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Nonsmooth Minimax Fractional Programming Problem with Exponential (p,r)-invexity*

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Abstract

We consider a nonsmooth minimax fractional programming problem under the exponential (p,r)-invexity.

Moreover, we establish the necessary and sufficient optimality conditions in minimax fractional programming problem involving exponential (p,r)-invex functions. By employing the optimality conditions, we constitute a parametric type dual model and prove that the duality theorems hold under the exponential (p,r)-invexity. The optimal value of the dual problem is equal to the optimal value of primary problem under the scheme of exponential (p,r)-invexity.

Key words: minimax fractional programming, locally Lipschitz function, generalized subdifferential, exponential (p,r)-invex function, optimality conditions, duality.

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1 Introduction

Convexity plays an important role for sufficient optimality theorems in programming problems. Invariant convexity will be regarded as invexity. In this note, we would like to consider a nonsmooth minimax fractional programming involving a kind of generalized invexity.

Throughout this paper, we let $X(\subset \mathbb{R}^n)$, $Y(\subset \mathbb{R}^m)$ and $(Z,C) \subset \mathbb{R}^p$ be separable reflexive Banach spaces with any norm, where C denotes a closed convex pointed cone used as an ordered cone in Z. Consider a minimax fractional programming problem as follows:

$$(P) \qquad \min_{x\in X} \sup_{y\in Y} \frac{f(x,y)}{g(x,y)} \equiv \varphi(x,y)$$

$$\begin{aligned} \text{subject to } & X = \{x \in \mathbb{R}^n \mid h(x) \in -C\}, \\ & C\text{: an ordered convex cone in } Z \subset \mathbb{R}^p, \\ & h\text{: } X \longrightarrow (Z,C), \\ & Y\text{: a given compact space in } \mathbb{R}^m. \end{aligned}$$

Without loss of generality, we may assume that $f(\cdot,\cdot) \geq 0$, $g(\cdot,\cdot) > 0$ and suppose that $f(\cdot,\cdot), g(\cdot,\cdot) \in C(X \times Y, \mathbb{R})$ are continuous functions on $X \times Y$. Furthermore, for each $y \in Y$, functions $f(\cdot,y), g(\cdot,y)$ and $h(\cdot)$ are locally Lipschitz functions. We will establish the sufficient optimality conditions involving exponential (p,r)-invex (Lipschitz) functions.

In a programming problem, if one has the necessary optimality conditions, then the existence of solution follows from the converse of necessary condition with extra assumptions; that is, we prove that there is no duality gap between the duality problem and primal problem with a view to searching the reasonable conditions. The main task of this note is to give the process to reach the solution is optimal.

Using the necessary optimality conditions with some extra assumptions to deduce sufficient optimality conditions are not unique.

Thus the sufficient optimality conditions are various depending on the extra assumptions, it may be including: differentiability, convexity, generalized convexity, as well as invexity and generalized invexity ... etc., which are often constituted by many authors as extra assumptions to smooth their theory in mathematical analysis from difference points of view.

After the existence of optimality solution is approved, it is naturally to employ the optimality conditions to investigate the duality models, and prove the duality theorems. A question arises to ask that whether the duality problem has the same optimal with the primal problem.

In order to get the sufficient optimality conditions, we will employ the exponential (p,r)-invexity to establish the existence of solutions for Problem (P). It is motivated from Antczak [1] in the case of differentiable function.

2 Exponential (p,r)-invexity

Invex means invariant convex. For a convex differentiable function $f : S \subset X \longrightarrow \mathbb{R}$, where S is convex and open, by Mean Value Theorem (cf. Carven [6]), we see that:

$$f(x) - f(u) \geq \nabla f(u)(x-u), \ \forall x \in S, \ \nabla f(u) \in X^*. \tag{2.1}$$

Suppose there exists a mapping $\eta : X \times X \longrightarrow X$ with property $\eta(x,u) = 0$ only if $x = u$. The differentiable function f is called η-invex at u if

$$f(x) - f(u) \geq \nabla f(u)\eta(x,u). \tag{2.2}$$

If the right-hand side of inequality (2.2) is replaced by

$$f(x) - f(u) \geq F(x,u;\nabla f(u)) \text{ for any } x \in X, \tag{2.3}$$
where $F : X \times X \times X^* \longrightarrow \mathbb{R}$ is sublinear on X^*,

we say that f is F-convex at u. (cf. Hanson and Mond [7])
(F-convex : F is sublinear with respect to the third variable.)

Definition 1. *The differentiable function f is said to be exponential (p,r)-invex with respect to η if (cf. Antczak [1])*

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1+\frac{r}{p}\nabla f(u)\left(e^{p\eta(x,u)}-\mathbf{1}\right)\right], \tag{2.4}$$

where $\mathbf{1}=(1,1,\cdots,1)\in\mathbb{R}^n$.

