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Demiclosedness Principles for Total Asymptotically Nonexpansive mappings

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Abstract

In this paper, we first are looking over the demiclosedness principles for nonlinear mappings. Next, we give the demiclosedness principle of a continuous non-Lipschitzian mapping which is called totally asymptotically nonexpansive by Alber et al. [Fixed Point Theory and Appl., 2006 (2006), article ID 10673, 20 pages]. This paper is a just survey for demiclosedness principles for nonlinear mappings.

Keywords: totally asymptotically nonexpansive mappings, demiclosedness principle

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1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. Let $\{x_n\}$ be a sequence in X, $x \in X$. We denote by $x_n \to x$ the strong convergence of $\{x_n\}$ to x and by $x_n \rightharpoonup x$ the weak convergence of $\{x_n\}$ to x. Also, we denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : \exists\, x_{n_k} \rightharpoonup x\}.$$

Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a mapping. Now let $Fix(T)$ be the fixed point set of T; namely,

$$Fix(T) := \{x \in C : Tx = x\}.$$

Recall that T is a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $\lim_{n\to\infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [15] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpanisve mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping of *asymptotically nonexpansive type* [18] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.2}$$

and T^N is continuous for some $N \geq 1$. The other is the stronger concept due to Brúck, Kuczumov and Reich [5]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is (uniformly) continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.3}$$

In this case, observe that if we define

$$\delta_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \tag{1.4}$$

(here $a \vee b := \max\{a, b\}$), then $\delta_n \geq 0$ for all $n \geq 1$, $\delta_n \to 0$ as $n \to \infty$, and thus (1.3) immediately reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + \delta_n \tag{1.5}$$

for all x, $y \in C$ and $n \geq 1$.

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [9]. They say that a mapping $T : C \to C$ is said to be *total asymptotically nonexpansive* (TAN, in brief) [1] (or [9]) if there exists two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with c_n, $d_n \to 0$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n, \tag{1.6}$$

for all x, $y \in K$ and $n \geq 1$, where $\mathbb{R}^+ := [0, \infty)$ and

$$\phi \in \Gamma(\mathbb{R}^+) \Leftrightarrow \phi \text{ is strictly increasing, continuous on } \mathbb{R}^+ \text{ and } \phi(0) = 0.$$

Remark 1.1. If $\varphi(t) = t$, then (1.6) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \|x - y\| + d_n$$

for all x, $y \in C$ and $n \geq 1$. In addition, if $d_n = 0$, $k_n = 1 + c_n$ for all $n \geq 1$, then the class of total asymptotically nonexpansive mappings coincides with the class of asymptotically nonexpansive mappings. If $c_n = 0$ and $d_n = 0$ for all $n \geq 1$, then (1.6) reduces to the class of nonexpansive mappings. Also, if we take $c_n = 0$ and $d_n = \delta_n$ as in (1.4), then (1.6) reduces to (1.5); see [9] for more details.

Let C be a nonempty closed convex subset of a real Banach space X, and let $T: C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Recall that the following Mann [21] iterative method is extensively used for solving a fixed point equation of the form $Tx = x$:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.7}$$

where $\{a_n\}$ is a sequence in $[0,1]$ and $x_0 \in C$ is arbitrarily chosen. In infinite-dimensional spaces, Mann's algorithm has generally only weak convergence. In fact, it is known [29] that if the sequence $\{\alpha_n\}$ is such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then Mann's algorithm (1.7) converges weakly to a fixed point of T provided the underlying space is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial's property. Furthermore, Mann's algorithm (1.7) also converges weakly to a fixed point of T if X is a uniformly convex Banach space such that its dual X^* enjoys the *Kadec-Klee property* (*KK-property*, in brief), i.e., $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| \;\Rightarrow\; x_n \to x$. It is well known [12] that the duals of reflexive Banach spaces with a Frechet differentiable norms have the KK-property. There exists uniformly convex spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the KK-property; see Example 3.1 of [14].

In this paper, we first are looking over the demiclosedness principles for nonlinear mappings. Next, we give the demiclosedness principle of continuous TAN mappings.

2 Geometrical properties of X

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle \cdot, \cdot \rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$. The normalized duality mapping J from X to X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

Now we summarize some well known properties of the duality mapping J for our further argument.

Proposition 2.1. [10, 30, 34]

(1) *for each $x \in X$, Jx is nonempty, bounded, closed and convex (hence weakly compact).*

(2) $J(0) = 0$.

