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VARIOUS FORMS OF THE KY FAN MINIMAX INEQUALITY IN CONVEX SPACES

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ABSTRACT. In this paper, for a convex space (X, D), we show that the KKM principle implies various forms of the Ky Fan minimax inequality. As an application, we give a direct proof of the Nash equilibrium theorem from a form of the inequality. Finally, we add some historical remarks.

1. Introduction

The KKM theory is originated from the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem in 1928 [10]. Since then, it has been found a large number of results which are equivalent to the KKM theorem; see [18,19]. Typical examples of the most remarkable and useful equivalent formulations are Ky Fan's KKM lemma in 1961 [5] and his minimax inequality in 1972 [6]. The inequality and its various generalizations are very useful tools in various fields of mathematical sciences.

The following is Fan's KKM lemma:

Lemma. [5] *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

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Five decades after the birth of this lemma, the above original form is still adopted by many authors in each year. But, it was found that the Hausdorffness is redundant quite long time ago by Lassonde [11]. Moreover, note that Y can be any convex subset of a topological vector space.

In the present paper, in order to present modern forms of Fan's KKM lemma and minimax inequality, we propose the use of the following term due to the author [16,17,25] instead of convex subsets:

Definition. A *convex space* (X, D) is a pair where X is a subset of a vector space with a nonempty subset $D \subset X$ such that $\operatorname{co} D \subset X$ and, for each nonempty finite subset A of D, its convex hull $\operatorname{co} A$ is equipped with the Euclidean topology. A subset $Y \subset X$ is said to be D-*convex* (or simply *convex*) if $\operatorname{co}(Y \cap D) \subset Y$. We denote $X = (X, X)$ if $X = D$.

This concept generalizes the one due to Lassonde for $X = D$; see [11]. Every convex subset X of a topological vector space with any nonempty subset $D \subset X$ becomes a convex space (X, D), but not conversely; see [4].

In this paper, we begin with the origin of the Fan minimax inequality (Section 2) and, for a convex space (X, D), we introduce a new KKM type theorem for maps having intersectionally closed values in the sense of Luc et al. [13] (Section 3). We show that this implies various forms of the Fan minimax inequality and analytic alternatives (Section 4). As an application, we give a direct proof of the Nash equilibrium theorem from a form of the inequality (Section 5). Finally, we add some related historical remarks (Section 6).

2. Preliminaries

Let A be a subset of a topological space X. We denote by $\overline{A}$ or $\operatorname{cl} A$ the closure of A in X and, by $\operatorname{Int} A$ the interior of A. Let Δ_n be the standard n-dimensional simplex in the Euclidean space $\mathbb{R}^{n+1}$. Let $\langle D \rangle$ be the set of all nonempty finite subsets of a set D.

Let (X, D) be a convex space.

Definition. If a multimap $G : D \multimap X$ satisfies

$$\operatorname{co} A \subset G(A) := \bigcup_{z \in A} G(z) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Then Fan's KKM lemma can be stated as follows:

The (partial) KKM principle. *For any closed-valued KKM map $G : D \multimap X$, the family $\{G(z)\}_{z \in D}$ has the finite intersection property. Further, if at least one of $G(z)$ is compact, then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

Example. (1) The original KKM theorem [10] is for the convex space (Δ_n, V), where V is the set of vertices of Δ_n.

(2) Fan's KKM lemma [5] is for (E, D), where D is a nonempty subset of a topological vector space E. His proof works for the above principle.

(3) For any (X, D), where X is a subset of a topological vector space and D is a nonempty subset of X such that $\operatorname{co} D \subset X$, the KKM principle works.

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower semicontinuous* (l.s.c.) if $\{x \in X \mid f(x) > r\}$ is open for each $r \in \overline{\mathbb{R}}$.

For a convex space (X, D), a function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* if $\{x \in X \mid f(x) > r\}$ is D-convex for each $r \in \overline{\mathbb{R}}$.

Similarly, the upper semicontinuity (u.s.c.) and the quasiconvexity can be defined.

