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Proximal Point Algorithm of the Zero Point Problems

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Abstract

In this paper, we apply the proximal point algorithm to study zero point problems in a reflexive, strictly convex and smooth Banach space and in its dual space. We obtain some existence theorems for zero point problems and some results on the boundedness and asymptotic behavior of the sequences generated by the proximal point algorithm without summability assumptions on the error sequences. Further we characterize the existence of the solutions of zero point problems of maximal monotone operators in a reflexive, strictly convex and smooth Banach space and in its dual space.

1 Introduction

Let E be a Banach space and E^* be its dual space. We consider the problems of finding points $u \in E$ and $v \in E^*$ such that

(1.1) $$0 \in A(u)$$

and

(1.2) $$0 \in B(v),$$

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where A and B are maximal monotone operator from E to 2^{E^*} and from E^* to 2^E, respectively. The problem of finding a solution of problem (1.1) has interesting interpretations in various fields. For example, saddle point problems, variational inequalities, and complementary problems can be written in (1.1) (see [1],[2],[14],[15],[18]). A variety of methods for solving problem (1.1) has been proposed and investigated (see [3],[4],[6],[7],[8],[9],[10],[13],[14],[16]). One of the most popular algorithms for solving problem (1.1) of a maximal monotone operator is the proximal point algorithm, which was first proposed by Martinet [11] in 1970. In a Hilbert space setting, Rockafellar [14] used the proximal point algorithm to show that problem (1.1) has at least one solution under some suitable assumptions.

Let H be a real Hilbert space, $\{t_n\}$ and $\{c_n\}$ be two sequences of positive numbers. Recently Khatibzadeh [5] proved a sufficient condition for the boundedness of the sequence generated by the following proximal point algorithm: for any starting point $x_0 \in H$,

$$x_n = (I + c_n A)^{-1}(x_{n-1} + e_n), \quad \forall n \geq 1, \tag{1}$$

where A is a maximal monotone operator on H, I is the identity mapping and $\{e_n\}$ is a sequence in H. Khatibzadeh [5] also consider the existence of solutions of problem (1.1) in the case that E is a real Hilbert space. On the other hand, Tian and Song [17] proposed a regularization method of proximal point algorithm: for any starting point $x_0 \in H$ and $u \in H$,

$$x_n = (I + c_n A)^{-1}(t_n u + (1 - t_n)x_{n-1} + e_n), \quad \forall n \geq 1, \tag{2}$$

where I, A and $\{e_n\}$ are the same as in (1). When E is a real Hilbert space, Tian and Song [17] obtain that the sequence $\{x_n\}$ generated by (2) converges strongly to a solution of problem (1.1) under some suitable assumptions. Motivated by [5] and [17], we proposed a regularization method of proximal point algorithm in a reflexive, strictly convex and smooth Banach space E. Let $G : E \to E^*$ and $H : E \to E$ be two mappings, we consider the following regularization method of proximal point algorithms: for any starting point $x_0 \in E$,

$$x_n = (J + c_n A)^{-1}(t_n G(x_{n-1}) + (1 - t_n)Jx_{n-1} + f_n) \quad \forall n \geq 1, \tag{3}$$

and

$$x_n = (I + c_n BJ)^{-1}(t_n H(x_{n-1}) + (1 - t_n)x_{n-1} + e_n) \quad \forall n \geq 1, \tag{4}$$

where $\{e_n\} \subseteq E$, $\{f_n\} \subseteq E^*$, A and B are maximal monotone mappings defined on E and E^*, respectively. The regularization methods of proximal point algorithm (3) and (4) are generalizations of (1) and (2). And the space E (a reflexive, strictly convex and smooth Banach space) is more general than the space H (a real Hilbert space) considered in [5] and [17]. In order to generalize the main results in [5] to Banach spaces, the assumption on $\{c_n\}$ ($\lim_{n\to\infty} c_n = +\infty$) in this paper is stronger than the one ($\sum_{n=1}^{\infty} c_n = +\infty$) in [5]. But the sequence $\{x_n\}$ generated by (2) with $\lim_{n\to\infty} c_n = +\infty$ has faster rate of convergence than the one with $\sum_{n=1}^{\infty} c_n = +\infty$. As a main result of this paper, we propose existence theorems of solutions of problems (1.1) and (1.2). Moreover we show that the set of all solutions of problem (1.1) (and (1.2)) is nonempty if and only if there exists a bounded sequence generated by our regularization method of proximal point algorithm with $\lim_{n\to\infty} c_n = +\infty$. The assumptions on $\{t_n\}$ and $\{c_n\}$ of (3) and (4) in this paper are different from the ones of (2) in [17], although the algorithm (2) is a special case of (3) and (4).