(3) $J(\lambda x) = \lambda J(x)$ *for all $x \in X$ and real λ.*

(4) *J is monotone, that is, $\langle x - y, j(x) - j(y) \rangle \geq 0$ for all $x, y \in X$, $j(x) \in J(x)$ and $j(y) \in J(y)$.*

(5) $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j(y)\rangle$ *for all* $x, y \in X$ *and* $j(x) \in J(y)$*; equivalently,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $x, y \in X$ *and* $j(x+y) \in J(x+y)$.

Remark 2.2. Note that (5) in Proposition 2.1 can be quickly computed by the well known Cauchy-Schwartz inequality:

$$2\langle x, j\rangle \leq 2\|x\|\|y\| \leq \|x\|^2 + \|y\|^2.$$

Recall that a Banach space X is said to be *strictly convex* (SC) [7] if any non-identically zero continuous linear functional takes maximum value on the closed unit ball at most at one point. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n+y_n)/2\| \to 1$.

We introduce some equivalent properties of strict convexity of X; see Proposition 2.13 in [7] for the detailed proof.

Proposition 2.3. ([7]) *A linear normed space X is strictly convex if and only if one of the following equivalent properties holds:*

(a) *if* $\|x + y\| = \|x\| + \|y\|$ *and* $x \neq 0$, $y = tx$ *for some* $t \geq 0$*;*

(b) *if* $\|x\| = \|y\| = 1$ *and* $x \neq y$*, then* $\|\lambda x + (1-\lambda)y\| < 1$ *for all* $\lambda \in (0,1)$*, namely, the unit sphere (or any sphere) contains no line segment;*

(c) *if* $\|x\| = \|y\| = 1$ *and* $x \neq y$*, then* $\|(x+y)/2\| < 1$*;*

(d) *the function* $x \to \|x\|^2$, $x \in X$*, is strictly convex.*

Remark 2.4. From (b), note that any three points x, y, z satisfying $\|x - z\| + \|y - z\| = \|x - y\|$ must lie on a line; specially, if $\|x - z\| = r_1$, $\|y - z\| = r_2$, and $\|x - y\| = r = r_1 + r_2$, then $z = \frac{r_1}{r}x + \frac{r_2}{r}y$; see [15] for more details. Indeed, taking $u := \frac{x-z}{r_1}$, $v := \frac{z-y}{r_2}$, we see $\|u\| = \|v\| = 1$ and

$$\|\lambda u + (1-\lambda)v\| = \|(x-y)/r\| = 1$$

for some $\lambda = \frac{r_1}{r} \in (0,1)$. Therefore, by (b), it must be $u = v \;\Leftrightarrow\; z = \frac{r_1}{r}x + \frac{r_2}{r}y$.

Let $S(X) := \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, $y \in S(X)$. In this case, the norm of X is said to be *Gâteaux differentiable.* The space X is said to be a *uniformly Gâteaux differentiable norm* if for each $y \in S(X)$, the limit (2.1) is attained uniformly for $x \in S(X)$. the norm of X is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit (2.1) is attained uniformly for $y \in S(X)$. The norm of X said to be *uniformly Fréchet differentiable* (or X is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $x, y \in S(X)$.

A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [10, 34] for more details.

Again, we introduce some well known properties of the duality mapping J relating to geometrical properties of X.

Proposition 2.5. ([10, 30, 34])

(1) *X is smooth if and only if J is single valued. In this case, J is norm-to-weak* continuous;*

(2) *if X is strictly convex, then J is one to one (or injective), i.e.,*

$$x \neq y \quad \Rightarrow \quad Jx \cap Jy = \emptyset.$$

(3) *X is strictly convex if and only if J is a strictly monotone operator, i.e.,*

$$x \neq y,\ j_x \in Jx,\ j_y \in Jy \quad \Rightarrow \quad \langle x - y, j_x - j_y \rangle > 0.$$

(4) *if X is reflexive, then J is a mapping of X onto X^*.*

(5) *if X^* is strictly convex (resp., smooth), then X is smooth (resp., strictly convex). Further, the converse is satisfied if X is reflexive.*

(6) *if X has a Fréchet differentiable norm, then J is norm-to-norm continuous.*

(7) *if X has a uniformly Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on each bounded subset of X.*

(8) *if X is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of X.*

Finally, we shall add the well-known properties between X and its dual X^*.

(9) X is uniformly convex if and only if X^* is uniformly smooth.

(10) X is reflexive, strictly convex, and has the Kadec-Klee property if and only if X^* has a Fréchet differentiable norm.