The following is the original form given by Fan [6]:

The Fan minimax inequality. *Let X be a compact convex set in a Hausdorff topological vector space. Let f be a real-valued function defined on $X \times X$ such that:*

(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.*

Then the minimax inequality

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

In [6], Fan applied his inequality to the following:

A variational inequality (extending Hartman-Stampacchia (1966) and Browder (1967)).

A geometric formulation of the inequality (equivalent to the Fan-Browder fixed point theorem (1968)).

Separation properties of u.d.c. multimaps, coincidence and fixed point theorems.

Properties of sets with convex sections (from which the Sion minimax theorem (1958), the equilibrium theorem of Nash (1951), and a variant of a theorem of Debrunner and Flor (1964) on extension of monotone sets easily follow).

A fundamental existence theorem in potential theory.

Further applications of the inequality appeared in various fields in mathematical sciences, for example, nonlinear analysis, especially in fixed point theory, variational inequalities, various equilibrium theory, mathematical programming, partial differential equations, game theory, impulsive control, and mathematical economics; see [12,31] and the references therein.

Moreover, the Fan minimax inequality has been followed by a large number of generalizations and applications in the KKM theory on convex subsets of topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, generalized convex spaces due to Park, and other types of spaces. Furthermore, many authors generalized the lower semicontinuity and quasiconcavity in the inequality or replaced them by another requirements. Therefore, even for convex spaces, it is necessary to establish proper forms of the Fan minimax inequality which unify as many particular cases as possible.

3. A new KKM type theorem

Recently, we obtained very general KKM type theorems. Consider the following related four conditions for a multimap $G : D \multimap X$ for a convex space (X, D):

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [13]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [LS], its authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

For a multimap $G : D \multimap X$, consider the following related four conditions:

(a) $\bigcup_{z\in D} G(z) = X$ implies $\bigcup_{z\in D} \operatorname{Int} G(z) = X$.

(b) $\operatorname{Int} \bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *unionly open-valued* [13]).

(c) $\bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *transfer open-valued*).

(d) G is open-valued.

Proposition 1. [13] *The multimap G is intersectionally closed-valued (resp., transfer closed-valued) if and only if its complement G^c is unionly open-valued (resp., transfer open-valued).*

Definition. For a convex space (X, D), a subset S of X is said to be *intersectionally closed* (resp., *transfer closed*) if there is an intersectionally (resp., transfer) closed-valued map $G : D \multimap X$ such that $S = G(z)$ for some $z \in D$. Similarly, we can define *unionly* (resp., *transfer*) *open* sets.

We have the following KKM type theorem from the corresponding ones in [20-22].

Theorem 1. *Let (X, D) be a convex space and $G : D \multimap X$ a map such that*

(1) *$\overline{G}$ is a KKM map [that is, $\operatorname{co} A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$]; and*

(2) *there exists a nonempty compact subset K of X such that either*

(i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact subset* L_N *of* X *such that* (L_N, D') *is a convex space for some* $D' \subset D \cap L_N$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Here (2) is called the compactness (or coercivity) condition. From now on, we deal with only the case (β) for simplicity. For the KKM type theorems for more general abstract convex spaces and their applications, see [19-24].

4. Minimax inequalities and analytic alternatives

From the KKM Theorem 1, we obtain the following prototype of minimax inequalities:

Theorem 2. *Let* (X, D) *be a convex space,* $\gamma \in \mathbb{R}$, *and* $f : D \times X \to \overline{\mathbb{R}}$ *an extended real-valued function. Suppose that*

(1) *for each* $z \in D$, $G(z) := \{y \in X \mid f(z,y) \leq \gamma\}$ *is intersectionally closed;*

(2) *for each finite subset* $N \subset D$, *we have*

$$\operatorname{co} N \subset \overline{G}(N) := \bigcup_{z \in N} \operatorname{cl}\{y \in X \mid f(z,y) \leq \gamma\}; \text{ and}$$

