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LIMIT THEOREMS FOR MONOTONE CONVOLUTION

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ABSTRACT. We survey some recent progress in limit theorems for monotone convolution. This note is based on the author's lecture at the RIMS workshop.

1. STATEMENT OF THE PROBLEM

The *monotone convolution* $\triangleright$ is an associative binary operation on $\mathcal{M}$, the set of Borel probability measures on the real line $\mathbb{R}$. Introduced by Muraki in [21, 22], this operation is based on his notion of monotonic independence, which is one of the five natural quantum stochastic independences coming from universal products [27, 23]. (The others are tensor, free, Boolean, and antimonotonic independences.) We begin by reviewing the construction of $\triangleright$.

Consider $\mathcal{B}(H)$ the C^*-algebra of bounded linear operators on a separable Hilbert space H and a unit vector $\xi \in H$. Let φ be the vector state associated with the vector ξ; i.e., $\varphi(a) = \langle a\xi, \xi\rangle$ for each $a \in \mathcal{B}(H)$. Two $*$-subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{B}(H)$ are said to be *monotonically independent* (with respect to ξ) if for every mixed moment $\varphi(a_1a_2\cdots a_n)$ (i.e., $a_j \in \mathcal{A}_{i_j}$, $i_j \in \{1,2\}$, and $i_1 \neq i_2 \neq \cdots \neq i_n$), one has that

$$\varphi(a_1a_2\cdots a_n) = \varphi(a_j)\varphi(a_1\cdots a_{j-1}a_{j+1}\cdots a_n) \tag{1.1}$$

whenever $a_j \in \mathcal{A}_2$.

Remark 1. Note first that the monotonic independence of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ does not necessarily imply the monotonic independence of $\mathcal{A}_2$ and $\mathcal{A}_1$. Secondly, monotonically independent subalgebras are not unital in general. For instance, if $\mathcal{A}_1$ contains the identity operator I on H, then the restriction of the sate φ on the algebra $\mathcal{A}_2$ has to be a homomorphism by (1.1), which is often not the case.

By a (noncommutative) *random variable* we mean a possibly unbounded self-adjoint operator X on the Hilbert space H. Let E_X be the spectral measure of X. The *distribution* μ_X of X is the Borel probability measure on $\mathbb{R}$ given by the composition $\mu_X = \varphi\circ E_X$. More generally, the distribution of an essentially self-adjoint operator X means the distribution of its operator closure $\overline{X}$.

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Two random variables X_1 and X_2 are said to be monotonically independent if the algebras $\mathcal{A}_i = \{f(X_i) : f \in C_b(\mathbb{R}),\ f(0) = 0\}$, $i = 1, 2$, are monotonically independent, where $C_b(\mathbb{R})$ is the algebra of bounded continuous functions from $\mathbb{R}$ to $\mathbb{C}$, and the normal operator $f(X) \in \mathcal{B}(H)$ is obtained via the functional calculus of spectral theory.

Given two measures $\mu, \nu \in \mathcal{M}$, their monotone convolution is constructed as follows. Consider the space $H = L^2(\mathbb{R} \times \mathbb{R}, \mu \otimes \nu)$ and the vector state $\varphi(\cdot) = \langle \cdot \mathbf{1}, \mathbf{1} \rangle$ on $\mathcal{B}(H)$, where $\mathbf{1}$ denotes the constant function $\mathbb{R}^2 \ni (x, y) \mapsto 1$. Let $\mathrm{Dom}(X)$ be the set of all functions $\psi \in H$ such that

$$\int_{-\infty}^{\infty} x^2 \left| \int_{-\infty}^{\infty} \psi(x,t)\, d\nu(t) \right|^2 d\mu(x) < \infty,$$

and let $\mathrm{Dom}(Y)$ be the set of all $\psi \in H$ so that the function $y\psi(x, y)$ is in H. For $\psi_1 \in \mathrm{Dom}(X)$ and $\psi_2 \in \mathrm{Dom}(Y)$, we introduce the self-adjoint operators X and Y by

$$X\psi_1(x,y) = x \int_{-\infty}^{\infty} \psi_1(x,t)\, d\nu(t) \qquad \text{and} \qquad Y\psi_2(x,y) = y\psi_2(x,y).$$

In this case we have $\mu_X = \mu$ and $\mu_Y = \nu$. Also, the sum $X + Y$ is densely defined and symmetric.