3 Demiclosedness for nonlinear mappings

Recall that a Banach space X is said to satisfy *Opial's condition* [25] if whenever a sequence $\{x_n\}$ in X converges weakly to x_0, then

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|, \qquad (x \neq x_0).$$

It is well known [16] that L^p spaces, $p \neq 2$, do not satisfy Opial's condition while all the ℓ^p spaces do ($1 < p < \infty$). Thus Opial's condition is independent of uniform convexity.

Spaces which satisfy Opial's condition have another nice property related to fixed point theory. Also, a function $f : D \subset X \to X$ is *demiclosed* at w if

$$x_n \rightharpoonup x, \quad \|f(x_n) - w\| \to 0 \quad \Rightarrow \quad x \in D, \quad f(x) = w.$$

The following theorem was well known; for an example, see Theorem 10.3 in [16].

Theorem 3.1. ([16]) *Let C be a nonempty closed convex subset of a reflexive Banach space X satisfying Opial's condition and let $T : C \to X$ be nonexpansive. Then $f = I - T$ is demiclosed on C.*

For the demiclosedness principle on uniformly convex spaces, We need the following useful lemmas; see Proposition 10.2 in [16].

Lemma 3.2. ([16]) *Let C be a bounded closed convex subset of a uniformly convex space X, and let $T : K \to X$ be nonexpansive such that* $\inf\{\|x - Tx\| : x \in K\} = 0$. *Then $F(T) \neq \emptyset$.*

Lemma 3.2 is a crucial tool to prove the following well known demiclosedness principle for nonexpansive mappings on uniformly convex Banach spaces; see Theorem 10.4 in [16] or [6].

Theorem 3.3. (Demiclosedness Principle; see [16] or [6]) *Let C be a nonempty closed convex subset of a uniformly convex space X and let $T : K \to X$ be nonexpansive. Then $f = I - T$ is demiclosed on C.*

We need the following notations.

$$\triangle^{n-1} = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \geq 0, \sum \lambda_i = 1\}.$$

and $\phi \in \Gamma_c$ if and only if $\phi \in \Gamma(\mathbb{R}^+)$ and ϕ is convex.

Recall that $T : C \to X$ is said to be *of type (γ)* [3] if $\gamma \in \Gamma_c$ and

$$\gamma(\|cTx + (1-c)Ty - T(cx + (1-c)y)\|) \leq \|x - y\| - \|Tx - Ty\| \tag{3.1}$$

for all $x, y \in C$ and $c \in [0, 1]$.

Remark 3.4. (a) Every type (γ) mapping is nonexpansive, and every affine nonexpansive mapping is of type (γ); but not every nonexpansive mapping is of type (γ) because $F(T)$ is obviously convex by (3.1) if $T : C \to X$ is of type (γ).

(b) Note that if T is nonexpansive, $F(T)$ is generally not convex; let us recall an example due to DeMarr [11]. Let X be the space of all ordered pairs (a, b) of real numbers. Define $\|x\| = \max\{|a|, |b|\}$ for $x = (a, b) \in X$. For $C := \{x : \|x\| \leq 1\}$, define $T : C \to C$ by

$$Tx = (|b|, b) \quad \forall x = (a, b) \in C.$$

Then T is nonexpansive because

$$\|Tx - Ty\| = \|(|b|, b) - (|d|, d)\| = |b - d| \le \max\{|a - c|, |b - d|\} = \|x - y\|$$

for all $x = (a, b)$ and $y = (c, d)$ in C. However, note that $x = (1, 1) < y = (1, -1) \in F(T)$ but $\frac{1}{2}(x + y) = (1, 0) \notin F(T)$.

(c) If X is uniformly convex and C is a *bounded* closed convex subset of X, there exists $\gamma \in \Gamma_c$ such that every nonexpansive mapping is of type (γ); moreover, γ can be chosen to depend only on $\mathrm{diam}(C)$ and not on T; see Lemma 1.1 in [3].

(d) If $T : C \to X$ is of type (γ), then $f = I - T$ is demiclosed on C; see Lemma 1.3 in [3].

Now recall the following subsequent results due to Bruck [4]; see Lemma 2.1 of [4] for the second lemma.

Lemma 3.5. ([4]) *Let C be a nonempty bounded closed convex subset of a uniformly convex X. Then there exists $\gamma \in \Gamma_c$ such that*

$$\|T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i T x_i\|) \le L\gamma^{-1}\Big(\max_{1 \le i,j \le n}(\|x_i - x_j\| - L^{-1}\|Tx_i - Tx_j\|)\Big) \quad (3.2)$$

for any Lipschitzian mapping $T : C \to X$ with its Lipschitz constant $L \ge 1$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$ and $x_1, \dots, x_n \in C$.