(3) *the compactness condition (2) in Theorem 1 holds.*

Then

(a) *there exists a point* $y^* \in X$ *such that*

$$f(z, y^*) \leq \gamma \quad \text{for all} \quad z \in D;$$

(b) *if* $\gamma = \sup_{z \in D} \inf_{y \in X} f(z,y)$, *then*

$$\inf_{y \in X} \sup_{z \in D} f(z,y) = \sup_{z \in D} \inf_{y \in X} f(z,y);$$

and

(c) *if* $X = D$ *and* $\gamma = \sup_{x \in X} f(x,x)$, *then*

$$\inf_{y \in X} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} f(x,x).$$

Proof. (a) Note that G is an intersectionally closed-valued KKM map satisfying the compactness condition. Therefore, by Theorem 1, $\{G(z)\}_{z\in D}$ has the nonempty intersection. Hence, there exists a $y^* \in \bigcap_{z\in D} G(z) \subset X$. So, $f(z,y^*) \le \gamma$ for all $z \in D$. This shows (a).

(b) From (a) we have

$$\inf_{y\in X}\sup_{z\in D} f(z,y) \le \gamma = \sup_{z\in D}\inf_{y\in X} f(z,y).$$

Since

$$\inf_{y\in X}\sup_{z\in D} f(z,y) \ge \sup_{z\in D}\inf_{y\in X} f(z,y)$$

is trivially true, we have (b).

(c) follows from (a) immediately. □

Remark. In case (c), if $X = D$ is compact and $y \mapsto f(x,y)$ is l.s.c. for each $x \in X$, then so is $y \mapsto \sup_{x\in X} f(x,y)$ and hence the conclusion becomes

$$\min_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

The following is a variant of Theorem 2:

Theorem 3. *In Theorem 2, condition (2) can be replaced by the following without affecting its conclusion:*

(2)′ *for each* $N \in \langle D\rangle$ *and* $y \in \operatorname{co} N$, $\min\{f(z,y) \mid z \in N\} \le \gamma$.

Lemma 1. *Under the hypothesis of Theorem 3, condition (2)′ holds if and only if the map* $G : D \multimap X$ *is a KKM map.*

Proof. (Necessity) Suppose, on the contrary, that there exists an $N \in \langle D\rangle$ such that $\operatorname{co} N \not\subset G(N)$. Choose a $y \in \operatorname{co} N$ such that $y \notin G(N)$, whence $f(z,y) > \gamma$ for all $z \in N$. Then $\min_{z\in N} f(z,y) > \gamma$, which contradicts (2)′. Therefore, G is a KKM map.

(Sufficiency) Since G is a KKM map, for any $N \in \langle D\rangle$, we have $\operatorname{co} N \subset G(N)$. If $y \in \operatorname{co} N$, then $y \in G(z)$ or $f(z,y) \le \gamma$ for some $z \in N$. Therefore, $\min\{f(z,y) \mid z \in N\} \le \gamma$. □

Proof of Theorem 3. Note that G is intersectionally closed-valued and a KKM map by Lemma 1. Note that G satisfies all of the requirements of Theorem 2 and hence the conclusion follows. □

The following is a prototype of minimax inequalities for two functions:

Theorem 4. *Let* (X, D) *be a convex space,* $f : D\times X \to \overline{\mathbb{R}}$, $g : X \times X \to \overline{\mathbb{R}}$ *be extended real-valued functions and* $\gamma \in \overline{\mathbb{R}}$ *such that*

(0) *for each* $(z,y) \in D \times X$, $f(z,y) > \gamma$ *implies* $g(z,y) > \gamma$ *[or* $f(z,y) \le g(z,y)$*];*

(1) *for each* $z \in D$, $G(z) := \{y \in X \mid f(z,y) \leq \gamma\}$ *is intersectionally closed;*
(2) *for each* $N \in \langle D \rangle$ *and* $y \in \operatorname{co} N$, $\min\{g(z,y) \mid z \in N\} \leq \gamma$; *and*
(3) *the compactness condition (2) in Theorem 1 holds.*