By a result of Franz [13], the random variables X and Y are monotonically independent with respect to $\mathbf{1}$, and the operator $X + Y$ is essentially self-adjoint. Thus it makes sense to give the following

Definition 1. The monotone convolution $\mu \triangleright \nu$ for two measures $\mu, \nu \in \mathcal{M}$ is defined as the distribution of $X + Y$.

Note that if μ and ν are compactly supported probability measures, then it is easy to see that both X and Y are actually bounded operators, and hence the probability measure $\mu \triangleright \nu$ is also compactly supported.

Example 1. [21, 22] Denote by δ_c the Dirac point mass at $c \in \mathbb{R}$, and by γ the standard arcsine law whose density is $\pi^{-1}(2 - x^2)^{-1/2}$ on the interval $(-\sqrt{2}, \sqrt{2})$. For $\mu \in \mathcal{M}$, its *dilation* $D_b\mu$ by a factor $b > 0$ is defined by $D_b\mu(A) = \mu(b^{-1}A)$ for Borel subsets $A \subset \mathbb{R}$. Note that if a random variable X has distribution μ, then the scalar product bX has distribution $D_b\mu$.

(1) For $a \in \mathbb{R}$, the measure $\mu \triangleright \delta_a$ is a translation of μ, i.e., $d\mu \triangleright \delta_a(t) = d\mu(t - a)$.

(2) Let $\mathcal{S}$ be the standard semicircular law with density $\sqrt{4 - x^2}/2\pi$ on the interval $[-2, 2]$. Then we have

$$[(\delta_{-1} + \delta_1)/2] \triangleright \mathcal{S} = \gamma \triangleright \gamma = D_{\sqrt{2}}\, \gamma.$$

The definition of the measure $\mu \triangleright \nu$ does not rely on the particular realization of the variables X and Y. Precisely, let X_1 and Y_1 be two random variables on some Hilbert space H_1 such that X_1 and Y_1 are monotonically independent with respect to a unit vector

$\xi \in H_1$, and $\mu_{X_1} = \mu$, $\mu_{Y_1} = \nu$. Discarding an irrelevant subspace if necessary, we assume further that the vector ξ is cyclic for the algebra generated by X_1 and Y_1; i.e.,

$$\overline{\mathrm{alg}\{f(X_1),\, f(Y_1) : f \in C_b(\mathbb{R})\}\xi} = H_1.$$

Then it was proved in [13] that there exists a unitary map $U : H \to H_1$ such that $U\mathbf{1} = \xi$, $X_1 U = UX$, and $Y_1 U = UY$. Moreover, the operator $X_1 + Y_1$ is essentially self-adjoint and has distribution $\mu \triangleright \nu$.

We say of arbitrary probability measures μ_n and μ on $\mathbb{R}$ that μ_n *converges weakly* to μ, which we indicate by writing $\mu_n \Rightarrow \mu$, if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, d\mu_n(t) = \int_{-\infty}^{\infty} f(t)\, d\mu(t)$$

for every $f \in C_b(\mathbb{R})$. The limit distributional theory for sums of monotonically independent random variables is concerned with the study of the following

Problem 1. Let k_n be a sequence of positive integers, and let $\{\mu_{nj} : n \geq 1,\ 1 \leq j \leq k_n\}$ be an *infinitesimal* triangular array of probability measures on $\mathbb{R}$, that is, to each $\varepsilon > 0$ one has

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} \mu_{nj}(\{t \in \mathbb{R} : |t| \leq \varepsilon\}) = 1. \tag{1.2}$$

Suppose that the measures

$$\mu_{n1} \triangleright \mu_{n2} \triangleright \cdots \triangleright \mu_{nk_n}, \qquad n \geq 1, \tag{1.3}$$

converge weakly to a measure $\nu \in \mathcal{M}$. It is asked what properties this limit law ν must possess, and when does such a convergence take place?