If f is not differentiable, we modify (2.4) by assuming that f is locally Lipschitz, and ∇f is replaced (cf. Clarke [5]) by generalized subdifferential $\partial^c f(u)$ of f at u. We call the exponential (p,r)-invex as the next definition.

Definition 2. *The locally Lipschitz function f is said to be exponential (p,r)-invex (strictly) at u if there exists a function $\eta : X\times X \longrightarrow X$ with property $\eta(x,u)=0$ only if $u=x$, such that for each $x\in X$,*

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1+\frac{r}{p}\left\langle\xi,\ \left(e^{p\eta(x,u)}-\mathbf{1}\right)\right\rangle\right],\ \text{for } p\neq 0,\ r\neq 0 \tag{2.5}$$

$$(>\ \text{if } x\neq u)$$

where $\xi\in\partial^c f(u)$, $\mathbf{1}=(1,1,\cdots,1)\in\mathbb{R}^n$, $\eta=(\eta_1,\cdots,\eta_n)$, $e^{p\eta(x,u)}-\mathbf{1}=(e^{p\eta_1(x,u)}-1,\cdots,e^{p\eta_n(x,u)}-1)\in\mathbb{R}^n$, *and* $\langle\cdot,\cdot\rangle$ *denotes an dual pair or inner product in* $\mathbb{R}^n$.

There are some special results of exponential (p,r)-invex:

(1) If $r\neq 0$, $p\rightarrow 0$ in (2.5), then the limit implies :

$$\frac{e^{rf(x)}-e^{rf(u)}}{r} \geq e^{rf(u)}\langle\xi,\eta(x,u)\rangle,\ \ \xi\in\partial^c f(u)\subset X^*.$$

$$(>\ \text{if } x\neq u)$$

(2) If $p\neq 0$, $r\rightarrow 0$ in (2.5), then the limit becomes :

$$f(x)-f(u) \geq \frac{1}{p}\left\langle\xi,\left(e^{p\eta(x,u)}-\mathbf{1}\right)\right\rangle.$$

$$(>\ \text{if } x\neq u)$$

(3) In (2), if $p \to 0$, then the limit turns to :

$$f(x) - f(u) \geq \langle \xi, \eta(x,u) \rangle \text{ for all } x \in X,\ \xi \in \partial^c f(u).$$
$$(> \text{ if } x \neq u)$$

This is usually called nonsmooth η-invex function at u.

3 Equivalence of minimax fractional and parametric nonfractional programming

From Problem (P), let $x \in X$ and $\sup_{y\in Y} \frac{f(x,y)}{g(x,y)} = \lambda (= \lambda(x))$, then

$$f(x,y) - \lambda g(x,y) \leq 0 \text{ with } \lambda = \lambda(x), \text{ a real parameter.}$$

Since the continuous function on the compact space Y is attainable to its maximum, Problem (P) deduces a nonfractional parametric problem as the following problem :

$$(P_\lambda) \qquad v(\lambda) = \min_{x\in X} \sup_{y\in Y} (f(x,y) - \lambda g(x,y))$$
$$\text{subject to } f(x,y) - \lambda g(x,y) \leq 0.$$

Then we could obtain the following result without any further assumptions on f and g.

Lemma 1. *(cf. Lai and Liu [10]) Problem (P) has an optimal solution* x_0 *with optimal value* λ^* *if and only if* $v(\lambda^*) = 0$ *and* x_0 *is an optimal solution of Problem* (P_{λ^*}). *That is,*

$$\lambda^* = \frac{f(x_0, \tilde{y})}{g(x_0, \tilde{y})}\ ;\ \tilde{y} \text{ is the maximal point of } \frac{f(x_0, y)}{g(x_0, y)}.$$

This lemma is useful to establish the necessary optimality conditions as follows.

4 Necessary and Sufficient Optimality Conditions

Theorem 1. (Necessary Optimality) *Let* x^* *be a (P)-optimal solution and it has generalized slater type constraint qualification at* x^* ($\exists x \in X$ *such* *that* $h_j(x) < 0,\ \forall j = 1,2,\cdots,p$). *Then there corresponds a triplet* $(s^*,t^*,y^*) \in K(x^*)$, $\lambda^* \in \mathbb{R}_+$ *and a p-vector Lagrange multiplier* $\mu^* \in \mathbb{R}_+^p$ *such that*

$$0 \in \sum_{i=1}^{s^*} t_i^* \{\partial^c f(x^*,y_i^*) + \lambda^* \partial^c [-g(x^*,y_i^*)]\} + \sum_{j=1}^{p} \mu_j^* \partial^c h_j(x^*) \tag{4.1}$$

$$f(x^*,y_i^*) - \lambda^* g(x^*,y_i^*) = 0,\ i = 1,2,\cdots,s^* \tag{4.2}$$

$$\mu^* h_j(x^*) = 0,\ j = 1,2,\cdots,p \tag{4.3}$$

where

$$K(x^*) = \left\{ (s^*,t^*,y^*) \,\middle|\, t_i^* \geq 0,\ \sum_{i=1}^{s^*} t_i^* = 1,\ y_i^* \in Y(x^*),\ i = 1,2,\cdots,s^* \right\} \tag{4.4}$$

