovered if we take a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{A} := L^\infty(\Omega, \mathcal{F}, P)$, acting on the Hilbert space $L^2(\Omega, \mathcal{F}, P)$. For a real-valued random variable $X \in \mathcal{A}$, the probability measure μ_X coincides with the distribution of X: $\mu_X(B) = P(X \in B)$ for Borel sets B of $\mathbb{R}$. The set of self-adjoint random variables affiliated to $\mathcal{A}$ is now equal to the set of real-valued random variables.

1.2 Free and monotone independences

Independence can be understood as a rule for calculating mixed moments. We now define two kinds of independences. In probability theory, independence can be defined for σ-fields which correspond to $*$-subalgebras in non-commutative probability. Hence we formulate independence in terms of $*$-subalgebras.

Definition 1.2. Let $\{\mathcal{A}_i\}_{i\in I}$ be $*$-subalgebras of $\mathcal{A}$ containing the unit $1_{\mathcal{A}}$, where the index set I is arbitrary. They are said to be **free** (or free independent) if the following property holds:

(F) $\varphi(a_1 \cdots a_n) = 0$ if $i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{n-1} \neq i_n$, $a_k \in \mathcal{A}_{i_k}$ and $\varphi(a_k) = 0$ for any k.

Using this definition, we can calculate mixed moments of random variables which belong to different free subalgebras.

Example 1.3. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be free $*$-subalgebras of $\mathcal{A}$. For random variables $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$ and $d \in \mathcal{D}$, we have

$$\begin{aligned}
&\varphi(bc) = \varphi(b)\varphi(c), \quad \varphi(bcb') = \varphi(bb')\varphi(c), \\
&\varphi(bcb'c') = \varphi(bb')\varphi(c)\varphi(c') + \varphi(b)\varphi(b')\varphi(cc') - \varphi(b)\varphi(b')\varphi(c)\varphi(c'), \\
&\varphi(bcd) = \varphi(b)\varphi(c)\varphi(d), \quad \varphi(bcdc'b') = \varphi(bb')\varphi(cc')\varphi(d).
\end{aligned} \tag{1.1}$$

Now we define monotone independence, another famous independence in non-commutative probability theory.

Definition 1.4. Let I be a totally ordered set. A sequence of $*$-subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ is said to be **monotonically independent** if the following property holds:

(M) $\varphi(a_1 \cdots a_n) = \varphi(a_j)\varphi(a_1 \cdots a_{j-1}a_{j+1} \cdots a_n)$ if $a_k \in \mathcal{A}_{i_k}$, $1 \le k \le n$ and $i_{j-1} < i_j > i_{j+1}$ (if $j = 1$ or $j = n$, one of the inequalities are eliminated).

From now on we only consider $I = \mathbb{N} := \{1, 2, 3, \cdots\}$.

The definition of monotone independence shows asymmetry of independent subalgebras: the monotone independence of $(\mathcal{A}_1, \mathcal{A}_2)$ does not imply the monotone independence of $(\mathcal{A}_2, \mathcal{A}_1)$. Hence we need to consider a sequence of subalgebras, not a set of subalgebras.

Example 1.5. Let $(\mathcal{B}, \mathcal{C}, \mathcal{D})$ be monotonically independent $*$-subalgebras of $\mathcal{A}$. For random variables $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$ and $d \in \mathcal{D}$, we have

$$\begin{aligned} &\varphi(bc) = \varphi(b)\varphi(c), \quad \varphi(bcb') = \varphi(bb')\varphi(c), \\ &\varphi(cbc') = \varphi(c)\varphi(b)\varphi(c'), \quad \varphi(bcb'c') = \varphi(bb')\varphi(c)\varphi(c'), \\ &\varphi(bcd) = \varphi(b)\varphi(c)\varphi(d), \quad \varphi(bcdc'b') = \varphi(bb')\varphi(cc')\varphi(d). \end{aligned} \tag{1.2}$$

A difference can be observed in the calculations of $\varphi(bcb')$ and $\varphi(cbc')$. This reflects the asymmetry of monotone independence.

2 Generalization of free independence

The original definition of free independence requires every element to have a zero expectation in order that the product of elements has a zero expectation. However, Example 1.3 indicates that the assumption of the zero expectation of every element is too much. For example, if $\varphi(c) = 0$, then $\varphi(bcb') = 0$; the assumptions $\varphi(b) = \varphi(b') = 0$ are not needed; if $\varphi(b) = 0$, $\varphi(c) = 0$ or $\varphi(d) = 0$, then $\varphi(bcd) = 0$. Thus, we can weaken the assumptions on the conditions of zero expectations.

If $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ are free and $a_k \in \mathcal{A}_{i_k}$ $(i_1 \ne i_2, \cdots, i_{n-1} \ne i_n)$, a crucial structure in this article is the graph of $(i_k)_{k=1}^n$ as in Fig. 1. If we regard i_k as a function of k, then it attains local extrema at some points. For example in Fig. 1, the function i_k attains local extrema at $k = 1, 3, 6, 7, 8, 9, 10, 11, 12, 13$. Let $E(i_1, \cdots, i_n)$ denote the points at which the function i_k attains local extrema.[3] Then we have an equivalent definition of free independence.