The motivation behind Problem 1 comes from the most general setting for limit theorems of sums of independent infinitesimal (commuting) random variables. The condition (1.2) of infinitesimality is introduced to exclude the possibility that in each row one single measure μ_{nj} plays the dominating role. Denote by $\mu * \nu$ the classical convolution for measures $\mu, \nu \in \mathcal{M}$; or, in probabilistic terms, $\mu * \nu$ stands for the distribution of $X+Y$, where X and Y are two independent real-valued random variables with distributions μ and ν, respectively. If one replaces the monotone convolution $\triangleright$ by the classical convolution $*$ in (1.3), then the same questions asked in Problem 1 have been answered completely by the work of Lévy, Khintchine, Kolmogorov, and others . It turns out that in the classical case if for a suitable choice of constants $a_n \in \mathbb{R}$ the measures $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ converge weakly to a law ν, then the law ν has to be $*$-*infinitely divisible*, i.e., to each $k \geq 1$ there exists a measure $\nu_k \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_k * \nu_k * \cdots * \nu_k}_{k \text{ times}}.$$

Conversely, any infinitely divisible law can be realized as the weak limit for an infinitesimal array of probability measures. Necessary and sufficient conditions for the convergence of $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ to a specific infinitely divisible law are also known; in particular, when the limit is the Gaussian distribution (resp., the point mass) these conditions imply the central limit theorem (resp., the weak law of large numbers). We refer to the monograph of Gnedenko and Kolmogorov [15] for the details.

In the context of Voiculescu's free probability, the analogous free convolutions

$$\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$$

have been also the subject of several investigations. In a striking contribution [4] Bercovici and Pata proved, in case $a_n = 0$ and $\mu_{n1} = \mu_{n2} = \cdots = \mu_{nk_n}$, that the measures $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ have a weak limit if and only if the measures $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ do. This convergence result is referred as the *Bercovici-Pata Bijection*, for it establishes a one to one correspondence between the free and classical limit laws for an infinitesimal array of measures with identical rows. Moreover, the free limit laws are infinitely divisible [5] and are related to the classical limit laws through a quite explicit formula [6]. In particular, the bijection shows that the free and classical domains of partial attraction for infinitely divisible laws coincide, as well as the free and classical domains of attraction for stable laws. The Bercovici-Pata bijection was extended to arbitrary arrays and centering constants a_n by Chistyakov and Götze in [11] (see also [7] for a different approach).

Clearly, a monotonic analogue of these convergence results will provide a full solution to Problem 1. To the author's best knowledge, the literature lacks a general treatment of limit theorems for monotone convolution; results like the Bercovici-Pata bijection or the characterization of infinitely divisible laws as weak limits of infinitesimal arrays are not available at this point. Nevertheless, in what follows we shall survey some results proved for certain arrays with identical rows.

2. RESULTS FOR IDENTICAL SUMMANDS

In this section we are concerned with the study of limit laws for the measures

$$\mu_n = \underbrace{D_{1/B_n}\mu \rhd D_{1/B_n}\mu \rhd \cdots \rhd D_{1/B_n}\mu}_{n \text{ times}}, \tag{2.1}$$

where $\mu \in \mathcal{M}$ and B_n is a positive sequence. This pattern of convergence corresponds to the limit theorems for sums of monotonically independent and identically distributed random variables. Thus, we are dealing with a triangular array $\{\mu_{nj}\}_{n,j}$ of the form: $k_n = n$ and $\mu_{nj} = D_{1/B_n}\mu$ for $j = 1, \cdots, n$. If $B_n \to \infty$, then the array is infinitesimal. Moreover, the following result shows that the infinitesimality of $\{\mu_{nj}\}_{n,j}$ is always guaranteed whenever there is a nonzero weak limit for the sequence μ_n.

Proposition 1. [28] *Let ν be a measure in $\mathcal{M}$ with $\nu \neq \delta_0$, and let μ_n be defined as in* (2.1). *If the weak convergence $\mu_n \Rightarrow \nu$ holds for some constants $B_n > 0$, then we must have* $\lim_{n\to\infty} B_n = \infty$.