Then (a) *there exists a* $\hat{y} \in X$ *such that*

$$f(z,\hat{y}) \leq \gamma \quad \textit{for all} \quad z \in D; \textit{ and}$$

(b) *if* $\gamma = \sup_{x \in X} g(x,x)$, *then we have the minimax inequality:*

$$\inf_{y \in X} \sup_{z \in D} f(z,y) \leq \sup_{x \in X} g(x,x).$$

Proof. Note that (0) and (2) imply that f also satisfies condition (2), that is, condition (2)$'$ of Theorem 3 holds. Note that other requirements of Theorem 3 are assumed. Therefore Theorem 4 follows from Theorem 3. □

The following is a prototype of analytic alternatives:

Theorem 5. *Let* (X,D) *be a convex space,* $\alpha, \beta \in \mathbb{R}$, *and* $f : D \times X \to \overline{\mathbb{R}}$, $g : X \times X \to \overline{\mathbb{R}}$ *extended real-valued functions. Suppose that*
(1) *for each* $z \in D$, $G(z) := \{y \in X \mid f(z,y) \leq \alpha\}$ *is intersectionally closed;*
(2) *for each* $y \in X$, *we have*

$$\operatorname{co}\{z \in D \mid f(z,y) > \alpha\} \subset \{x \in X \mid g(x,y) > \beta\}; \textit{ and}$$

(3) *the compactness condition (2) in Theorem 1 holds.*

Then either
(i) *there exists a* $y_0 \in X$ *such that* $f(z,y_0) \leq \alpha$ *for all* $z \in D$; *or*
(ii) *there exists an* $\hat{x} \in X$ *such that* $g(\hat{x},\hat{x}) > \beta$.

Lemma 2. *Under the hypothesis of Theorem 5, assume (2) and the negation of (ii). Then the map* $G : D \multimap X$ *is a KKM map.*

Proof. The negation of (ii) is that $g(x,x) \leq \beta$ for all $x \in X$. Suppose, on the contrary, that there exists a finite $N \subset D$ such that $\operatorname{co} N \not\subset G(N)$. Then there exist a $y \in \operatorname{co} N$ such that $y \notin G(z)$ or $f(z,y) > \alpha$ for all $z \in N$. Hence $N \subset \{z \in D \mid f(z,y) > \alpha\}$ and, by (2), we have $\operatorname{co} N \subset \{x \in X \mid g(x,y) > \beta\}$. Since $y \in \operatorname{co} N$, we have $g(y,y) > \beta$. This contradicts our supposition. □

Proof of Theorem 5. Suppose (ii) does not hold. Then, by Lemma 2, G is a KKM map. Therefore, all the requirements of Theorem 2 with $\gamma = \alpha$ are satisfied. Hence, there exists a $y_0 \in X$ such that $f(z,y_0) \leq \alpha$ for all $z \in D$. This is (i). □

Corollary 5.1. *Under the hypothesis of Theorem 5 with $\alpha = \beta = 0$, if $g(x,x) \le 0$ for all $x \in X$, then*

(i)$'$ *there exists a $y_0 \in X$ such that $f(z, y_0) \le 0$ for all $z \in D$.*

Definition. For a convex space (X, D), an extended real function $f : D \times X \to \overline{\mathbb{R}}$ is said to be *generally lower semicontinuous* (g.l.s.c.) on D if for each $z \in D$, $\{y \in X \mid f(z,y) > r\}$ is unionly open for each $r \in \overline{\mathbb{R}}$.

This is a generalization of the transfer l.s.c. due to Tian [29]. Similarly, we can define generally u.s.c.

From Corollary 5.1, we obtain the following:

Corollary 5.2. *Let X be a compact convex space and $f, g : X \times X \to \overline{\mathbb{R}}$ two functions such that*

(1) *$f(x,y) \le g(x,y)$ for every $(x,y) \in X \times X$ and $g(x,x) \le 0$ for all $x \in X$;*
(2) *$y \mapsto f(x,y)$ is g.l.s.c. on X for every $x \in X$; and*
(3) *$x \mapsto g(x,y)$ is quasiconcave on X for every $y \in X$.*

Then there exists a $y_0 \in X$ such that $f(x, y_0) \le 0$ for all $x \in X$.