Proposition 2.1. *Let $\mathcal{A}_i$ be $*$-subalgebras of $\mathcal{A}$ containing the unit $1_{\mathcal{A}}$. Then they are free if and only if the following holds:*

(F') $\varphi(a_1 \cdots a_n) = 0$ *if* $i_1 \ne i_2, i_2 \ne i_3, \cdots, i_{n-1} \ne i_n$, $a_k \in \mathcal{A}_{i_k}$ *for* $1 \le k \le n$ *and* $\varphi(a_k) = 0$ *for* $k \in E(i_1, \cdots, i_n)$.

Proof. The implication (F') $\Rightarrow$ (F) is immediate, and hence we will prove (F) $\Rightarrow$ (F'). Let us assume that $\mathcal{A}_i$ are free and $a_k \in \mathcal{A}_{i_k}$, $i_1 \ne i_2, i_2 \ne i_3, \cdots, i_{n-1} \ne i_n$ and $\varphi(a_k) = 0$ for $k \in E(i_1, \cdots, i_n)$. We have to prove that $\varphi(a_1 \cdots a_n) = 0$. If $k \notin E(i_1, \cdots, i_n)$, then $i_{k-1} < i_k < i_{k+1}$ or $i_{k-1} > i_k > i_{k+1}$. In both cases, let us decompose $a_{k-1}a_k a_{k+1}$ as $\varphi(a_k)a_{k-1}a_{k+1} + a_{k-1}(a_k - \varphi(a_k)1_{\mathcal{A}})a_{k+1}$. By iterating this procedure, we can write $a_1 \cdots a_n$

[3] We include the edge points $1, n$ in $E(i_1, \cdots, i_n)$.

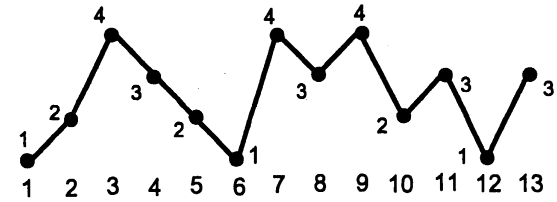

Figure 1: $i_1 = i_6 = i_{12} = 1$, $i_2 = i_5 = i_{10} = 2$, $i_4 = i_8 = i_{11} = i_{13} = 3$, $i_3 = i_7 = i_9 = 4$.

as a sum $\sum_{j=1}^{p} a_1^{(j)} \cdots a_{n_j}^{(j)}$, where $a_k^{(j)} \in \mathcal{A}_{i_k(j)}$ $(1 \le k \le n_j)$, $i_1(j) \ne i_2(j), \cdots, i_{n_j-1}(j) \ne i_{n_j}(j)$ and $\varphi(a_k^{(j)}) = 0$ for any j, k. Then the free independence implies $\varphi(a_1 \cdots a_n) = 0$. □

Take $i_k \in \mathbb{N}$, $k = 1, \cdots, n$ so that $i_1 \ne i_2, \cdots, i_{n-1} \ne i_n$. Now we know that the points where the function $k \mapsto i_k$ takes local extrema are important to define free independence. Moreover, such points can be classified into two subsets: the points at which i_k takes local maxima and the points at which i_k takes local minima.[4] Let us denote the former points by $\mathrm{Max}(i_1, \cdots, i_n)$ and the latter by $\mathrm{Min}(i_1, \cdots, i_n)$. In Fig. 1, $\mathrm{Max}(i_1, \cdots, i_{13}) = \{3, 7, 9, 11, 13\}$ and $\mathrm{Min}(i_1, \cdots, i_{13}) = \{1, 6, 8, 10, 12\}$.

To distinguish these two classes, we introduce another state ψ on $\mathcal{A}$ and arrive at a new definition of independence.

Definition 2.2. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $(\mathcal{A}_i)_{i \in \mathbb{N}}$ be a sequence of $*$-subalgebras of $\mathcal{A}$ containing the unit $1_{\mathcal{A}}$. Assume that there is another state ψ on $\mathcal{A}$. The sequence $(\mathcal{A}_i)$ is said to be **ordered free independent** if the following condition holds:

(OF) $\varphi(a_1 \cdots a_n) = 0, \psi(a_1 \cdots a_n) = 0$ if $i_1 \ne i_2, i_2 \ne i_3, \cdots, i_{n-1} \ne i_n$, $a_k \in \mathcal{A}_{i_k}$ for $1 \le k \le n$, $\varphi(a_k) = 0$ for $k \in \mathrm{Max}(i_1, \cdots, i_n)$ and $\psi(a_k) = 0$ for $k \in \mathrm{Min}(i_1, \cdots, i_n)$.

This definition enables us to calculate mixed moments by using both φ and ψ. Some examples are shown below.