Remark 1. *When* $Y \subset \mathbb{R}^m$ *is compact, for each feasible point* $x \in X$, *there exists an integer* $s \in \mathbb{N}$, *s points* $y_i \in Y(x) \subset Y$, $i = 1,2,\cdots,s$ *and* $t_i \geq 0$ *with* $\sum_{i=1}^{s} t_i = 1$ *so that the convex combination of s points* $y_i \in Y$, $i = 1,2,\cdots,s$ *exists. Then for convenience, we denote a set*

$$K(x) = \left\{ (s,t,y) \in \mathbb{N} \times \mathbb{R}_+^s \times Y^{sm} \,\middle|\, \begin{array}{l} \sum_{i=1}^{s} t_i = 1,\ t = (t_1,t_2,\cdots,t_s) \in \mathbb{R}^s, \\ y = (y_1,y_2,\cdots,y_s) \in Y^{sm} \end{array} \right\}.$$

While the sufficient optimality conditions, we can get from the converse of the necessary optimality conditions with extra assumptions. So the Sufficient Optimality Theorem is various.

This paper will employ the exponential (p,r)-invex (Lipschitz) function for nonsmooth functions to establish the following theorem.

Theorem 2. (Sufficient Optimality) *Assume that* (x,λ,μ,s,t,y) *instead of* $(x^*,\lambda^*,\mu^*,s^*,t^*,y^*)$ *to satisfy the necessary conditions* (4.1) $\sim$ (4.4) *in Theorem 1. Let* $\tilde{x}$ *be a feasible solution of Problem (P). Suppose further that any one of conditions (a) and (b) holds :*

(a) $A(\cdot) = \sum_{i=1}^{s} \tilde{t}_i(f(\cdot,\tilde{y}_i) - \tilde{\lambda} g(\cdot,\tilde{y}_i))$ *and* $B(\cdot) = \sum_{j=1}^{p} \tilde{\mu}_j h_j(\cdot)$ *are exponential* (p,r)*-invex w.r.t. the same* η *at* $\tilde{x} \in X$.

(b) $C(\cdot) = \sum_{i=1}^{s} \tilde{t}_i(f(\cdot,\tilde{y}_i) - \tilde{\lambda} g(\cdot,\tilde{y}_i)) + \sum_{j=1}^{p} \tilde{\mu}_j h_j(\cdot)$ *is exponential* (p,r)*-invex w.r.t.* η *at* $\tilde{x}$ *on* X.

Then $\tilde{x}$ *is an optimal solution of Problem (P).*

5 Parametric Dual Model

As in Section 3, a parameter $\lambda = \lambda(x) = \sup_{y \in Y} \frac{f(x,y)}{g(x,y)}$ deduces a one parametric dual problem (D) by using the feasible variable $x \in X$ replaced by $u \in X$, and use the conditions (4.1) ~ (4.4) modified to be (5.1) ~ (5.4) as the following expressions :

$$\sum_{i=1}^{s} t_i \{\partial^c f(u,y_i) + \lambda \partial^c [-g(u,y_i)]\} + \sum_{j=1}^{p} \mu_i \partial^c h_j(u) = 0 \tag{5.1}$$

$$\sum_{j=1}^{p} \mu_j h_j(u) \geq 0 \tag{5.2}$$

$$\sum_{i=1}^{s} t_i [f(u,y_i) - \lambda g(u,y_i)] \geq 0 \tag{5.3}$$

$$(s,t,y) \in K_\lambda(u) \tag{5.4}$$

which are employed as the constraints of a parametric dual problem (D) :

$$(D) \quad \max_{u \in X} \sup_{(s,t,y) \in K_\lambda(u)} \lambda = \lambda(u)$$

$$\text{subject to } (s,t,y,\lambda,u) \in H,$$

where H is the set of all (s,t,y,λ,u) satisfying conditions (5.1) ~ (5.4).

We should reduce that (D) is a dual problem of (P), and will proceed it as next section.

6 Duality Theorems

Once we establish a dual model (D), we could establish three theorems:

Weak, Strong and Strict Converse Duality Theorem,

to prove the duality theorem has no duality gap w.r.t. the primary Problem (P) under some reasonable conditions.

For details, one can refer the full paper intitules "Sufficient optimality and parametric duality on minimax fractional programming problem with generalized exponential (p,r)-invexity" composed by Lai and Ho which will appear elsewhere.

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