From Theorem 5, we clearly have the following:

Theorem 6. *Under the hypothesis of Theorem 5, if $\alpha = \beta = \sup_{x \in X} g(x,x)$, then*

(a) *there exists a $y_0 \in X$ such that*

$$f(z, y_0) \le \sup_{x \in X} g(x,x) \quad \text{for all } z \in D; \text{ and}$$

(b) *we have the following minimax inequality*

$$\inf_{y \in X} \sup_{z \in D} f(z,y) \le \sup_{x \in X} g(x,x).$$

Corollary 6.1. *Let X be a compact convex space and $f, g : X \times X \to \overline{\mathbb{R}}$ two functions such that*

(1) *$f(x,y) \le g(x,y)$ for every $(x,y) \in X \times X$;*
(2) *$y \mapsto f(x,y)$ is l.s.c. on X for every $x \in X$;*
(3) *$x \mapsto g(x,y)$ is quasiconcave on X for every $y \in X$.*

Then

$$\min_{y \in X} \sup_{x \in X} f(x,y) \le \sup_{x \in X} g(x,x).$$

Proof. Observe that $y \mapsto \sup_{x \in X} f(x,y)$ is l.s.c. by (2), and so its minimum on the compact space X exists. $\square$

For $f = g$, Corollary 6.1 reduces to the following:

Corollary 6.2. *Let X be a compact convex space and $f : X \times X \to \overline{\mathbb{R}}$ a function satisfying*

(1) *$y \mapsto f(x, y)$ is l.s.c. on X for every $x \in X$; and*

(2) *$x \mapsto f(x, y)$ is quasiconcave on X for every $y \in X$.*

Then

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).$$

This reduces to the Fan inequality when X is a compact convex subset of a t.v.s.

Finally, in this section, note that the KKM Theorem 1 in this paper can be extended to various types of abstract convex spaces without any linear structure and to other compactness condition; see [19-24]. Each of such extended KKM theorem implies many Fan type minimax inequalities as shown in this section.

5. The Fan minimax inequality and the Nash equilibrium theorem

Recall that the original Nash equilibrium theorem was proved by the Brouwer or the Kakutani fixed point theorem; see [14,15]. Later Fan [7] proved it by applying his result on sets with convex sections. Nowadays it is known to be one of the most important applications of the Fan minimax inequality; see [19]. Note that, in a wide sense, the Brouwer theorem, the KKM theorem, the Kakutani theorem, the Nash theorem, Fan's theorem on sets with convex sections, the Fan inequality, the Fan-Browder fixed point theorem, and many others are mutually equivalent; see [18].

In this section, we apply Theorem 5 or Corollary 5.1 to a direct proof of the Nash theorem in [7].

Let $I = \{1, 2, \ldots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple

$$\Lambda := \{X_1, \ldots, X_n; u_1, \ldots, u_n\},$$

where the nonempty set X_i is the ith player's pure strategy space and $u_i : X = \prod_{i=1}^n X_i \to \mathbb{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Let $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$ and denote by x and x_{-i} an element of X and X_{-i}, resp. A strategy n-tuple $(y_1^*, \ldots, y_n^*) \in X$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(y_i^*, y_{-i}^*) \geq u_i(x_i, y_{-i}^*) \quad \text{for all } x_i \in X_i \text{ and } i \in I.$$

The following is the Nash theorem in [7, Theorem 4]:

Theorem 7. *Let $\Lambda := \{X_1, \dots, X_n; u_1, \dots, u_n\}$ be a game where each X_i is a compact convex space and each u_i is continuous. If for each $i \in I$ and for any given point $x_{-i} \in X_{-i}$, $x_i \mapsto u_i(x_i, x_{-i})$ is a quasiconcave function on X_i, then there exists a Nash equilibrium for Λ.*

Proof. Fix an element $a = (a_1, \dots, a_n) \in X$. For each $i \in I$, let $e_i : X_i \hookrightarrow X$ be the embedding such that $e_i : x_i \in X_i \mapsto (x_i, a_{-i}) \in X$. Let $D_i := e_i(X_i) \subset X$. Then D_i is convex since so is X_i, and $z \in D_i$ implies $z = (z_i, a_{-i}) \in X$.