Lemma 3.6. ([4]) *Let C be a nonempty bounded closed convex subset of a uniformly convex X, $\gamma \in \Gamma_c$, and let $T : C \to X$ be of type (γ). Then there exists $\gamma_p \in \Gamma_c$ such that*

$$\gamma_p(\|T(\sum \lambda_i x_i) - \sum \lambda_i T x_i\|) \le \max_{1 \le i,j \le p}(\|x_i - x_j\| - \|Tx_i - Tx_j\|)$$

for $\lambda = (\lambda_1, \dots, \lambda_p) \in \Delta^{p-1}$ and $x_1, \dots, x_p \in C$.

Using Lemma 3.5 (Bruck), Xu [36] also established the following subsequent results for asymptotically nonexpansive mappings; see Theorem 2 of [36], Lemma 2.3 of [32], respectively.

Theorem 3.7. ([36]) *Let C be a nonempty bounded closed convex subset of a uniformly convex space X and let $T : C \to C$ ne a asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero.*

Theorem 3.8. ([32]) *Let C be a nonempty bounded closed convex subset of a uniformly convex space X and let $T : C \to C$ ne a asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero in the sense that whenever $x_n \rightharpoonup x$ and*

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \|x_n - T^k x_n\| = 0$$

it follows that $x = Tx$.

In 2001, Chang et all [8] removed the assumption of *boundedness* of C in Theorem 3.7; see Theorem 1 of [8].

Theorem 3.9. ([8]) *Let C be a nonempty closed convex subset of a uniformly convex space X and let $T : C \to C$ ne a asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero.*

More generally, we easily observe the following demiclosedness principle for asymptotically nonexpansive mappings.

Theorem 3.10. *Let C be a nonempty closed convex subset of a uniformly convex space X and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero in the sense that whenever $x_n \rightharpoonup x$ and*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0 \tag{3.3}$$

it follows that $x = Tx$.

Proof. Let $x_n \rightharpoonup x$ and $\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0$. Then, since $\{x_n\}$ is bounded, $\exists\ r > 0$ such that $\{x_n\} \subset K := C \cap B_r$, where B_r denotes the closed ball of X with center 0 and radius r. Then K is a nonempty bounded closed convex subset in C. For arbitrary $\epsilon > 0$, choose k_0 such that

$$\limsup_{n\to\infty} \|x_n - T^k x_n\| < \epsilon, \quad k \geq k_0 \tag{3.4}$$

by (3.15). Since $x \in \overline{co}(\{x_n\})$, for each $n \geq 1$, we can also choose a convex combination

$$y_n := \sum_{i=1}^{m(n)} \lambda_i^{(n)} x_{i+n}, \ \lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_{m(n)}^{(n)}) \in \Delta^{m(n)-1}$$

such that

$$\|y_n - x\| < \frac{1}{n}. \tag{3.5}$$

Now for any (fixed) $k \geq k_0$, using (3.4), we can choose n_0 such that

$$\|x_n - T^k x_n\| < \epsilon, \quad n \geq n_0. \tag{3.6}$$

Since $T^k : K \to X$ is AN (hence Lipschitzian with its Lipschitz constants $L_k := 1 + c_k$), use to Bruck's Lemma 3.5 (with $d :=$diam K) to derive

$$\begin{aligned}
&\|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\| \\
\leq\ & L_k \gamma^{-1}\big(\max_{1\leq i.j\leq m(n)} (\|x_{i+n} - x_{j+n}\| - L_k{}^{-1}\|T^k x_{i+n} - T^k x_{j+n}\|)\big) \\
\leq\ & L_k \gamma^{-1}\Big(\max_{1\leq i.j\leq m(n)} (\|x_{i+n} - T^k x_{i+n}\| + \|x_{j+n} - T^k x_{j+n}\| \\
& +(1 - L_k{}^{-1})\|T^k x_{i+n} - T^k x_{j+n}\|)\Big) \\
\leq\ & L_k \gamma^{-1}(2\epsilon + (1 - L_k{}^{-1})d) \qquad \text{by (3.6).}
\end{aligned}$$