In the sequel the symbol $\mu^{\triangleright n}$ denotes the *n-th monotone convolution power* $\mu \triangleright \mu \triangleright \cdots \triangleright \mu$ of a measure $\mu \in \mathcal{M}$, and the n-fold classical convolution μ^{*n} is defined analogously. Note that we have $D_b(\mu \triangleright \nu) = D_b\mu \triangleright D_b\nu$ for any $\mu, \nu \in \mathcal{M}$. Thus, (2.1) becomes $D_{1/B_n}\mu^{\triangleright n}$.

2.1. **Central limit theorem.** The earliest limit theorem for (2.1) was an analogue of the central limit theorem (CLT) proved by Muraki [22], where the support of the measure μ was assumed to be bounded and the limit law was the standard arcsine law γ. The result below shows that the monotonic CLT actually holds under the same conditions as the classical CLT. Recall that the centered measure $\mu * \delta_a = \mu \triangleright \delta_a$ means a shift of μ by the amount of a, and that a probability measure μ is said to be *nondegenerate* if $\mu \neq \delta_a$ for $a \in \mathbb{R}$.

Theorem 1. [29] (Monotone CLT) *Let μ be any nondegenerate probability measure on $\mathbb{R}$, and let $a \in \mathbb{R}$ and $b > 0$. Then the following statements are equivalent:*

(1) *the weak convergence $D_{1/b\sqrt{n}}(\mu \triangleright \delta_{-a})^{\triangleright n} \Rightarrow \gamma$ holds;*
(2) *the measure μ has finite variance.*

If (1) *and* (2) *are satisfied, then the constants a and b can be chosen as a to be the mean of the measure μ and b to be the standard deviation of μ.*

In particular, denoting by $\mathcal{N}$ the standard Gaussian law, for a nondegenerate measure μ with finite mean a and standard deviation b Theorem 1 shows that the weak convergences

$$D_{1/b\sqrt{n}}(\mu \triangleright \delta_{-a})^{\triangleright n} \Rightarrow \gamma \quad \text{and} \quad D_{1/b\sqrt{n}}(\mu * \delta_{-a})^{*n} \Rightarrow \mathcal{N}$$

are equivalent.

Note that one has the obvious identity

$$D_{1/b\sqrt{n}}(\mu * \delta_{-a})^{*n} = \delta_{-a\sqrt{n}/b} * D_{1/b\sqrt{n}}\mu^{*n} = D_{1/b\sqrt{n}}\mu^{*n} * \delta_{-a\sqrt{n}/b},$$

because $\mu * \delta_{-a} = \delta_{-a} * \mu$. In monotone probability theory, however, we have in general $\mu \triangleright \delta_c \neq \delta_c \triangleright \mu$ (see [22, 13]), and hence it is not always possible to write $D_{1/b\sqrt{n}}(\mu \triangleright \delta_{-a})^{\triangleright n}$ as $\delta_{-a\sqrt{n}/b} \triangleright D_{1/b\sqrt{n}}\mu^{\triangleright n}$ or $D_{1/b\sqrt{n}}\mu^{\triangleright n} \triangleright \delta_{-a\sqrt{n}/b}$. This phenomenon reflects the facts that the monotonic independence does not behave well with respect to the centering process of measures and that it is a notion depending on the order of subalgebras, as indicated in Remark 1. From this perspective, the theory of stable laws in classical probability does not seem to have a good analogue in monotone probability. Theorem 1 can be generalized further to include measures without finite variance, see Theorem 3 below.

2.2. **Strictly stable laws.** Let $\mu, \nu \in \mathcal{M}$. We say that μ is *of the same strict type* as ν if $\mu = D_b\nu$ for some constant $b > 0$ (and we write $\mu \sim \nu$). The relation $\sim$ is an equivalence relation for measures in $\mathcal{M}$, and hence the set $\mathcal{M}$ partitions into disjoint classes of measures belonging to the same strict type. The degenerate measures constitute three strict types: those at negative points, those at positive points, and the single delta measure at 0.

The self-reproducing property of the arcsine law γ described in Example 1 suggests our next definition.

Definition 2. [28] A law $\nu \in \mathcal{M} \setminus \{\delta_0\}$ is said to be *$\triangleright$-strictly stable* if $\mu_1 \triangleright \mu_2 \sim \nu$ whenever $\mu_1 \sim \nu \sim \mu_2$. In other words, ν is $\triangleright$-strictly stable if and only if for arbitrary positive a and b there exists $c > 0$ such that $D_a\nu \triangleright D_b\nu = D_c\nu$.