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers. Let X, Y be two topological spaces and let $T : X \multimap Y$ be a multivalued mapping, we denote $D(T) := \{x \in X : Tx \neq \emptyset\}$ the domain of T and $R(T) := \bigcup_{x\in D(T)} Tx$ the range of T. Let E be a reflexive, strictly convex and smooth Banach space and let E^* be its dual space. A mapping $T : D(T) \subseteq E \multimap E^*$ is called a monotone operator if $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$, for all $y_i \in Tx_i$, $i = 1, 2$. The monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. The monotone operator T is called coercive if $\lim_{\|x\|\to 0} \frac{\langle y, x\rangle}{\|x\|} = +\infty$, for all $(x, y) \in G(T)$. Let $A : D(A) \subseteq E \to E^*$ and $B : D(B) \subseteq E^* \to E$ be maximal monotone operators. Let $G : E \to E^*$ and

$H : E \to E$ be two mappings. Let $\{c_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers with $\{t_n\} \subseteq [0,1]$ and $c_n > 0$, $\{e_n\}$ and $\{f_n\}$ be sequences in E and E^*, respectively.

3 Bounded sequences

Theorem 3.1. Let A be a coercive maximal monotone operator. If the sequences $\{\frac{t_n}{c_n}\}$ and $\{\frac{\|f_n\|}{c_n}\}$ are bounded. Suppose at least one of the following conditions is satisfied:

(*i*) $R(G)$ is bounded;

(*ii*) $\|Gx\| \leq \|x\|$ for all $x \in E$.

Then for each $x_0 \in E$, the sequence $\{x_n\}$ generated by (3) is bounded.

Theorem 3.2. Let E be a real Hilbert space. Suppose that $\{x_n\}$ be the sequence generated by (1) with $f_n \equiv 0$ and $A = \partial\varphi$, where φ is a proper, convex and lower semicontinuous function. If $\sum\limits_{n=1}^{+\infty} c_n = +\infty$, then $\varphi(x_n) - \varphi(p) = o((\sum\limits_{i=1}^{n} c_i)^{-1})$, where p is a minimum point of φ.

Theorem 3.3. Let A be a coercive maximal monotone operator. If the sequences $\{\frac{t_n}{c_n}\}$ and $\{\frac{\|f_n\|}{c_n}\}$ are bounded, then for each $x_0 \in E$ and $v \in E^*$, the sequence $\{x_n\}$ generated by

$$x_n = J_{c_n}(t_n v + (1 - t_n)Jx_{n-1} + f_n) \tag{5}$$

is bounded.

Theorem 3.4. Let A be a coercive maximal monotone operator. If $\lim\limits_{n\to\infty} c_n = \infty$ and the sequence $\{\frac{\|f_n\|}{c_n}\}$ is bounded. Suppose at least one of the following conditions is satisfied:

(*i*) $R(G)$ is bounded;

(*ii*) $\|Gx\| \leq \|x\|$ for all $x \in E$.

Then for each $x_0 \in E$ and $v \in E^*$, the sequence $\{x_n\}$ generated by

$$x_n = J_{c_n}(Gx_{n-1} + f_n) \tag{6}$$

is bounded.

Theorem 3.5. Let B be a coercive maximal monotone operator. If the sequences $\{\frac{t_n}{c_n}\}$ and $\{\frac{\|e_n\|}{c_n}\}$ are bounded. Suppose at least one of the following conditions is satisfied:

(*i*) $R(H)$ is bounded;

(*ii*) $\|H(x)\| \leq \|x\|$ for all $x \in E$.

Then for each $x_0 \in E$, the sequence $\{x_n\}$ generated by (4) is bounded.

Theorem 3.6. Let B be a coercive maximal monotone operator. If the sequences $\{\frac{t_n}{c_n}\}$ and $\{\frac{\|e_n\|}{c_n}\}$ are bounded, then for each $x_0 \in E$ and $u \in E$, the sequence $\{x_n\}$ generated by

$$x_n = Q_{c_n}(t_n u + (1 - t_n)x_{n-1} + e_n) \tag{7}$$

is bounded.