Also, for $k \geq k_0$ and $n \geq n_0$, it follows that

$$\begin{aligned} & \|T^k y_n - y_n\| \\ \leq \; & \|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\| + \sum_{i=1}^{m(n)} \lambda_i^{(n)} \|T^k x_{i+n} - x_{i+n}\| \\ \leq \; & L_k \gamma^{-1}\big(2\epsilon + (1 - L_k{}^{-1})d\big) + \epsilon \qquad \text{by (3.6) again.} \end{aligned}$$

Taking $\limsup_{n\to\infty}$ firstly on both sides, we have

$$\limsup_{n\to\infty} \|T^k y_n - y_n\| \leq L_k \gamma^{-1}\Big(2\epsilon + (1 - L_k{}^{-1})d\Big) + \epsilon \tag{3.7}$$

for all $k \geq k_0$. Therefore, for $k \geq k_0$,

$$\begin{aligned} \|T^k x - x\| &\leq \|T^k x - T^k y_n\| + \|T^k y_n - y_n\| + \|y_n - x\| \\ &\leq (1 + L_k)\|y_n - x\| + \|T^k y_n - y_n\| \end{aligned}$$

By virtue of (3.7) and $\|y_n - x\| \to 0$, we see

$$\|T^k x - x\| \leq L_k \gamma^{-1}\Big(2\epsilon + (1 - L_k{}^{-1})d\Big) + \epsilon \to 0$$

as $k \to \infty$ and $\epsilon \to 0$. This shows $x = \lim_{k\to\infty} T^k x$. Hence $Tx = x$ by continuity of T. The proof is complete. □

Recall that X is said to satisfy the *uniform Opial property* [28] if for each $c > 0$, $\exists r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\| \tag{3.8}$$

for each $x \in X$ with $\|x\| \geq c$ and each weakly null sequence $\{x_n\}$ in X with $\liminf_{n\to\infty} \|x_n\| \geq 1$.

Remark 3.11. (a) It suffices to take the weakly null sequence $\{x_n\}$ with $\|x_n\| = 1$ for all $n \geq 1$ instead of asking that $\liminf_{n\to\infty} \|x_n\| \geq 1$ in (3.8).

(b) We can substitute both lim inf by lim sup in (3.8).

Proof. (a) Let $x \in X$ with $\|x\| \geq c$ and $\liminf_n \|x_n\| \geq 1$. Assume $\liminf_n \|x + x_n\| = \lim_m \|x + x_m\|$ for some subsequence $\{m\}$ of $\{n\}$. Also, assume without loss of generality that $\liminf_m \|x_m\| = \lim_k \|x_{m_k}\| = 1$; put $y_k := x_{m_k}/\|x_{m_k}\|$ for all sufficient large $k \geq k_0$ (otherwise, i.e., if $d := \liminf_m \|x_m\| > 1$, consider $z_m := x_m/d$; put $y_k := z_{m_k}/\|z_{m_k}\| = x_{m_k}/\|x_{m_k}\|$). Since $\{y_k\}$ is a weakly null sequence with $\|y_k\| = 1$ for all $k \geq k_0$, it follows from assumption that

$$\begin{aligned} 1 + r &\leq \liminf_{k\to\infty} \|x + y_k\| \\ &= \liminf_{k\to\infty} \Big\|(x + x_{m_k}) - \Big(1 - \frac{1}{\|x_{m_k}\|}\Big) x_{m_k}\Big\| \\ &\leq \liminf_{k\to\infty} \|x + x_{m_k}\| + \limsup_{k\to\infty} \Big|1 - \frac{1}{\|x_{m_k}\|}\Big| \cdot \|x_{m_k}\| \\ &= \lim_{k\to\infty} \|x + x_{m_k}\| = \lim_{m\to\infty} \|x + x_m\| = \liminf_{n\to\infty} \|x + x_n\|. \end{aligned}$$

Hence (3.8) is required.

(b) Given $c > 0$, $\exists r > 0$ such that the inequality (3.8) replaced with $\limsup_n$ is satisfied. Let $x \in X$ with $\|x\| \geq c$ and $\|x_n\| = 1$ for all $n \geq 1$. Assume $\liminf_n \|x + x_n\| = \lim_k \|x + x_{n_k}\|$ for some subsequence $\{n_k\}$ of $\{n\}$. Then it follows from (a) (with $y_k := x_{n_k}$) and hypothesis that

$$\begin{aligned} 1 + r &\leq \limsup_{k\to\infty} \|x + y_k\| \\ &= \lim_{k\to\infty} \|x + x_{n_k}\| = \liminf_{n\to\infty} \|x + x_n\|. \end{aligned}$$

Hence (3.8) is satisfied with $\liminf_n$. □

Recall also that X satisfies the liminf-locally uniform Opial condition (in brief, liminf-LUO) [19] if for any weakly null sequence $\{x_n\}$ in X with $\liminf_{n\to\infty} \|x_n\| \geq 1$ and any $c > 0$, $\exists r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\| \tag{3.9}$$

for all $x \in X$ with $\|x\| \geq c$.