The analogous $*$-strict stability was introduced and studied thoroughly by Lévy in his 1925 monograph [20]. He made the first fundamental step toward understanding the role of strictly stable laws in limit theorems. Precisely, Lévy proved that the limit law for $D_{1/B_n}\mu^{*n}$ must be $*$-strictly stable, and conversely, any $*$-strictly stable law can be realized as a limit law in this way. These limit theorems motivate the concept below.

Definition 3. [28] Let ν be a measure in $\mathcal{M} \setminus \{\delta_0\}$. We say that a measure $\mu \in \mathcal{M}$ is *strictly attracted* to the law ν if there exist constants $B_n > 0$ such that the weak convergence $D_{1/B_n}\mu^{\triangleright n} \Rightarrow \nu$ holds. The set of all probability measures that are strictly attracted to ν is called the *strict domain of attraction* of ν and is denoted by $\mathcal{D}_{\triangleright}[\nu]$.

The strict domain of attraction $\mathcal{D}_{*}[\nu]$ relative to the convolution $*$ is defined analogously. Of course, Definition 3 could be extended to accommodate the case of δ_0. Indeed, we will do so when we treat the weak law of large numbers in Subsection 2.3. Here we shall require the limit to be different from δ_0, and we have the following Lévy type characterization for $\triangleright$-strictly stable laws.

Theorem 2. [28] *Given $\nu \in \mathcal{M}$ with $\nu \neq \delta_0$, the following statements are equivalent:*

(1) *for each positive integer k, the measure $\nu^{\triangleright k}$ is of the same strict type as ν;*
(2) *there exist $\mu \in \mathcal{M}$ and constants $B_n > 0$ such that $D_{1/B_n}\mu^{\triangleright n} \Rightarrow \nu$;*
(3) *the measure ν is $\triangleright$-strictly stable.*

Moreover, if these equivalent conditions are satisfied, then associated with ν there exists a unique number $\alpha \in (0, 2]$ such that

$$\nu^{\triangleright k} = D_{k^{1/\alpha}}\nu, \qquad k \geq 1,$$

$$D_a\nu \triangleright D_b\nu = D_{(a^\alpha + b^\alpha)^{1/\alpha}}\nu, \qquad a, b > 0. \tag{2.2}$$

Thus, just like in the classical case, $\triangleright$-strictly stable laws, and only these, can appear as the limit distributions for $D_{1/B_n}\mu^{\triangleright n}$. We shall call the number α the *stability index* for the strictly stable law. The strict type of the arcsine law γ is the only strict type of $\triangleright$-strictly stable laws with index $\alpha = 2$. Similarly, in the usual probability, any $*$-strictly stable law of index 2 is of the same strict type as the Gaussian law $\mathcal{N}$.

Remark. All possible norming constants B_n in Theorem 2 (2) are also characterized in [28]. The sequence B_n, as a function on $\mathbb{N}$, extends to a *regularly varying* function $B(x)$ on $(0, \infty)$ with index $1/\alpha$ (i.e., $\lim_{x\to\infty} B(x)^{-1}B(cx) = c^{1/\alpha}$ for every constant $c > 0$). By Karamata's theory of regular variation [9], one obtains an integral representation:

$$B(x) = x^{1/\alpha}c(x)\exp\left(\int_1^x t^{-1}\varepsilon(t)\,dt\right), \qquad x \geq 1,$$

where $c(x)$ and $\varepsilon(x)$ are measurable and $c(x) \to c \in (0, +\infty)$, $\varepsilon(x) \to 0$ as $x \to \infty$. It is worth mentioning that this result also has a classical counterpart, namely, if the measures $D_{1/B_n}\mu^{*n}$ converge weakly to a $*$-strictly stable law ν, then the sequence B_n extends to a regularly varying function on $(0, \infty)$ (see [10]).