For $u_i : X \to \mathbb{R}$, define $f_i : D_i \times X \to \mathbb{R}$ and $g_i : X \times X \to \mathbb{R}$ by

$$f_i(z, y) := u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) \text{ and } g_i(x, y) := u_i(x_i, y_{-i}) - u_i(y_i, y_{-i}),$$

resp. Then $f_i(z, y) = g_i(z, y)$ on $D_i \times X$ and $g_i(x, x) = 0$ for all $x \in X$. Moreover, note that f_i and g_i do not depend on the point a.

Now we apply Theorem 5 for the convex space (X, D_i) with $\alpha = \beta = 0$.

(1) Since each u_i is continuous, for each $z \in D_i$, the set

$$\{y \in X \mid f_i(z, y) > 0\} = \{y \in X \mid u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) > 0\}$$

is open.

(2) For each $y \in X$, $z \mapsto u_i(z_i, y_{-i})$ is quasiconcave. Therefore $\{z \in D_i \mid u_i(z_i, y_{-i}) > r\}$ is convex for each $r \in \mathbb{R}$ and hence

$$\{z \in D_i \mid f_i(z, y) = u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) > 0\}$$

is convex and contained in $\{x \in X \mid g_i(x, y) > 0\}$.

(3) X is compact.

Consequently, all requirements (1)-(3) of Theorem 5 are satisfied. Moreover, the conclusion (ii) does not hold since $g_i(x, x) = 0$ for all $x \in X$. Therefore, we have

(i) there exists a $y^i \in X$ such that $f_i(z, y^i) \leq 0$ for all $z \in D$; that is,

$$u_i(y_i^i, y_{-i}^i) \geq u_i(z_i, y_{-i}^i) \quad \text{for all } z_i \in X_i \text{ and } i \in I.$$

Then $y^* := (y_1^1, \dots, y_n^n)$ is the required Nash equilibrium. $\square$

Remark. 1. Ziad [33] indicated that the Nash theorem follows from the Fan inequality. The above proof completes this matter.

2. Since the Nash theorem follows from the Fan inequality and the latter has a large number of generalizations for various abstract convex spaces, our argument works for corresponding generalizations of the Nash theorem. More refined versions of this matter, see [19].

6. Some historical notes

The concepts of lower semicontinuity, quasiconcavity, and compactness in the Fan minimax inequality are extended in various stages. In this section we give just a few steps in such development:

(I) Most of early works on KKM theory for convex subsets of t.v.s. are collected in the classical monograph of Granas [8]; see also Park [18].

(II) In 1981, Yen [30] obtained Corollary 6.1 for a compact convex subset in a t.v.s. and applied it to variational inequalities.

(III) In 1983, Lassonde [11] established the KKM theory on his convex spaces.

(IV) The origin of Theorem 3 is Zhou and Chen in 1988 [32, Theorem 2.11 and Corollary 2.13], where $X = D$ is a compact convex subset and quasiconcavity is extended to γ-diagonal quasiconcavity (γ-DQCV). These are applied to a variation of the Fan inequality, a saddle point theorem, and a quasi-variational inequality.

(V) Motivated by Lassonde's convex spaces, in 1992, Park [16] introduced his concept of convex spaces. Early applications were given in Park 1994 [17], Park and Kim 1993 [25].

(VI) In 1992, Tian [29] obtained the following particular form of Theorem 2:

Theorem. [29] *Let Y be a nonempty convex subset of a Hausdorff t.v.s. E, let $\emptyset \neq X \subset Y$, let $\gamma \in \mathbb{R}$, and let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a function such that*

(1) *it is γ-transfer l.s.c. in y [that is, for each $x \in X$, $\{y \in Y \mid \phi(x, y) \leq \gamma\}$ is transfer closed];*

(2) *for each $N \in \langle X \rangle$, $\operatorname{co} N \subset \bigcup_{x \in N} \operatorname{cl}_Y \{y \in Y \mid \phi(x, y) \leq \gamma\}$;*

(3) *there exists a nonempty subset $C \subset X$ such that for each $y \in Y \setminus C$ there exists a point $x \in C$ with $y \in \operatorname{Int}_Y \{z \in Y \mid \phi(x, z) > \gamma\}$ and C is contained in a compact convex subset of Y.*

Then there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$.