Definition 3.12. ([13]) A Banach space X has the limsup-LUO if for any weakly null sequence $\{x_n\}$ in X with $\limsup_{n\to\infty} \|x_n\| \geq 1$ and any $c > 0$, $\exists r > 0$ such that (3.8) replaced with $\limsup_n$ holds for all $x \in X$ with $\|x\| \geq c$.

Remark 3.13. Note that (UO) $\Rightarrow$ (liminf-LUO) $\Rightarrow$ (limsup-LUO). But the converse implications don't remain true in general. Consider $X = (\sum_{i=2}^{\infty} \ell_i)_{\ell_2}$. Then $X =$ (limsup-LUO) but it lacks (UO); see [37]. Also, by [13], $X \neq$ (liminf-LUO). If we take $X = (\sum_{i=2}^{\infty} \ell_i)_{\ell_1}$, it has liminf-LUO but not (UO); see [13].

For the following lemma, see Lemma 2.3 of Oka [23] or Lemma 1.5 of [38].

Lemma 3.14. *Let C be a nonempty bounded closed convex subset of a uniformly convex space X and let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. For each $\epsilon > 0$, $\exists K_\epsilon > 0$ and $\delta_\epsilon > 0$ such that if $k \geq K_\epsilon$, $z_1, \ldots, z_n \in C$ $(n \geq 2)$ and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\epsilon$ for $1 \leq i, j \leq n$, then*

$$\Big\|T^k\Big(\sum_{i=1}^{n} t_i z_i\Big) - \sum_{i=1}^{n} t_i T^k z_i\Big\| \leq \epsilon$$

for all $t = (t_1, \ldots, t_n) \in \Delta^{n-1}$.

Using Lemma 3.14, Yang, Xie, Peng and Hu [38] recently proved the following demiclosedness principle of $I - T$ for a mapping T which is asymptotically nonexpansive in the intermediate sense.

Theorem 3.15. ([38]) *Let C be a nonempty closed convex subset of a uniformly convex space X and let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that whenever $x_n \rightharpoonup x$ and*

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$$

it follows that $x = Tx$.

Remark 3.16. Note that Theorem 3.15 is the slight modification of Lemma 2.5 in [23].

Here we give an easy example of an asymptotically nonexpansive mapping which is not nonexpansive.

Example 3.17. *Let $X = \mathbb{R}$, $C = [0, 1]$, and $1/2 < k < 1$. For each $x \in C$, define*

$$Tx = \begin{cases} kx, & \text{if } 0 \le x \le 1/2; \\ \frac{-k}{2k-1}(x-k), & \text{if } 1/2 \le x \le k; \\ 0, & \text{if } k \le x \le 1. \end{cases}$$

Then $T : C \to C$ is asymptotically nonexpansive but not nonexpansive.

Now we shall give the demiclosedness principle of $I - T$ for a TAN mapping T. We first begin with following slight modification of Lemma 2.1 in [38].

Lemma 3.18. *Let C be a nonempty closed convex subset of a uniformly convex X and let $T : C \to C$ be a TAN mapping and let K a nonempty bounded closed convex subset of C. Then, for each $\epsilon > 0$, $\exists N_\epsilon \ge 1$ and $\delta_{2,\epsilon}$ with $0 < \delta_{2,\epsilon} \le \epsilon$ such that if $k \ge N_\epsilon$, $x_1, x_2 \in K$ and if $\|x_1 - x_2\| - \|T^k x_1 - T^k x_2\| \le \delta_{2,\epsilon}$, then*

$$\left\| T^k(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T^k x_1 - \lambda_2 T^k x_2 \right\| \le \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$.