One of the fundamental problems in the study of strictly stable laws should be the determination of their strict domains of attraction. Here we present a complete solution for the arcsine law γ, which corresponds to the most general form of CLT for identical summands. The strict domain of attraction $\mathcal{D}_\triangleright[\gamma]$ is characterized completely in [29], and surprisingly, the set $\mathcal{D}_\triangleright[\gamma]$ coincides with the classical strict domain of attraction for the Gaussian law $\mathcal{N}$. To explain this result in detail, we first recall that $f : (0, \infty) \to (0, \infty)$ is a *slowly varying* function if $\lim_{x\to\infty} f(x)^{-1}f(cx) = 1$ for every $c > 0$.

Theorem 3. [29] (General Monotone CLT) *A measure $\mu \in \mathcal{M}$ is in $\mathcal{D}_\triangleright[\gamma]$ if and only if μ belongs to $\mathcal{D}_*[\mathcal{N}]$ if and only if μ has mean zero and its truncated variance*

$$H_\mu(x) = \int_{-x}^{x} t^2\,d\mu(t), \qquad x > 0,$$

is slowly varying.

This result implies immediately that $D_{1/B_n}\mu^{\triangleright n} \Rightarrow \gamma$ for some constants $B_n > 0$ if and only if $D_{1/C_n}\mu^{*n} \Rightarrow \mathcal{N}$ for some $C_n > 0$. We remark here that we can actually choose the same constants for both weak convergences; precisely, we can take $B_n = C_n$ to be the classical cutoff constants $\inf\{y > 0 : nH_\mu(y) \leq y^2\}$ (see [12], Section IX.8).

Finally, the Bercovici-Pata bijection gives us the following result.

Corollary 1. *One has that $\mathcal{D}_\triangleright[\gamma] = \mathcal{D}_*[\mathcal{N}] = \mathcal{D}_\boxplus[\mathcal{S}]$.*

Here $\mathcal{S}$ is the standard semicircle law, and the symbol $\mathcal{D}_\boxplus[\mathcal{S}]$ means its free strict domain of attraction.

2.3. **Weak law of large numbers.** We now address the issue of convergence to the point masses, that is, the law of large numbers. Let μ be a probability measure on $\mathbb{R}$, and let

$\{b_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $b_1 \leq b_2 \leq \cdots$ and $\lim_{n\to\infty} b_n = \infty$. The classical counterpart of the following theorem was found by Kolmogorov for the special case $b_n = n$ and by Feller for arbitrary sequence $\{b_n\}_{n=1}^{\infty}$ (see [15, 12]).

Theorem 4. [28] (WLLN) *Let $a \in \mathbb{R}$. We shall have*

$$D_{1/b_n}\mu^{\triangleright n} \Rightarrow \delta_a$$

if and only if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{nb_nt}{b_n^2+t^2}\,d\mu(t) = a \qquad \textit{and} \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{nt^2}{b_n^2+t^2}\,d\mu(t) = 0. \tag{2.3}$$

When a measure $\mu \in \mathcal{M}$ has finite mean a, Theorem 4 shows that the monotone convolutions $D_{1/n}\mu^{\triangleright n}$ converge weakly to δ_a, which justifies the name law of large numbers. Apparently, Theorem 4 can also be applied to certain measures without expectation, and the condition (2.3) shows us how to select the norming constants in order to obtain the weak convergence. For instance, if μ is purely atomic with $\mu(\{2^k\}) = 2^{-k}$ for $k \geq 1$ (The St. Petersburg Game), then (2.3) implies that

$$D_{1/(n\,\mathrm{Log}\,n)}\mu^{\triangleright n} \Rightarrow \delta_1,$$

where $\mathrm{Log}\,n$ is the logarithm of n to the base 2. In other words, a law of large numbers still exists, but, with a different normalization.