As Tian [29] noted that, (1) is satisfied if $\phi(x, y)$ is l.s.c. in y, (2) is satisfied if ϕ is γ-diagonally quasiconcave in $x \in X$, and (3) is satisfied if $X = Y$ and Y is compact. This theorem generalizes previously obtained results due to Fan, Allen, Zhou-Chen, and Tian.

Note that (Y, X) is a convex space in our sense.

(VII) In 1993, Lin and Tian [12, Theorem 3] defined γ-DQCV in slightly more general form than [32] as follows:

Let Y be a convex subset of a Hausdorff t.v.s. E and let $\emptyset \neq X \subset Y$. A functional $\varphi(x,y) : X \times Y \to \overline{\mathbb{R}}$ is said to be *γ-diagonally quasi-concave* (*γ-DQCV*) in x if, for any finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_\lambda \in \mathrm{co}\{x_1, \ldots, x_m\}$, we have $\min_{1 \leq j \leq m} \varphi(x_j, x_\lambda) \leq \gamma$.

Adopting this concept, they obtain a particular form of the above theorem as in our Theorem 3. Note that (Y, X) is also a convex space in our sense.

(VIII) Let E be a topological vector space, X a nonempty convex subset of E, D a nonempty subset of X. In 2000, Song [26,27] deduced particular forms of Theorem 1 for the convex space (X, D) and applied them as follows:

In [S1], Song [26] obtained a vector and set-valued generalization of the Ky Fan minimax inequality. This is applied to several existence theorems for generalized vector variational inequalities involving certain set-valued operators. Moreover, he gave a certain relationship between a kind of generalized vector variational inequality and a vector optimization problem.

In [27], Song obtained an existence result for a generalized vector equilibrium problem was obtained. This was applied to existence results for vector equilibrium problems and vector variational inequalities. He adopted artificial and impractical concepts like compactly closed and compact closure, which can be eliminated by adopting the compactly generated extension of the original topology.

In 2002, another form of Theorem 1 for a convex space was given by Song [28] and applied to similar problems.

(IX) In 2005, Balaj and Muresan [1] applied a Fan-Browder type fixed point theorem that is equivalent to a KKM type theorem to several minimax inequalities as follows:

Theorem. [1] *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \mathbb{R}$ be a function quasiconvex in y and transfer upper semicontinuous in x. Then* $\inf_{x \in X} f(x,x) \leq \sup_{x \in X} \inf_{y \in X} f(x,y)$.

Note that this follows from Theorem 6 with $f = g$.

(X) In 2009, Cho, Kim, and Lee [3] obtained Theorem 6 for $X = D$ and $f = g$.

(XI) Moreover, a number of authors gave some generalizations or variants of the concavity; see [9]. In 2009, Hou [9] defined $\mathcal{C}$-quasiconcavity which unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) and the $\mathcal{C}$-concavity (weaker than concavity) due to other authors.

(XII) In 2010, S.-Y. Chang [2] extended the $\mathcal{C}$-quasiconcavity [9] to the following 0-pair-concavity:

Let X be a nonempty set and Y be a topological space, and $D \subset X$. A function $f : X \times Y \to \mathbb{R}$ is said to be *0-pair-concave* on D, if for any $\{x^0, \ldots, x^n\} \in \langle A \rangle$, there is a continuous map $\phi_n : \Delta_n \to Y$, where Δ_n is the n-simplex, such that

$$\min_{i \in I(\lambda)} f(x^i, \phi_n(\lambda)) \leq 0$$

for all $\lambda = \{\lambda_0, \dots, \lambda_n\} \in \Delta_n$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$.

(XIII) In our forthcoming work [22], this concept is further generalized and we obtain more generalized versions of the Fan minimax inequality. More general and detailed approaches to generalizations of the inequality will appear elsewhere.

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