Proof. We employ the method of the proof in [23]. Since X is uniformly convex, the modulus of convexity δ is a continuous and strictly increasing function on $[0, 2]$ (see [16] for more details). Then the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$F(x) = \begin{cases} \frac{1}{2}\int_0^x \delta(t)dt, & \text{if } 0 \le x \le 2; \\ \frac{1}{2}(x-2) + F(2), & \text{if } x > 2. \end{cases}$$

is clearly strictly increasing, continuous and convex on $\mathbb{R}^+$. Obviously, since $F(x) \le \delta(x)$ $(0 \le x \le 2)$, the uniform convexity of X implies that

$$2\lambda_1\lambda_2 F(\|x - y\|) \le 1 - \|\lambda_1 x + \lambda_2 y\| \tag{3.10}$$

for $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$, $\|x\| \le 1$ and $\|y\| \le 1$. If either λ_1 or λ_2 is 1 or 0, our conclusion is clearly satisfied. So assume that $0 < \lambda_1, \lambda_2 < 1$ and let $\epsilon > 0$ be arbitrary given. Set

$$M := \operatorname{diam} K \vee \sup_{x,y \in K} \phi(\|x - y\|) < \infty.$$

(Note that $\sup_{x,y\in K} \phi(\|x - y\|) \le \phi(\operatorname{diam} K)$ because ϕ is strictly increasing.)

Choose $d_\epsilon > 0$ such that $\frac{M}{2}F^{-1}\left(\frac{2d_\epsilon}{M}\right) < \epsilon$ and put $\delta_{2,\epsilon} = \min\left\{\epsilon, d_\epsilon, \frac{M}{4}\right\}$. For $\bar{\delta}_{2,\epsilon} = \min\{\lambda_i \delta_{2,\epsilon} : i = 1, 2\} > 0$, since c_n, $d_n \to 0$, there exists an integer $N_\epsilon \ge 1$ (depending on the set K) such that if $k \ge N_\epsilon$,

$$c_k < \bar{\delta}_{2,\epsilon}/2M \quad \text{and} \quad d_k < \bar{\delta}_{2,\epsilon}/2.$$

Then, by (1.6), we have

$$\begin{aligned} \|T^k x - T^k y\| &\leq \|x-y\| + c_k\,\phi(\|x-y\|) + d_k \\ &\leq \|x-y\| + c_k\,M + d_k \\ &< \|x-y\| + \bar{\delta}_{2,\epsilon} \end{aligned} \tag{3.11}$$

for all $k \geq N_\epsilon$, $x, y \in K$. Now let $k \geq N_\epsilon$ and let x_1, $x_2 \in K$ with $\|x_1 - x_2\| - \|T^k x_1 - T^k x_2\| \leq \delta_{2,\epsilon}$. On letting

$$x := \frac{T^k x_2 - T^k(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1(\|x_1 - x_2\| + \delta_{2,\epsilon})} \text{ and } y := \frac{T^k(\lambda_1 x_1 + \lambda_2 x_2) - T^k x_1}{\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon})},$$

we have $\|x\| \leq 1$, $\|y\| \leq 1$ by with help of (3.11) and

$$\lambda_1 x + \lambda_2 y = \frac{T^k x_2 - T^k x_1}{\|x_1 - x_2\| + \delta_{2,\epsilon}}.$$

From these facts, on letting

$$0 < t := \frac{2}{M}\lambda_1\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon}) \leq \frac{2}{M}\frac{1}{4}\left(M + \frac{M}{4}\right) < 1,$$

we have

$$\frac{2}{M}\|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k(\lambda_1 x_1 + \lambda_2 x_2)\| = t\|x - y\| \tag{3.12}$$

and

$$\begin{aligned} \frac{1}{2\lambda_1\lambda_2}(1 - \|\lambda_1 x + \lambda_2 y\|) &= \frac{\|x_1 - x_2\| - \|T^k x_1 - T^k x_2\| + \delta_{2,\epsilon}}{2\lambda_1\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon})} \\ &\leq \frac{2\delta_{2,\epsilon}}{tM}. \end{aligned} \tag{3.13}$$

Using (3.10), (3.12), (3.13) and the convexity of F with $F(0) = 0$, we have

$$\begin{aligned} & F\left(\frac{2}{M}\|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k(\lambda_1 x_1 + \lambda_2 x_2)\|\right) \\ = \;& F(t\|x-y\|) = F\big(t\|x-y\| + (1-t)0\big) \\ \leq \;& tF(\|x-y\|) + (1-t)F(0) \\ = \;& \frac{t}{2\lambda_1\lambda_2}(1 - \|\lambda_1 x + \lambda_2 y\|) \leq \frac{2\delta_{2,\epsilon}}{M} \leq \frac{2d_\epsilon}{M} \end{aligned}$$

and so we have

$$\|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k(\lambda_1 x_1 + \lambda_2 x_2)\| \leq \frac{M}{2}F^{-1}\left(\frac{2d_\epsilon}{M}\right) < \epsilon$$

from the choice of d_ϵ and the proof is complete. □

Now on mimicking Lemma 2.2 and 2.3 in Oka [23] we have the following result.