Theorem 4 gives a complete description of the strict domain of attraction for a degenerate limit type. Here is another surprise. By the Bercovici-Pata bijection, the convergence condition (2.3) is equivalent to the weak convergence

$$\underbrace{D_{1/b_n}\mu * D_{1/b_n}\mu * \cdots * D_{1/b_n}\mu}_{n\text{ times}} \Rightarrow \delta_a$$

or

$$\underbrace{D_{1/b_n}\mu \boxplus D_{1/b_n}\mu \boxplus \cdots \boxplus D_{1/b_n}\mu}_{n\text{ times}} \Rightarrow \delta_a.$$

In particular, we obtain the following

Corollary 2. *A degenerate measure has the same classical, free, and monotonic strict domains of attraction.*

3. PROOFS AND OPEN QUESTIONS

Results in the preceding section support the existence of the Bercovici-Pata type convergence result between $\triangleright$ and $*$. Therefore, it is natural to ask:

Problem 2. Let $\alpha \in (0, 2)$, and let $\nu_\triangleright$ and ν_* be two nondegenerate strictly stable laws of index α relative to the convolutions $\triangleright$ and $*$, respectively. Do we always have $\mathcal{D}_\triangleright[\nu_\triangleright] = \mathcal{D}_*[\nu_*]$?

This question remains unsolved. Some necessary conditions for a measure μ to belong to a strict domain of attraction were obtained in [28].

Theorem 5. [28] *Let ν be a nondegenerate $\rhd$-strictly stable law of index $\alpha \in (0,2)$. If a measure $\mu \in \mathcal{M}$ is strictly attracted to the law ν, then the integral*

$$\int_{-\infty}^{\infty} |t|^p \, d\mu(t) \begin{cases} < \infty & \text{if } 0 \le p < \alpha; \\ = \infty & \text{if } p > \alpha. \end{cases} \tag{3.1}$$

Since every $\rhd$-strictly stable law belongs to its own strict domain of attraction, a $\rhd$-strictly stable law of index $\alpha > 1$ has finite mean, and among all $\rhd$-strictly stable laws only the arcsine law ($\alpha = 2$) has finite variance. For $0 < \alpha \le 1$, the nondegenerate $\rhd$-strictly stable laws have neither mean nor variance. No sufficient conditions are known for strict attraction to a $\rhd$-strictly stable law. (The paper [17] shows a weak convergence to the Cauchy law for the monotone convolutions $D_{1/n}\mu^{\rhd n}$.) Finally, it is well known that a nondegenerate $*$-strictly stable law of index $\alpha \in (0,2)$ also satisfies the moment condition (3.1) (see [12, Chapter VIII]).

Most proofs of limit theorems for monotone convolution in the literature are of combinatorial nature [22, 26, 18]. This is because the computation of monotone convolution of measures involves the composition of analytic functions in the complex upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$. Precisely, the *Cauchy transform* of a measure $\mu \in \mathcal{M}$ is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} \, d\mu(t), \qquad z \in \mathbb{C}^+,$$

so that the reciprocal Cauchy transform $F_\mu = 1/G_\mu$ is an analytic self-map of $\mathbb{C}^+$. Since the imaginary part of $-G_\mu$ is the Poisson integral of the measure μ up to a scalar multiple, the measure μ is completely determined by its Cauchy transform G_μ (and hence by the function F_μ). Given two measures $\mu, \nu \in \mathcal{M}$, we have that

$$F_{\mu \rhd \nu}(z) = F_\mu(F_\nu(z)), \qquad z \in \mathbb{C}^+. \tag{3.2}$$

(See [21, 3, 13] for the proof.)

Weak convergence of probability measures is equivalent to the pointwise convergence for their F-functions (e.g., see [14]). Thus, understanding the distributional behavior of the measures $\mu_1 \rhd \mu_2 \rhd \cdots \rhd \mu_n$ amounts to the understanding of the limiting behavior of the compositions $F_{\mu_1} \circ F_{\mu_2} \circ \cdots \circ F_{\mu_n}$. In the case of identical summands, this is reduced to the study of iterations $\{F_\mu^{\circ n}\}_{n=1}^{\infty}$ on $\mathbb{C}^+$.

When the measure μ has a bounded support (meaning that it can be realized as a distribution of a bounded random variable), the Cauchy transform G_μ has a power series expansion at ∞:

$$G_\mu(z) = 1/z + m_1/z^2 + m_2/z^3 + \cdots,$$

where m_n means the n-th moment of μ. Then (3.2) becomes merely a composition of power series, and the combinatorial approach to limit theorems seems natural in this case. Indeed, methods based on the monotonic independence (1.1) and the combinatorics of non-crossing partitions had been developed and used to prove the monotone CLT and the Poisson type limit theorem [22, 26, 18]. This approach has the advantage that it can treat limit theorems for operator-valued random variables, as shown in [25].