Theorem 3.7. Let B be a coercive maximal monotone operator. If $\lim_{n\to\infty} c_n = \infty$ and the sequence $\{\frac{\|e_n\|}{c_n}\}$ is bounded. Suppose at least one of the following conditions is satisfied:

(*i*) $R(H)$ is bounded;

(*ii*) $\|Hx\| \leq \|x\|$ for all $x \in E$.

Then for each $x_0 \in E$, the sequence $\{x_n\}$ generated by

$$x_n = Q_{c_n}(H(x_{n-1}) + e_n) \tag{8}$$

is bounded.

4 Main results

In this section, we study the existence of solutions of problems (1.1) and (1.2). The following theorem is one of the main results in this paper and it is an existence result of solutions of problem (1.1).

Theorem 4.1. Let $\{x_n\}$ be a bounded sequence generated by (3). If $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|f_n\|}{c_n} = 0$. Suppose at least one of the following conditions is satisfied:

(*i*) $R(G)$ is bounded;

(*ii*) $\|Gx\| \leq \|x\|$ for all $x \in E$.

Then $A^{-1}(0) \neq \emptyset$. Moreover, every weak cluster point of the sequence $\{w_n\}$ belongs to $A^{-1}(0)$, where $w_k = \frac{\sum_{n=1}^{k} c_n x_n}{\sum_{n=1}^{k} c_n}$.

Theorem 4.2. Let $\{x_n\}$ be a bounded sequence generated by (3). If $\sum_{k=1}^{n-1} c_k = o(c_n)$ (the small O of c_n) and $\lim_{n\to\infty} \frac{\|f_n\|}{c_n} = 0$. Suppose at least one of the following conditions is satisfied:

(*i*) $R(G)$ is bounded;

(*ii*) $\|Gx\| \leq \|x\|$ for all $x \in E$.

Then $A^{-1}(0) \neq \emptyset$. Moreover, every weak cluster point of the sequence $\{x_n\}$ belongs to $A^{-1}(0)$.

Theorem 4.3. Let A be a coercive maximal monotone operator. Then $A^{-1}(0) \neq \emptyset$. Moreover, Suppose at least one of the following conditions is satisfied:

(*i*) $R(G)$ is bounded;

(*ii*) $\|Gx\| \leq \|x\|$ for all $x \in E$.

Let $\{x_n\}$ be a sequence generated by (3) and sequence $\{w_k\}$ be the same as in Theorem 4.1. Then we have the following conclusions:

(i) If $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|f_n\|}{c_n} = 0$, then every weak cluster point of the sequence $\{w_k\}$ belongs to $A^{-1}(0)$.

(ii) If $\sum_{k=1}^{n-1} c_k = o(c_n)$ and $\lim_{n\to\infty} \frac{\|f_n\|}{c_n} = 0$, then every weak cluster point of the sequence $\{x_n\}$ belongs to $A^{-1}(0)$.

Theorem 4.4. Let A be a maximal monotone operator. Then the following are equivalent:

(i) $A^{-1}(0) \neq \emptyset$.

(ii) There exists a bounded sequence $\{x_n\}$ generated by (17) with $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|f_n\|}{c_n} = 0$.

In this case, every weak cluster point of $\{w_k\}$ belongs to $A^{-1}(0)$, where $w_k = \frac{\sum_{n=1}^{k} c_n x_n}{\sum_{n=1}^{k} c_n}$. Moreover if $\sum_{k=1}^{n-1} c_k = o(c_n)$, then every weak cluster point of the sequence $\{x_n\}$ also belongs to $A^{-1}(0)$.

Theorem 4.5. Let $\{x_n\}$ be a bounded sequence generated by (4). If $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|e_n\|}{c_n} = 0$. Suppose at least one of the following conditions is satisfied:

(i) $R(H)$ is bounded;

(ii) $\|Hx\| \leq \|x\|$ for all $x \in E$.

Then $B^{-1}(0) \neq \emptyset$. Moreover, every weak cluster point of the sequence $\{w_k\}$ belongs to $B^{-1}(0)$, where $w_k = \frac{\sum_{n=1}^{k} c_n J x_n}{\sum_{n=1}^{k} c_n}$.

Theorem 4.6. Let $\{x_n\}$ be a bounded sequence generated by (4). If $\sum_{k=1}^{n-1} c_k = o(c_n)$ and $\lim_{n\to\infty} \frac{\|e_n\|}{c_n} = 0$. Suppose at least one of the following conditions is satisfied:

(*i*) $R(H)$ is bounded;

(*ii*) $\|Hx\| \le \|x\|$ for all $x \in E$.