Example 2.3. Let $(\mathcal{B}, \mathcal{C}, \mathcal{D})$ be ordered free independent. Then, for $b, b' \in \mathcal{B}, c, c' \in \mathcal{C}, d \in \mathcal{D}$,

$$\begin{aligned}
&\varphi(bc) = \varphi(b)\varphi(c), \quad \psi(bc) = \psi(b)\psi(c), \quad \varphi(bcd) = \varphi(b)\varphi(c)\varphi(d), \\
&\varphi(bcb') = \varphi(c)\varphi(bb'), \quad \varphi(cbc') = \psi(b)\varphi(cc') + \varphi(c)(\varphi(b) - \psi(b))\varphi(c'), \\
&\psi(bcb') = \varphi(c)\psi(bb') + \psi(b)(\psi(c) - \varphi(c))\psi(b'), \quad \psi(cbc') = \psi(b)\psi(cc'), \\
&\varphi(bcb'c') = \varphi(bb')\varphi(c)\varphi(c') + \varphi(b)\psi(b')\varphi(cc') - \varphi(b)\varphi(c)\psi(b')\varphi(c'), \\
&\varphi(cbc'b') = \varphi(cc')\psi(b)\varphi(b') + \varphi(c)\varphi(c')\varphi(bb') - \varphi(c)\psi(b)\varphi(c')\varphi(b'), \\
&\psi(bcb'c') = \psi(bb')\varphi(c)\psi(c') + \psi(b)\psi(b')\psi(bb') - \psi(b)\varphi(c)\psi(b')\psi(c'), \\
&\psi(cbc'b') = \psi(cc')\psi(b)\psi(b') + \psi(c)\varphi(c')\psi(bb') - \psi(c)\psi(b)\varphi(c')\psi(b').
\end{aligned} \tag{2.1}$$

[4]We include edge points also. For example, if $i_1 > i_2$, then $1 \in \mathrm{Max}(i_1, \cdots, i_n)$.

This new independence extends free independence and also monotone independence. A connection to free independence can be observed as follows: if we set $\varphi = \psi$, the above condition (OF) is equivalent to the free independence. For monotone independence, we have to assume an additional structure on the algebra $\mathcal{A}$.

Proposition 2.4. *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. If $\mathcal{A}$ has a decomposition $\mathcal{A} = \mathbb{C}1_{\mathcal{A}} \oplus \mathcal{A}^0$ as von Neumann algebras, we can define a canonical state $\delta(\lambda 1_{\mathcal{A}} + a^0) := \lambda$, where $\lambda \in \mathbb{C}$ and $a^0 \in \mathcal{A}^0$. Let $\mathcal{A}_i^0$ be $*$-subalgebras of $\mathcal{A}^0$. Then the sequence $(\mathcal{A}_i^0)_{i\in\mathrm{N}}$ with respect to φ is monotonically independent if and only if the sequence $(\mathbb{C}1_{\mathcal{A}} \oplus \mathcal{A}_i^0)_{i\in\mathrm{N}}$ is ordered free independent with respect to the states (φ, δ).*

Proof. We prove only one implication. Assume that the sequence $(\mathbb{C}1_{\mathcal{A}} \oplus \mathcal{A}_i^0)_{i\in\mathrm{N}}$ is ordered free independent with respect to the states (φ, δ). Take natural numbers $i_1, \cdots, i_n$ such that $i_1 \neq i_2, \cdots, i_{n-1} \neq i_n$ and $a_k \in \mathcal{A}_{i_k}^0$. If we use the associative law of ordered free independence [4], it is sufficient to consider only two subalgebras: $i_k \in \{1, 2\}$, $1 \leq k \leq n$.

If $i_k = 2$, let us define $a_k' := a_k - \varphi(a_k)1_{\mathcal{A}}$ and if $i_k = 1$, $a_k' := a_k$. Then we have $\varphi(a_1' \cdots a_n') = 0$. This implies that

$$\varphi(a_1 \cdots a_n) = \varphi\left(\prod_{k:i_k=1} a_k\right) \prod_{k:i_k=2} \varphi(a_k),$$

where the product symbol $\prod_i$ preserves the order of the index i. This means the monotone independence of $(\mathcal{A}_1^0, \mathcal{A}_2^0)$. □

Therefore, if one set $\psi(b) = \psi(c) = \psi(d) = \psi(b') = \psi(c') = 0$ in Example 2.3, results of Example 1.5 can be obtained. We mention below further information on ordered free independence. See [4] for detalis.

Remark 2.5. (1) The independence (OF) can be generalized more to include Boolean independence [1, 7]. In that case, we need three states.

(2) Ordered free independence of two subalgebras ($\mathcal{A}_1$, $\mathcal{A}_2$) is related to c-free independence [2, 3].

(3) We can formulate central limit theorem as follows. Let $(X_n)_{i\in\mathrm{N}}$ be a sequence of i.i.d. self-adjoint random variables with $\varphi(X_n) = \psi(X_n) = 0$ and $\varphi(X_n^2) = \alpha^2, \psi(X_n^2) = \beta^2$ (independence is in the sense of (OF)). Then the distribution of $\frac{X_1 + \cdots + X_N}{\sqrt{N}}$ converges to a limit distribution (μ, ν) with respect to the states (φ, ψ). In fact the limit distributions μ, ν are Kesten distributions.

(4) Anti-monotone independence [6] is also included: it is realized by the ordered free independence with respect to the states (δ, φ).

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