However, the combinatorial approach is not suitable for general measures. In fact, the proofs of the results in Section 2 do not make use of the combinatorics of monotone convolution at all. They are based on the free harmonic analysis tools developed in [6]. A key ingredient is the adoption of the Bernstein blocking technique from classical probability (see the book [10] for a full account of this technique).

Finally, we return to the class of infinitely divisible laws. Carrying the analogy with $*$-infinite divisibility, a measure $\nu \in \mathcal{M}$ is said to be $\triangleright$-*infinitely divisible* if for each positive integer k, there exists a measure $\nu_k \in \mathcal{M}$ such that $\nu = \nu_k^{\triangleright k}$. Thus, Theorem 2 (1) shows that every $\triangleright$-strictly stable law is $\triangleright$-infinitely divisible. In addition, given a $\triangleright$-strictly stable law ν of index α, let us introduce the measures

$$\nu_t = D_{t^{1/\alpha}}\nu, \qquad t > 0,$$

and $\nu_0 = \delta_0$. Then, by (2.2), we have $\nu_s \triangleright \nu_t = \nu_{s+t}$ for $s, t \geq 0$; and hence the family $\{\nu_t\}_{t\geq 0}$ forms a *convolution semigroup*. Also, note that the map $t \mapsto \nu_t$ is weakly continuous. Consequently, the family $\{F_{\nu_t}\}_{t\geq 0}$ of the corresponding reciprocal Cauchy transforms forms a *composition semigroup* of analytic maps from $\mathbb{C}^+$ into itself (cf. [8]).

In general, every infinitely divisible measure embeds into a unique weakly continuous convolution semigroup (see [21, 22] and [2]). Thus, by taking Theorem 2 (1) as the definition of $\triangleright$-strict stability and using the theory of composition semigroups, it is proved in [16] that for a $\triangleright$-strictly stable law ν of index $\alpha \in (0, 2]$, one has

$$F_\nu(z) = (z^\alpha + w)^{1/\alpha}, \qquad z \in \mathbb{C}^+.$$

Here the power $z^p = \exp(p \log z)$ is defined in $\mathbb{C} \setminus [0, \infty)$, where the range of the argument of z is chosen to be $0 < \arg z < 2\pi$. The complex number w satisfies the conditions: (i) $0 \leq \arg w \leq \alpha\pi$ if $\alpha \in (0, 1]$; (ii) $(\alpha - 1)\pi \leq \arg w \leq \pi$ if $\alpha \in (1, 2]$.

Unlike in the usual probability theory, we know every little about the connections of $\triangleright$-infinitely divisible measures with limit theorems of monotone convolution. To illustrate, recall the result of Lêvy that the weak limit for $\mu_n = D_{1/B_n}\mu^{*n}$ must be $*$-strictly stable. It could happen that the measures μ_n do not converge for any choice of the constants B_n, but that for some subsequence $n_1 < n_2 < \cdots < n_k < \cdots$ a weak convergence holds. From the general theory described in Section 1, we only know that this limit distribution is necessarily $*$-infinitely divisible. In this regard, Khintchine [19] proved a rather deep converse proposition, which says that every $*$-infinitely divisible law can appear as the weak limit for μ_{n_k}.

We shall say that a law $\mu \in \mathcal{M}$ belongs to the $\triangleright$-*strict domain of partial attraction* of $\nu \in \mathcal{M}$ if there exists a subsequence $n(k)$, $k \geq 1$, such that the weak convergence $D_{1/B_{n(k)}}\mu^{\triangleright n(k)} \Rightarrow \nu$ $(k \to \infty)$ holds for suitably chosen constants $B_n > 0$. We pose the following open question, which may serve as a starting point for the further investigation of $\triangleright$-strict domains of partial attraction.

Problem 3. Does every $\triangleright$-infinitely divisible law have a (non-empty) $\triangleright$-strict domain of partial attraction?

Note that every $\boxplus$-infinitely divisible law does have a non-empty $\boxplus$-strict domain of partial attraction, see [24].

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