Then $B^{-1}(0) \ne \emptyset$. Moreover, every weak cluster point of the sequence $\{x_n\}$ belongs to $B^{-1}(0)$.

Theorem 4.7. Let B be a coercive maximal monotone operator. Then $B^{-1}(0) \ne \emptyset$. Moreover, suppose at least one of the following conditions is satisfied:

(*i*) $R(H)$ is bounded;

(*ii*) $\|Hx\| \le \|x\|$ for all $x \in E$.

Let $\{x_n\}$ be a sequence generated by (4) and sequence $\{w_k\}$ be the same as in Theorem 4.5. Then we have the following conclusions:

(*i*) If $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|e_n\|}{c_n} = 0$, then every weak cluster point of the sequence $\{w_k\}$ belongs to $B^{-1}(0)$.

(*ii*) If $\sum_{k=1}^{n-1} c_k = o(c_n)$ and $\lim_{n\to\infty} \frac{\|e_n\|}{c_n} = 0$, then every weak cluster point of the sequence $\{x_n\}$ belongs to $B^{-1}(0)$.

Theorem 4.8. Let B be a maximal monotone operator. Then $B^{-1}(0) \ne \emptyset$ if and only if there exists a bounded sequence $\{x_n\}$ generated by (31) with $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \frac{\|e_n\|}{c_n} = 0$. In this case, every weak cluster point of the sequence $\{w_k\}$ belongs to $B^{-1}(0)$, where $w_k = \frac{\sum_{n=1}^{k} c_n J x_n}{\sum_{n=1}^{k} c_n}$. Moreover if $\sum_{k=1}^{n-1} c_k = o(c_n)$, then every weak cluster point of the sequence $\{x_n\}$ also belongs to $B^{-1}(0)$.

References

[1] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, Wiley and Sons, New York, 1984.

[2] V. Barbu, T. Precupanu, *Convexity and Optimization in Banach Spaces*, second edition, D. Reidel, Dordrecht, 1986.

[3] H. H. Bauschke, P. L. Combettes, *Constriction of best Bregman approximations in reflexive Banach spaces*, Proc. Amer. Math. Soc., 131 (2003), 3757-3766.

[4] H. Brézis, P.-L. Lions, *Produits infinis de résolvantes*, Israel J. Math., 29 (1978), 329-345.

[5] H. Khatibzadeh, *Some remarks on the proximal point algorithm*, J. Optim. Theory Appl., in press, Doi:10.1007/s10957-011-9973-5.

[6] S. Kamimura, F. Kohsaka, W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal., 12 (2004), 417-429.

[7] G. Kassay, *The proximal point algorithm for reflexive Banach spaces*, Studia Univ. Babes Bolyai Math., 30 (1985), 9-17.

[8] K. Kido, *Strong convergence of resolvents of monotone operators in Banach spaces*, Proc. Amer. Math. Soc., 103 (1988), 755-758.

[9] F. Kohsaka, W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal., 2004 (2004), 239-249.

[10] F. J. Luque, *Asymptotic convergence analysis of the proximal point algorithm*, SIAM J. Control Optim., 22 (1984), 277-293.

[11] B. Martinet, *Régularisationtion d'inéquations variationnelles par approximations successives*, Rev. Francaise Informat. Recherche Opérationnelle, 4 (1970), 154-158.

[12] S. Matsushita, L. Xu, *On convergence of the proximal point algorithm in Banach spaces*, Proc. Amer. Math. Soc., 139 (2011), 4087-4095.

[13] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied Nonlinear Analysis (V. Lakshmikantham, ed.), Academic Press, New York, 1979, 335-345.

[14] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., 14 (1976), 877-898.

[15] R. T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[16] M. V. Solodov, B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program, 87 (2000), 189-202.

[17] C. Tian, Y. Song, *Strong convergence of a regularization method for Rockafellar's proximal point algorithm*, J. Glob. Optim., in press, Doi:10.1007/s10898-011-9827-6.

[18] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[19] W. Takahashi, *Convex Analysis and Approximation of fixed points (Japanese)*, Yokohama Publishers, Yokohama, Japan, 2000.

[20] W. Takahashi, J.-C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math., 15 (2011), 787-818.