Lemma 3.19. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a TAN mapping and let K a bounded closed convex subset of C. Then, for $\epsilon > 0$ there exists an integers $N_\epsilon \geq 1$ and δ_ϵ with $0 < \delta_\epsilon \leq \epsilon$ such that $k \geq N_\epsilon$, $x_1, x_2, \cdots, x_n \in K$ and if $\|x_i - x_j\| - \|T^k x_i - T^k x_j\| \leq \delta_\epsilon$ for $1 \leq i, j \leq n$, then*

$$\left\| T^k \left(\sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i T^k x_i \right\| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \triangle^{n-1}$.

As a direct application of Lemma 3.19, we have the following demiclosedness principle for continuous TAN mapping.

Theorem 3.20. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a continuous TAN mapping. Then $I - T$ is demiclosed at zero in the sense that whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \,(\in C)$ and it satisfies (3.15), namely,*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0.$$

Then $x \in F(T)$.

Proof. First, we claim that $\lim_{k\to\infty} T^k x = x$. For this end, since $\{x_n\}$ is bounded in C, take the bounded set K in Lemma 3.19 by the closed convex hull of $\{x_n : n \geq 1\}$. For $\epsilon > 0$, take $N_\epsilon \geq 1$ and δ_ϵ with $0 < \delta_\epsilon \leq \epsilon$ as in Lemma 3.19. From (3.15), there exists an integer $k_0 \,(\geq N_\epsilon)$ such that

$$\limsup_{n\to\infty} \|x_n - T^k x_n\| < \delta_\epsilon/2 \qquad (k \geq k_0).$$

Also, we can choose an integer $n_0 \,(\geq k_0)$ such that

$$\|x_n - T^k x_n\| \leq \delta_\epsilon/2 \qquad (k, n \geq n_0). \tag{3.14}$$

Since $x_n \rightharpoonup x$ and $x \in \overline{co}\{x_i : i \geq n\}$ for each $n \geq 1$, we can choose for each $n \geq 1$ a convex combination

$$y_n = \sum_{i=1}^{m(n)} \lambda_i^{(n)} x_{i+n}, \quad \text{where } \lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_{m(n)}^{(n)}) \in \triangle^{m(n)-1}$$

such that $\|y_n - x\| \to 0$. Let $k, n \geq n_0$. Then it follows from (3.14) that, for $1 \leq i, j \leq m(n)$,

$$\begin{aligned} & \|x_{i+n} - x_{j+n}\| - \|T^k x_{i+n} - T^k x_{j+n}\| \\ \leq\ & \|x_{i+n} - T^k x_{i+n}\| + \|x_{j+n} - T^k x_{j+n}\| \\ \leq\ & \delta_\epsilon/2 + \delta_\epsilon/2 = \delta_\epsilon \end{aligned}$$

and so applying Lemma 3.19 yields

$$\left\|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\right\| < \epsilon$$

and hence

$$\begin{aligned} \|T^k y_n - y_n\| &\leq \left\|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\right\| + \left\|\sum_{i=1}^{m(n)} \lambda_i^{(n)} (T^k x_{i+n} - x_{i+n})\right\| \\ &< \epsilon + \delta_\epsilon/2 \leq (3/2)\epsilon \end{aligned}$$

for $k, n \geq n_0$. Since $\Im = \{T_n : C \to C\}$ is TAN on C, this implies that, for $k, n \geq n_0$,

$$\begin{aligned} \|T^k x - x\| &\leq \|T^k x - T^k y_n\| + \|T^k y_n - y_n\| + \|y_n - x\| \\ &\leq \|x - y_n\| + c_k\, \phi(\|x - y_n\|) + d_k + (3/2)\epsilon + \|y_n - x\| \\ &= 2\|y_n - x\| + c_k\, \phi(\|x - y_n\|) + d_k + (3/2)\epsilon. \end{aligned} \tag{3.15}$$

Taking the lim sup as $n \to \infty$ at first and next the lim sup as $k \to \infty$ in both sides of (3.15), we have $\limsup_{k\to\infty} \|T^k x - x\| \leq (3/2)\epsilon$ and since ϵ is arbitrary given, $T^k x \to x$. Therefore $x \in F(T)$ by continuity of T. The proof is complete. □

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